# Deformation constraints- Applications to solitions, D branes and spatial modulation

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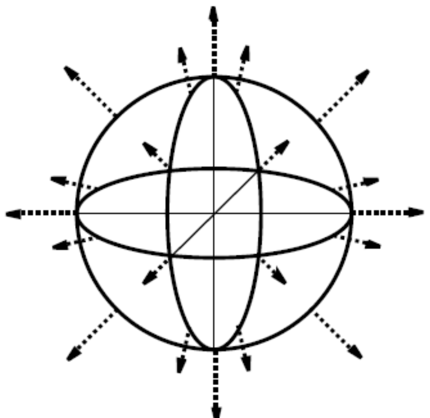
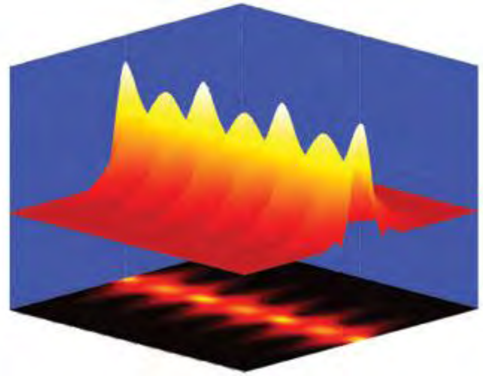
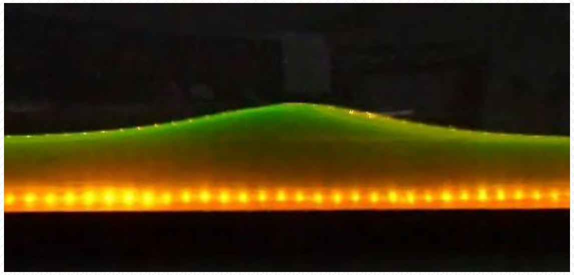
# Introduction

- **Solitons-** classical static configurations of finite energy show up in a wide range of physical systems
- Solitons are known for instance in hydrodynamics and non linear optics .
- In field theory we have encountered sine-Gordon solitons, ‘t Hooft Polyakov monopoles , Skyrmions and Instantons ( solitons of 5d YM theory)
- In recent years solitons take the form of Wilson-lines, Dbranes etc.







# Introduction

- Determining soliton solutions typically means solving non linear differntial equations.

- One would like to find tools to handle such configurations without solving for them explicitly.

- Two important issues are:

(i) **Existence proofs**

(ii) **Stability** of the solutions.

# Derrick theorem

- Consider a scalar field in d+1 dimensions with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi).$$

non negative,
vanishes for $\phi$=0

;

- The energy associated with a static configuration

$$E = \int d^dx \left[\frac{1}{2}(\nabla\phi(\vec{x}))^2 + V(\phi(\vec{x}))\right]$$

- Consider a scaling deformation ( not a symmetry)

$$\phi(x^i) \rightarrow \phi(\lambda x^i)$$

# Derrick theorem

- The energy of the rescaled configuration

$$E(\lambda) = \int d^d x \left[ \frac{1}{2} (\nabla \phi(\lambda \vec{x}))^2 + V(\phi(\lambda \vec{x})) \right]$$

- The minimum of the energy is for the un-rescaled soliton with λ=1

$$dE(\lambda = 1)/d\lambda = 0$$

$$d^2 E / d\lambda^2 > 0$$

# Derrick theorem

- We now change the integration variable $x^\mu \to x^\mu/\lambda$

- The re-scaled energy is

$$= \lambda^{-d} \int d^d x \left[ \frac{1}{2} \lambda^2 (\nabla \phi(\vec{x}))^2 + V(\phi(\vec{x})) \right]$$

- The variation of the energy has to obey

$$\frac{dE(\lambda)}{d\lambda}|_{\lambda=1} = \int d^d x \left[ \frac{1}{2}(d-2)(\nabla \phi(\vec{x}))^2 + dV(\phi(\vec{x})) \right] = 0$$

- For d>2 each term has to vanish separately and for d=2 the potential has to vanish. Both cases occur only for the vacuum.

- **Solitons can exist only for d=1**

# Manton's integral constraints

- For a static configuration the conservation of the energy-momentum tensor implies a spatial conservation of the stress tensor

$$\partial^j T_{ij} = 0.$$

- Define the vector

$$P_i = V^j T_{ij}$$

- Then

$$\int d^d x (\partial^i P_i) = \oint d^{d-1} \sigma^i P_i = \int d^d x (\partial^i V^j) T_{ij}$$

# Manton's integral constraints

- Let's take $V^j = A^j_k x^k$

- For this choice we get

$$A^j_k \int d^d x (\partial^i x^k) T_{ij} = A^{ji} \int d^d x T_{ij} = A^j_k \oint d^{d-1} \sigma^i x^k T_{ij}$$

- In particular when the surface term vanishes we get

**Manton integral constraint**

$$\boxed{\int d^d x T_{ij}(x) = 0}$$

# Introduction- questions

- The questions that we have explored are
- Can Derrick's theorem and Manton's integral constraints be unified?
- Can one generalize these constraints to other types of deformations?
- What are their implications on Solitons, Wilson lines, static solutions of gravity , D branes and spatially modulated configurations.

# Outline

- Part I- General formalism
- **A. Geometrical deformations** of solitons
- B. Deformations by global transformations
- C. Deformation, and stress forces of periodic solutions
- D. " Elasticity requirements" ( or minimizing and not only extreemizing)

# Outline

Part II- Applications-

- (i) Higher derivative actions and sigma models
- (ii) Current constraints on known solitons
- (iii) Solitons of non linear **(DBI)Electromagnetism**

# Outline

- (v) Constraints on D brane  and string actions
- (vii) Probe branes in brane backgrounds
- (vii) D3 brane with electric and magnetic fields
- (viii) Adding  Wess –Zumino terms
- (ix)Flavor branes in M- theory MQCD
-  (x) Application to the Ooguri Park  spatial modulation models

# Part I-

# General formalism

# (i) Geometrical deformations

- Consider a theory of several scalar fields $\phi^a$
- Take $\phi_0^a(x)$ to be a **soliton** with ( finite) energy

$$E[\phi_0^a] = \int d^d x\, \mathcal{E}(\phi_0^a, \partial_i \phi_0^a)$$

- We now deform the soliton

$$\phi_\Lambda^a(x) = \phi_0^a(\Lambda x)$$

- We expand the **geometrical deformation**

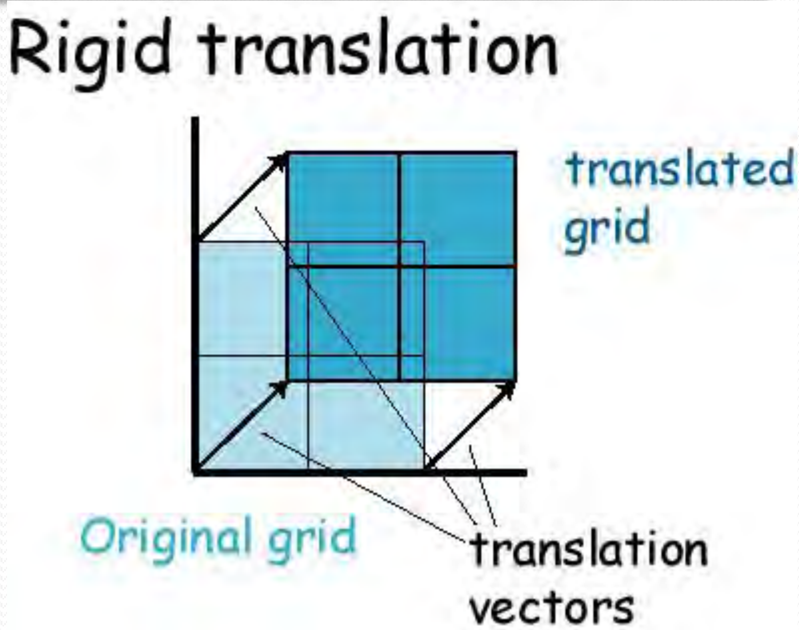
$$(\Lambda x)^i \simeq x^i + \xi^i(x).$$

- We take it to be linear

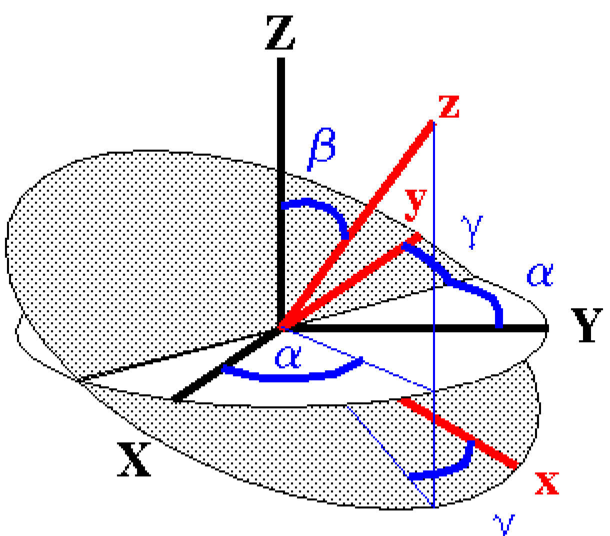
$$(\Lambda x)^i = \Lambda^i{}_j x^k + b^i \simeq x^i + \lambda^i{}_k x^k + b^i$$

# Geometrical deformations

- $b^i$ - rigid translations

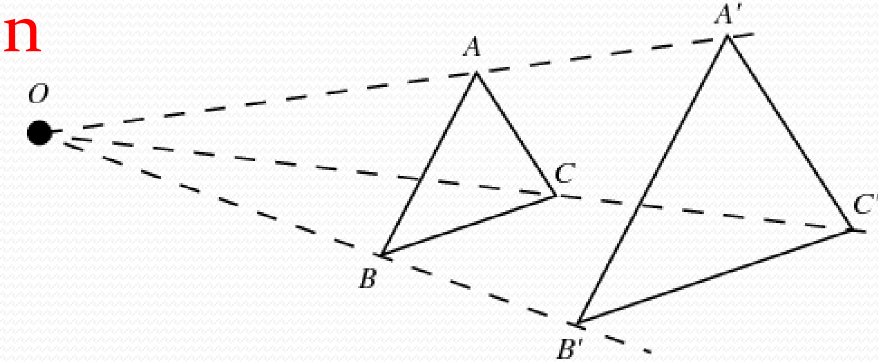



- antisymmetic $\lambda^i{}_j$ -rotations

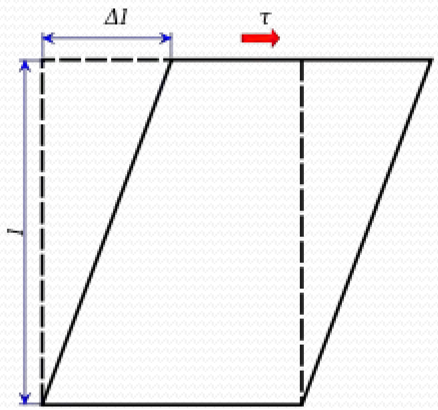


# Geometrical deformations

- diagonal $\lambda^i{}_j$ -dilatation (not necessarily isotropic)




- Symmetric $\lambda^i{}_j$ no diagonal components - shear



# Geometrical deformation

- The energy of the deformed soliton is

$$
\begin{aligned}
E[\phi^a_\Lambda] &= \int d^dx\, \mathcal{E}\left(\phi^a_\Lambda(x), \partial_i \phi^a_\Lambda(x)\right) \\
&= \int d^dx' \left|\left|\frac{\delta x^i}{\delta x^{j\prime}}\right|\right| \mathcal{E}\left(\phi^a_0(x'), \frac{\partial x^{j\prime}}{\partial x^i}\partial'_j \phi^a_0(x')\right) \\
&\simeq \int d^dx' \mathcal{E}\left(\phi^a_0(x'), \partial_i \phi^a_0(x')\right) + \int d^dx'\, \partial_i \xi^j \left[\delta^i{}_j \mathcal{E} - \frac{\delta \mathcal{E}}{\delta \partial_i \phi^a} \partial_j \phi^a_0\right] \\
&= E[\phi^a_0] - \int d^dx'\, \partial_i \xi^j \Pi^i{}_j(\phi^a_0).
\end{aligned}
$$

- The Stress tensor is

$$
\Pi^i{}_j = \frac{\delta \mathcal{E}}{\delta \partial_i \phi^a} \partial_j \phi^a - \delta^i_j \mathcal{E}
$$

# Geometrical deformation

- Thus the variation of the energy relates to the stress tensor as

$$E[\phi^a_\Lambda] - E[\phi^a_0] = \int d^d x \delta\mathcal{E} = -\int d^d x\, \partial_i \xi^j \Pi^i{}_j$$
$$= -\int d^d x\, \lambda^j{}_i \Pi^i{}_j$$

- For theories with scalars and no gauge fields $\mathcal{E} = -\mathcal{L}$
- Hence the stress and energy momentum tensors are related via $\Pi^i{}_j = -T^i{}_j = -\frac{\delta\mathcal{L}}{\delta\partial_i\phi^a}\partial_j\phi^a + \delta^i_j\mathcal{L}$

# Geometrical deformation

- Since $\lambda^i{}_j$ are arbitrary we get **Manton's integral conditions**

$$\int d^d x \, \Pi^i{}_j = 0$$

- More precisely we get

$$\int d^d x \partial_i \xi^j T^i{}_j = \oint d^{d-1} \sigma_i T^i{}_j \xi^j = \lim_{|x| \to \infty} \oint d^{d-1} x \hat{x}_i T^i{}_j \xi^j$$

- So that for vanishing surface term we get the constraint of vanishing integral of the stress tensor

# Geometrical deformation

- As is well known for Maxwell theory, the canonical energy momentum tensor is not gauge invariant and one has to add to it an improvement term

$$\mathcal{E} = T^{0}_{\mathrm{can}\,0} + \partial_i \Psi^{i0}{}_{0} \qquad T^{i}{}_{j} = T^{i}_{\mathrm{can}\,j} + \partial_k \Psi^{ki}{}_{j}$$

- Such that $\Psi^{\rho\mu}{}_{\nu} = -\Psi^{\mu\rho}{}_{\nu}$, which guarantees the conservation of the improved tensor
- For these cases we get that the variation of the energy

$$\delta\mathcal{E} = -\partial_i \xi^j \Pi^{i}{}_{j} = \partial_i \xi^j T^{i}{}_{j} - \partial_i \left( \delta\Psi^{i0}{}_{0} + \partial_k \Psi^{ki}{}_{j} \xi^j \right)$$

# Geometrical deformation

- For the modified case the integral constraint reads

$$\int d^d x\, \partial_i \xi^j \Pi^i{}_j = \int d^d x \partial_i \left[ -T^i{}_j \xi^j + \delta \Psi^{i0}{}_0 + \partial_k \Psi^{ki}{}_j \xi^j \right]$$
$$= \oint d^{d-1} \sigma_i \left[ -T^i{}_j \xi^j + \delta \Psi^{i0}{}_0 + \partial_k \Psi^{ki}{}_j \xi^j \right].$$

- Again when the surface term vanishes we get that the integral of the stress tensor vanishes

# BPS configurations and the vanishing of the stress tensor

- With right fall off we have

$$\int d^d x \, \Pi^i{}_j = 0$$

- What about the vanishing of the stress tensor itself?
- For 1+1 dim. solitions the virial theorem reads

$$V = \frac{1}{2}(\partial_1 \phi)^2$$

So the stress tensor

$$T_{11} = \frac{1}{2}(\partial_1 \phi)^2 - V = 0$$

# BPS configurations and the vanishing of the stress tensor

- This result can be related to a 1+1 supersymmetic model

$$S = \int d^2x \left( \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}F^2 + FW(\phi) - \frac{1}{2}W'(\phi)\bar{\psi}\psi \right)$$

for which $T_{01} = T_{10} = 0, \quad T_{11} = \frac{1}{2}(\partial_1\phi)^2 - \frac{1}{2}W^2 \quad = \frac{1}{2}(\partial_1\phi)^2 - V$

Supersymmetry relates the stress tensor $T_{ij}$ to the supercurrent

$$S^\mu_\alpha = (\gamma^\nu\partial_\nu\phi + iW)\gamma^\mu\psi$$

Via the susy Ward Identity

$$\{S^\mu_\alpha, \bar{Q}_\beta\} = 2i\gamma^\nu T_\nu{}^\mu + 2i\gamma_{3\alpha\beta}W'\epsilon^{\mu\nu}\partial_\nu\phi$$

From the fact that the BPS solutions are invariant under half of the supersymmetris → vanishing of $T_{ij}$
[Moreno Schaposnik]

# (ii) Deformations by global symmetry

- Suppose that our system is invariant under a global symmetry.
- The corresponding current conservation for static configurations reads $\partial_i J^i = 0.$
- Deforming the soliton $\delta_\theta \phi^a(x) = \theta^A T_{A}{}^{a}_{b}(\phi^b(x)).$

yields a variation of the energy

$$\delta_\theta E[\phi^a] = \int d^d x \, [\partial_i \theta^A J^i_A + \partial_i(\delta_\theta \Psi^{i0}{}_0)] = \int d^d x \, \partial_i \theta^A J^i_A$$

- For constant $\theta^A$ it is obviously a symmetry but again we take the transformation parameter

$$\theta^A = C^A_i x^i$$

# Deformations by global symmetry

- Thus we get the integral equation

$$\int d^d x \partial_i \theta^A J^i_A = \int d^d x \partial_i (\theta^A J^i_A) = \oint d^{d-1} \sigma_i \theta^A J^i_A$$

$$= \lim_{|x|\to\infty} \oint d^{d-1} x \hat{x}_i \theta^A J^i_A$$

- For vanishing surface term the integral of the space components of the global currents vanishes

$$\int d^d x \, J^i_A = 0.$$

# Deformations by global symmetry

- In order to have a finite surface integral the current should go as $\hat{x}_i J_a^i \sim \frac{1}{|x|^d}$
- At leading order for large radii the current reads

$$J_a^i = \frac{x^2 \delta^{ij} - d x^i x^j}{|x|^{d+2}} v_{aj} + \cdots \qquad J_a^i = \partial^i \left( \frac{x^j v_{aj}}{|x|^d} \right) = \partial^i \phi_a$$

- So there must be a massless mode $\partial^i \partial_i \phi_a = 0$
- This happens generically when the symmetry is spontaneously broken and the mode is the NG mode

# (iii)Geometric deformation of periodic solutions

- Apart from solitons there are also static solutions that break translational invariance but have divergent energy ( but finite energy density).
- The analysis of above does not apply but one can do a local analysis on some restricted region.
- For periodic configurations will take the unit cell
- The total force on the surface surrounding the unit cell of such a solid should be zero.
- The force on a face of the cell is

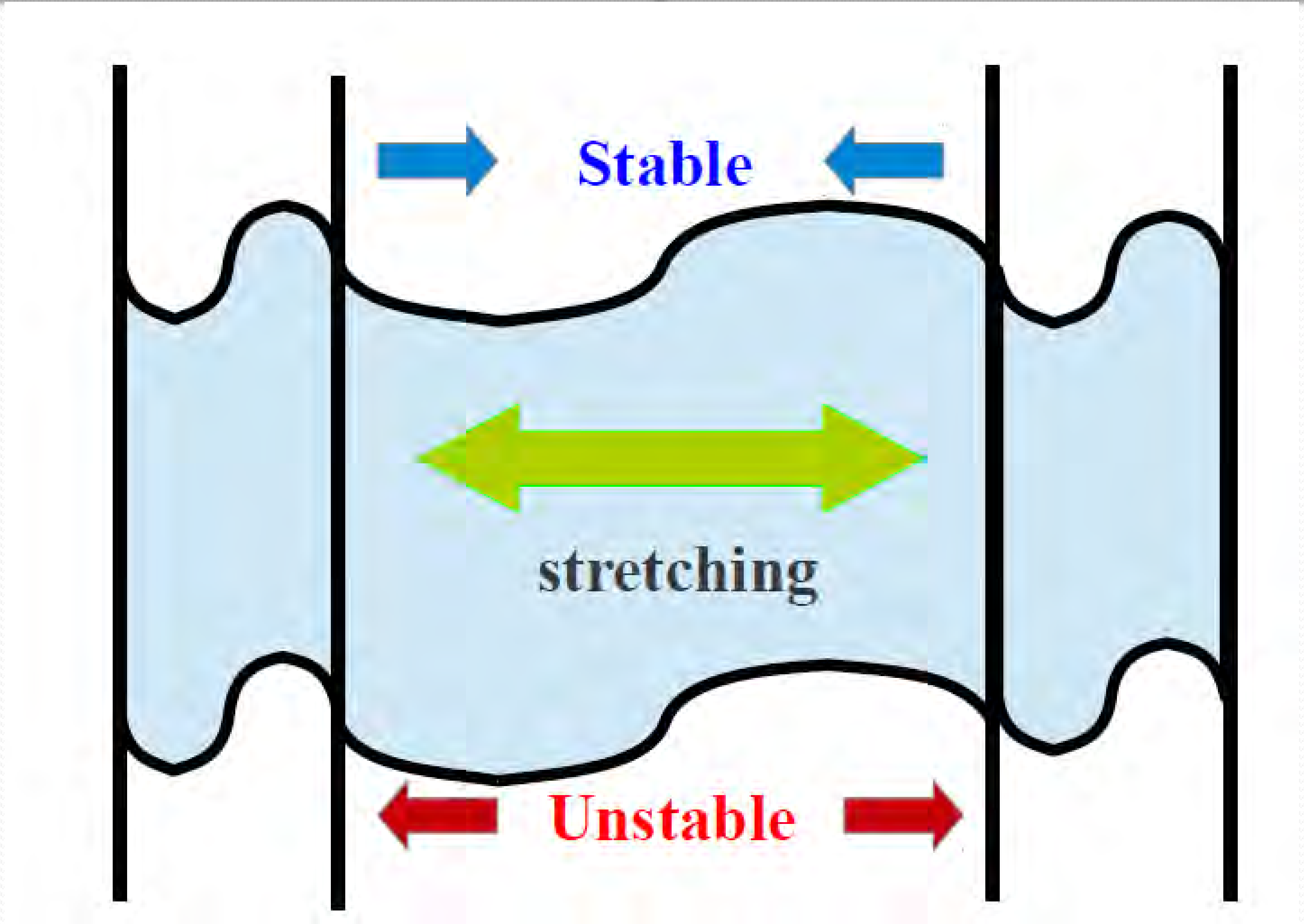
$$F_i = \int_{\text{face}} d^{d-1}\sigma f_i = -\int_{\text{face}} d^{d-1}\sigma_k T^k_{\ i}.$$

# Geometric deformation of periodic solutions

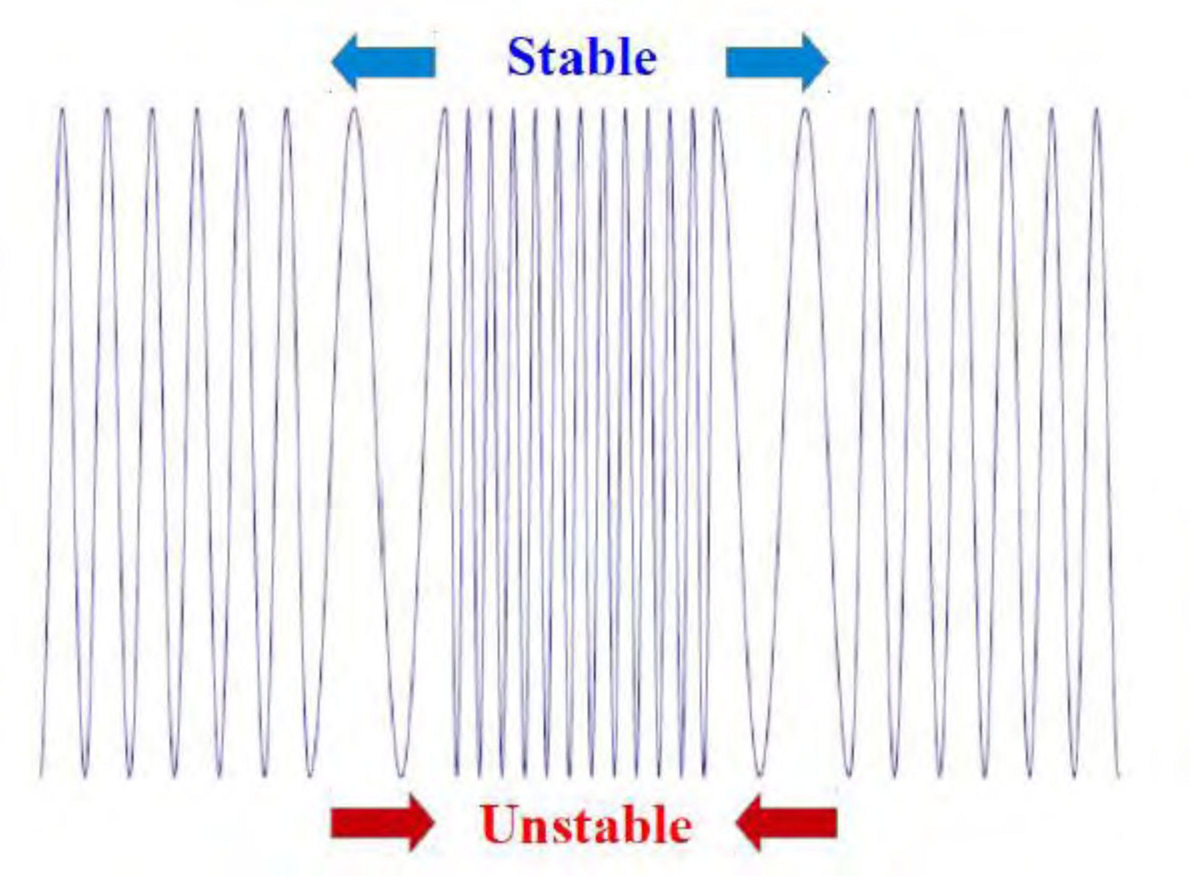
- We can increase the size of a unit cell and at the same time deform the neighboring cells so the periodic solution remains unchanged farther away.
- The forces on the faces of the unit cell no longer cancel:
- The net force on the surface after the transformation could be pointing

- (i) Out of the unit cell - unstable since the deformed cell will now continue increasing its size.
- (ii) Into the unit cell – restoring stability

# Geometric deformation of periodic solutions



# Geometric deformation of periodic solutions



# (iv) Elastic properties of inhomogeneous solutions

- We need to minimize and not only extreemize the energy.

- We vary the energy to second order.

- We use an analogy with elasticity theory and map the minimization to a positivity condition on the stiffness tensor.

# Elastic properties of inhomogeneous solutions

- In general in thermodynamics we have

$$d\varepsilon = T ds + \sigma^{ij} du_{ij}$$

Stress tensor

Strain $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$

- For ideal isotropic fluid $\sigma^{ij} = -p\delta^{ij}$ displacement vector

$$\sigma^{ij} = \left(\frac{\partial \varepsilon}{\partial u_{ij}}\right)_s$$

# Elastic properties of inhomogeneous solutions

- Hook's law for small deformations

$$\sigma^{ij} = C^{ijkl} u_{kl}$$

- The energy is minimized  if for any two unit vectors

a and b  the stiffness tensor obeys

$$C^{il}_{jm} \hat{a}_i \hat{a}_l \hat{b}^j \hat{b}^m > 0$$

# ``Elastic properties of inhomogeneous solutions

- Consider fluctuations of the coordinates

$$\tilde{x}^i = x^i - \xi^i(x),$$

- The variation of the energy of a scalar field theory to second order is

$$E[\phi^a_\Lambda] - E[\phi^a_0] = \frac{1}{2}\int d^d\tilde{x}\; C^{il}_{jm}\tilde{\partial}_i\xi^j\tilde{\partial}_l\xi^m$$

- The stiffness tensor is

$$C^{il}_{jm} = \delta^l_j\tilde{\partial}_m\phi^a\frac{\delta\mathcal{E}}{\delta\tilde{\partial}_i\phi^a} + \delta^i_m\tilde{\partial}_j\phi^a\frac{\delta\mathcal{E}}{\delta\tilde{\partial}_l\phi^a} + \left(\delta^i_j\delta^l_m - \delta^i_m\delta^l_j\right)\mathcal{E} + \tilde{\partial}_j\phi^a\tilde{\partial}_m\phi^b\frac{\delta^2\mathcal{E}}{\delta\tilde{\partial}_i\phi^a\delta\tilde{\partial}_l\phi^b}$$

# Elastic properties of inhomogeneous solutions

- For a gauge theory the stiffness tensor is

$$C^{il}_{jm} = \delta^l_j \pi^i{}_m + \delta^i_m \pi^l{}_j + \left(\delta^i_j \delta^l_m - \delta^i_m \delta^l_j\right) \mathcal{E}$$
$$+ \frac{\delta \mathcal{E}}{\delta F_{il}} F_{jm} + \frac{1}{2} \frac{\delta^2 \mathcal{E}}{\delta A_i \delta A_l} A_j A_m + 2 \frac{\delta^2 \mathcal{E}}{\delta F_{0i} \delta F_{0l}} F_{0j} F_{0m} + 2 \frac{\delta^2 \mathcal{E}}{\delta F_{in} \delta F_{ls}} F_{jn} F_{ms}$$

- Where

$$\pi^i{}_l = \frac{\delta \mathcal{E}}{\delta A_i} A_l + 2 \frac{\delta \mathcal{E}}{\delta F_{0i}} F_{0l} + 2 \frac{\delta \mathcal{E}}{\delta F_{ij}} F_{lj}$$

# Part II-
# Aplications

# 1. ( warm-up) Sigma models

- One can easily generalize Derrick's theorem to a case of a **sigma model**

$$\mathcal{L} = -\frac{1}{2} G^{ab}(\phi)\partial_\mu \phi_a \partial^\mu \phi_b - V(\phi_a)$$

- Repeating the procedure of above yields

$$\left.\frac{dE(\lambda)}{d\lambda}\right|_{\lambda=1} = -\int d^d x \left[\frac{1}{2}(d-2)G^{ab}\nabla\phi_a \nabla\phi_b + dV(\phi_a)\right] = 0$$

- When the signature of the metric $G^{ab}(\phi)$ is positive then the conclusions for the generalized case are the same .
- If the signature is not positive there is no constraints in any dimension.

# 2. Higher derivative actions

- Consider the **higher derivative** lagrangian density

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi)^2 + a(\partial_\mu\partial^\mu\phi)^2] - V(\phi)]$$

- The corresponding equations of motion

$$\partial_\mu\partial^\mu\phi - a\partial^2(\partial^2\phi) + \frac{\partial V(\phi)}{\partial\phi} = 0$$

- The conserved energy momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + a[\partial^2\phi\partial_\mu\partial_\nu\phi - \partial_\mu(\partial^2\phi)\partial_\nu\phi] - \eta_{\mu\nu}\mathcal{L}.$$

# Higher derivative actions

- The Hamiltonian of the static system

$$H = \int d^d x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}a(\nabla^2\phi)^2 + V(\phi)\right]$$

- Under isotropic re-scaling of the coordinates

$$x \to \lambda x$$

- Requiring extreemality for $\lambda = 1$

$$\frac{dH}{d\lambda}|_{\lambda=1} = \int d^d x \left[\frac{1}{2}(2-d)(\nabla\phi)^2 + \frac{1}{2}a(4-d)(\nabla^2\phi)^2 - dV(\phi)\right]$$

# Higher derivative actions

- The higher derivative terms thus ease the restriction on solitonic solutions for pure scalar field theories: we can get solitons for $d < 4$.
- That's the mechanism in the Skyrme model
- Generalizing this result to any higher order derivative Lagrangian density, where the derivative terms are quadratic in the fields of the form

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)^2 + \sum_n^N a_n (\partial^n \phi)^2] - V(\phi)],$$

- Now the constraint in principle allows solitons for any dimension $d < 2N$.

# 3. integral of the current constraint

- Let's examine this constraints on familiar solitons for: Topological currents, global and local currents.
- Topological currents are conserved without the use of equations of motion.
- The general structure of these currents in d space dimensions

$$J_\mu = \epsilon_{\mu\nu_1\ldots\nu_d}\tilde{J}^{\nu_1\ldots\nu_d} \qquad J =^* F_d$$

is a tensor of degree d composite of the underlying fields and their derivatives

- If the current is composed of only scalar fields, abelian or non-abelian, the spatial components have to include a time derivative
- So we conclude that $J_i = 0 \qquad i = 1, \ldots d$

# The current constraint in the ‘t Hooft Polyakov monople

- The system is based on SO(3) gauge fields and iso-vector scalars described by

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \frac{1}{2} D_\mu \vec{\phi} \cdot D^\mu \vec{\phi} - \frac{1}{4} \lambda (|\phi|^2 - v^2)^2$$

- The SO(3) current is

$$J_\mu^a = \epsilon_{bc}^a (D_\mu \phi)^b \phi^c \qquad (D_\mu \phi)^a = \partial_\mu \phi^a - e \epsilon_{bc}^a A_\mu^b \phi^c$$

- The equations of motion

$$(D_\nu F^{\nu\mu})^a = J_\mu^a \qquad (D^\mu D_\mu \phi)_a = -\lambda \phi^a (|\phi^2| - v^2)^2$$

# The current constraint in the ‘t Hooft Polyakov monople

- The relevant ansatz for the classical configurations

$$\phi^a(\vec{r}) = H(evr)\frac{r^a}{er^2} \qquad A^{ia}(\vec{r}) = -\epsilon^{ai}_{j}\frac{r^j}{er^2}[1 - K(evr)]$$

- Asymptotically they behave as

$$H \sim evr \text{ and } K \sim evre^{-evr}$$

- Substituting the ansatz to the current

$$J^a_i = \left(\frac{H(evr)}{er^2}\right)^2 K(evr)\epsilon^a_{ij}r^j$$

- It is obvious that

$$\int d^3x J^a_i = 0$$

- The current constraint is obeyed

# The constraint on the axial current of the skyrme model

- The two flavor Skyrme model is invariant under both the $SU(2)$ vector and axial flavor global transformations.
- The currents read

$$J_\mu^{\frac{V}{A}} = \frac{iN_c}{8\pi}[(g^{-1}\partial_\mu g \pm g\partial_\mu g^{-1}) + \epsilon_{\mu\nu}(g^{-1}\partial^\nu g \mp g\partial^\nu g^{-1}]$$

- plus higher derivative corrections that follow from the Skyrme term.
- The space integral of the non-abelian (axial)current

$$J^a_{i\ ax} = \frac{1}{4}F_\pi \frac{B}{r^3}[(\tau_i - 3\vec{\tau}\cdot\hat{x}\hat{x}_i)\tau_a] + \ldots$$

is

$$\int d^3x J^a_{i\ ax} = -\frac{2}{3}F_\pi^3 B\pi\delta_i^a$$

In accordance with the fact that there is an SSB

# 4. Soliton of non linear electromagnetism

- Let us analyze the constraints on EM expressed in terms of a DBI action.
- We check first the ordinary Maxwell theory
- The energy is $E = \frac{1}{2}\int d^d x[(\vec{B})^2 + (\vec{E})^2]$.
- The scaling of E and B are $\vec{E} \to \lambda\vec{E}$ and $\vec{B} \to \lambda^2\vec{B}$.
- The scaled energy $E(\lambda) = \frac{1}{2}\int d^d x \lambda^{-d}[\lambda^4(\vec{B})^2 + \lambda^2(\vec{E})^2]$
- Derrick's condition

$$\frac{dE(\lambda)}{d\lambda}|_{\lambda=1} = \frac{1}{2}\int d^d x[(4-d)(\vec{B})^2 - (d-2)(\vec{E})^2] = 0$$

- **No solitons apart from d=3 for self duals** $\vec{E} = \pm\vec{B}$

# Soliton of DBI non- linear electromagnetism

- The DBI action of EM in d+1 =4 is

$$S_{DBI} = T_3 \int d^4x \left[1 - \sqrt{1 - (\vec{E})^2 + (\vec{B})^2 - (\vec{E}\cdot\vec{B})^2}\right]$$

- The associated energy density

$$\mathcal{E} = \sqrt{1 + (\vec{B})^2 + (\vec{D})^2 + (\vec{B}\times\vec{D})^2} - 1,$$

- Derrick's constraint

$$\frac{dE(\lambda)}{d\lambda}\Big|_{\lambda=1} = \int d^3x \left[3 - \frac{3 + 4(\vec{B})^2 - 6(\vec{E}\cdot\vec{B})^2 - 4(\vec{E})^2 + (\vec{B})^2((\vec{B})^2 - 2(\vec{E}\cdot\vec{B})^2)}{\sqrt{(1 - (\vec{E})^2 + (\vec{B})^2 - (\vec{E}\cdot\vec{B})^2)^3}}\right]$$

- **Electric and magnetic solitons are not excluded**

# 5. Constraints on string and D-brane actions

- If the generalized constraints are fulfilled by some string or D brane configuration it may indicate about possible states apart from the trivial ones.

- The constraints are based on comparing configurations with the same boundary conditions
- For finite volume ones, the variation may change the boundary conditions.

- Satisfying the constraints is not a proof of existence
- The constraints my exclude classes of solitons

# Constraints on string and D-brane actions

- The action of the low energy dynamics of D-branes

$$S_{DBI} = -T_p \int d^{p+1}x e^{-\phi} \sqrt{-\det\left(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} + B_{\mu\nu}\right)}$$

Dp brane tension — dilaton — induced metric — pull back of the NS form

- D-branes can also carry charge that couples to a RR flux. This corresponds to a WZ (CS) action

$$S_{WZ} = T_p \int d^{p+1}x \sum_k C_k \wedge e^{2\pi\alpha' F + B}.$$

pullback of the RR k-form

# Constraints on strings

- Similarly the NG action describes the fundamental string

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2x\sqrt{-\det\left(g_{\mu\nu}\right)}$$

- The string is charged under the NS two form

$$S_B = \frac{1}{2\pi\alpha'}\int d^2x B$$

- The induced metric is

$$g_{\mu\nu} = G_{MN}\partial_\mu X^M \partial_\nu X^N$$

$X^M(x),\ M = 1, \cdots, D$ the embedding coordinates

# Fixing diffeomorphism

- The brane ( string) action is invariant under diffeomorphism hence the constraints are trivially satisfied.
- For instance for the NG string the energy

$$E(\lambda) = -S_{NG} = \frac{1}{2\pi\alpha'}\int dtdx\sqrt{-G_{MN}\partial_x X^M(\lambda x)\partial_x X^N(\lambda x)}$$
$$= \frac{1}{2\pi\alpha'}\int dtdx'\lambda^{-1}\sqrt{-\lambda^2 G_{MN}\partial_{x'} X^M(x')\partial_{x'} X^N(x')} = E(1).$$

- To get non-trivial constraints we must gauge fixed the diffeomorphis invariance

# Fixing diffeomorphism

- We use the usual static gauge.
- We split the coordinates

$X^{\mu}(x) = x^{\mu}$ $X^{\mu}$ $(\mu = 0, 1, \ldots, p)$ -worldvolume

$X^{a}(x) = Y^{a}(x)$ $Y^{a}$ $(a = p+1, \ldots, D-1)$ -transverse

We impose

(i) space-time translation invariance on the worlvolume

(ii) $\partial_t Y^a = 0$ truly static

# Constraints on D brane action without gauge fields

- In the static gauge the pull-back metric reads

$$g_{00} = G_{00}, \quad g_{0i} = G_{0i}, \quad g_{ij} = G_{ij} + G_{ab}\partial_i Y^a \partial_j Y^b$$

- The energy is

$$E(\{\lambda_i\}) = T_p \int d^p x\, e^{-\epsilon\phi} \left(\prod_{i=1}^{p} \lambda_i^{-1}\right) \left[\sqrt{-\det(g[\{\lambda_i\}])} - \sqrt{-\det G}\right]$$

. $\epsilon = 1$ -Dbrane

. $\epsilon = 0$ -string

.

subtraction for

$\partial_i Y^a = 0$

E=0

# Constraints on D brane action without gauge fields

- Derrick's condition is now

$$0 = \frac{dE}{d\lambda_i} = -T_p \int d^p x \, e^{-\epsilon\phi} \left( \sqrt{-\det g} - \sqrt{-\det G} - \sqrt{-\det g} \sum_l g^{il}(g_{il} - G_{il}) \right)$$

- Where we have used

$$\frac{dg_{kl}}{d\lambda_i} = 2\delta^i_k \lambda_l G_{ab} \partial_i Y^a \partial_l Y^b = 2\delta^i_k \lambda_l (g_{il} - G_{il})$$

- After some algebra we find that the condition is

$$p - \mathrm{tr}(M^{-1}) + \delta > 0$$

$$M = I + \hat{G}^{-1}(Y \cdot Y) \qquad \delta = \frac{\hat{g}^{kl} G_{0k} G_{0l} - G_{ij} \hat{g}^{ik} \hat{g}^{jl} G_{0k} G_{0l}}{G_{00} - \hat{g}^{kl} G_{0k} G_{0l}}$$

- This can be obeyed so we can not exclude Dbrane solitons

# Constraints on D brane action without gauge fields

- However for $Y^a$ depending only on a single x

$$\left.\frac{dE}{d\lambda}\right|_{\lambda=1} = 0 = T_p \int d^p x \sqrt{\frac{-\det G}{1+\hat{G}^{xx} G_{ab}\partial_x Y^a \partial_x Y^b}} \left\{ \sqrt{1+\hat{G}^{xx} G_{ab}\partial_x Y^a \partial_x Y^b} - 1 \right\}$$

- Since for non-trivial $\partial_x Y^a$ the integrand is positive the constraints cannot be satisfied.
- **There are no solitons D-branes ( even for p=1) that depend on only one coordinate**

# Probe brane in Dp brane background

- The near horizon background has the metric

$$ds^2 = \left(\frac{R}{r}\right)^{(7-p)/2} dr^2 + \left(\frac{r}{R}\right)^{(7-p)/2} \eta_{\mu\nu} dx^\mu dx^\nu + R^2 \left(\frac{r}{R}\right)^{(p-3)/2} d\Omega_{8-p}^2$$

- The dilaton

$$e^\phi = e^{\phi_0} \left(\frac{R}{r}\right)^{(7-p)(3-p)/4}$$

- A RR form

$$C_{p+1} = \frac{1}{g_s} \left(\frac{r}{R}\right)^{7-p} dx^0 \wedge \cdots \wedge dx^p$$

- The DBI+ CS actions read

$$S = -T_p \int d^{p+1}x \left[e^{-\phi}\sqrt{-g} - C_{p+1}\right] = -\frac{T_p}{g_s} \int d^{p+1}x \left[\sqrt{-g} - \left(\frac{r}{R}\right)^{7-p}\right]$$

# Probe brane in Dp brane background

- Derrick's condition is now

$$0 = \left.\frac{dE}{d\lambda}\right|_{\lambda=1} = \int d^p x \left( -pL + \frac{r'^2}{\sqrt{1 + r^{p-7} r'^2}} \right)$$

$$L \to r^{7-p}\left[\sqrt{1 + r^{p-7} r'^2} - 1\right]$$

- The second derivative condition is

$$\left.\frac{d^2 E}{d\lambda^2}\right|_{\lambda=1} = \int d^p x \left( -(p-2)\frac{r'^2}{\sqrt{1 + r^{p-7} r'^2}} - \frac{r^{p-7} r'^4}{(1 + r^{p-7} r'^2)^{3/2}} \right)$$

- **There are no soliton solutions for any p that obey the stronger condition** of vanishing of the integrand.

# Generalized conditions for Branes with gauge fields

- When electric field is turned on the energy is not just $-L_{DBI}$ but rather the Legendre transform

$$E_{\rm can} = \int d^p x \, \frac{\delta S_{DBI}}{\delta \partial_0 A_i} \partial_0 A_i - S_{DBI}.$$

- It is convenient to define M such that

$$g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} = g_{\alpha\nu} M_{\mu}{}^{\alpha}. \qquad M_{\mu}{}^{\nu} = \delta_{\mu}^{\nu} + 2\pi\alpha' F_{\mu}{}^{\nu}$$

- The energy can be written as

$$E = \frac{T_p}{2} \int d^p x e^{-\phi} \sqrt{-\det g \det M} \left[ 1 + \frac{\det \hat{M}}{\det M} \right]$$

# Deformation constraints on D branes with gauge fields

- Rather than deriving Derrick's condition let's look this time on Manton's constraints

$$\int d^d x\, \Pi^i{}_j = 0.$$

- The explicit form of the stress tensor reads

$$\Pi^i{}_j = \mathcal{E}\left(g^{ik} G_{ab}\partial_k Y^a \partial_j Y^b - \delta^i{}_j\right)$$
$$+\frac{1}{2}\left[\frac{\delta S_{DBI}}{\delta F_{0l}}\left[1-\frac{\det \hat{M}}{\det M}\right]-2\frac{S_{DBI}}{\det M}\frac{\delta \det \hat{M}}{\delta F_{0l}}\right]\left((F_0{}^i\delta^k_l + F_0{}^k\delta^i_l)G_{ab}\partial_k Y^a\partial_j Y^b - \delta^i_l F_{0j}\right)$$
$$+\frac{1}{2}\left[\frac{\delta S_{DBI}}{\delta F_{nl}}\left[1-\frac{\det \hat{M}}{\det M}\right]-2\frac{S_{DBI}}{\det M}\frac{\delta \det \hat{M}}{\delta F_{nl}}\right]\left((F_n{}^i\delta^k_l + F_n{}^k\delta^i_l)G_{ab}\partial_k Y^a\partial_j Y^b - 2\delta^i_n F_{jl}\right)$$

# Adding the WZ terms

- Again like the DBI action we have first to gauge fix
- The pullback of the RR fields is

$$S_{WZ} = T_p \int d^{p+1}x \sum_{n=0}^{p} \frac{1}{(p-1+n)!} \epsilon^{\mu_1\cdots\mu_n\alpha_1\cdots\alpha_{p+1-n}} C_{\mu_1\cdots\mu_n a_1\cdots a_{p+1-n}} \partial_{\alpha_1} Y^{a_1} \cdots \partial_{\alpha_{p+1}} Y^{a_{p+1-n}}$$

- For instance for D1 brane the WZ action reads

$$S_{WZ,D1} = T_1 \int d^2x \left[ \frac{1}{2}\epsilon^{\mu\nu} C^{(2)}_{\mu\nu} + 2\frac{1}{2}\epsilon^{\mu\alpha} C^{(2)}_{\mu a} \partial_\alpha Y^a + \frac{1}{2}\epsilon^{\alpha\beta} C^{(2)}_{ab} \partial_\alpha Y^a \partial_\beta Y^b + \frac{1}{2} C^{(0)} \epsilon^{\mu\nu} \tilde{F}_{\mu\nu} \right]$$

$$= T_1 \int d^2x \left[ C^{(2)}_{01} + C^{(2)}_{0a} \partial_1 Y^a + C^{(0)} \tilde{F}_{01} \right], \qquad (4.89)$$

- The contribution to the stress tensor is

$$\Delta\Pi^1{}_1 = -T_1 \left[ C^{(2)}_{0a} \partial_1 Y^a + 2\pi\alpha' C^{(0)} F_{01} \right] + T_1 \left[ C^{(2)}_{01} + C^{(2)}_{0a} \partial_1 Y^a + 2\pi\alpha' C^{(0)} F_{01} \right] = T_1 C^{(2)}_{01}.$$

- In the absence of gauge fields $0 < \int dx\, \Pi^1{}_1$
- Hence we see again that **there is no D1 soliton solution**

# The D3 brane case

- For the D3 brane case the WZ term is

$$
\begin{aligned}
S_{WZ,D3} = T_3 \int d^4x \Big[ & C^{(4)}_{0123} + \frac{1}{2}\epsilon^{ijk} C^{(4)}_{0ajk}\partial_i Y^a + \frac{1}{2}\epsilon^{ijk} C^{(4)}_{0abk}\partial_i Y^a \partial_j Y^b \\
& + \frac{1}{3!}\epsilon^{ijk} C^{(4)}_{0abc}\partial_i Y^a \partial_j Y^b \partial_k Y^c + \frac{1}{2\cdot 3!}\epsilon^{ijk} C^{(2)}_{0i}\tilde{F}_{jk} + \frac{1}{2\cdot 3!}\epsilon^{ijk} C^{(2)}_{0a}\partial_i Y^a \tilde{F}_{jk} \\
& + \frac{1}{2\cdot 3!}\epsilon^{ijk} C^{(2)}_{ij}\tilde{F}_{0k} + \frac{1}{3!}\epsilon^{ijk} C^{(2)}_{ia}\partial_j Y^a \tilde{F}_{0k} + \frac{1}{2\cdot 3!}\epsilon^{ijk} C^{(2)}_{ab}\partial_i Y^a \partial_j Y^b \tilde{F}_{0k} + \frac{1}{2\cdot 3!} C^{(0)}\epsilon^{ijk}\tilde{F}_{0i}\tilde{F}_{jk} \Big]
\end{aligned}
$$

- The contribution of the WZ term to the stress tensor

$$
\begin{aligned}
\Delta\Pi^i{}_j = -T_3 \Big[ & \frac{1}{2}\epsilon^{ikl} C^{(4)}_{0akl}\partial_j Y^a + \epsilon^{ikl} C^{(4)}_{0abl}\partial_j Y^a \partial_k Y^b + \frac{1}{2}\epsilon^{ikl} C^{(4)}_{0abc}\partial_j Y^a \partial_k Y^b \partial_l Y^c \\
& + \frac{1}{3!}\epsilon^{ikl} C^{(2)}_{0l}\tilde{F}_{jk} + \frac{1}{3!}\epsilon^{ikl} C^{(2)}_{0a}\partial_l Y^a \tilde{F}_{jk} + \frac{1}{2\cdot 3!}\epsilon^{ikl} C^{(2)}_{0a}\partial_j Y^a \tilde{F}_{kl} \\
& + \epsilon^{kli} E_j \left( \frac{1}{2\cdot 3!} C^{(2)}_{kl} + \frac{1}{3!} C^{(2)}_{la}\partial_l Y^a + \frac{1}{2\cdot 3!} C^{(2)}_{ab}\partial_l Y^a \partial_l Y^b + \frac{1}{2\cdot 3!} C^{(0)}\tilde{F}_{kl} \right) \\
& + \epsilon^{kil} E_l \left( \frac{1}{3!} C^{(2)}_{ka}\partial_j Y^a + \frac{1}{3!} C^{(2)}_{ab}\partial_k Y^a \partial_j Y^b - \frac{1}{3!} C^{(0)}\tilde{F}_{jk} \right) \Big] - \delta^i{}_j \mathcal{E}_{WZ}.
\end{aligned}
$$

# Application to gravitational backgrounds

- Upon gauge fixing the diffeomorphism and parameterizing the metric the dilaon and fluxes we get an action of a bunch of scalar fields with a potential.
- In case that there is a dependence only on the radial direction it is a 1+1 dimensional action.
- Generically the ``kintic terms” are not positive definite.
- It turns out that the integrand of Derrick’s condition translates to the ``null energy condition”.
- Let’s demonstrate this

# Application to gravitational backgrounds

- Consider the DC on d brane solutions of gravity
- The bosonic part of the SUGRA action in D dimensions

$$S = \int d^D x \sqrt{G} e^{-2\phi} \left( R + 4(\partial\phi)^2 + \frac{c}{\alpha'} \right) - \frac{e^{-2\phi}}{2} \int H_{(3)} \wedge \star H_{(3)} - \sum_p \frac{1}{2} \int F_{(p+2)} \wedge \star F_{(p+2)},$$

$\frac{c}{\alpha'} = \frac{10-D}{\alpha'}$

$H_{(3)}$ is the NS three form

$F_{p+2}$ is a RR form

- We take the metric in the string frame

$$l_s^{-2} ds^2 = d\tau^2 + e^{2\lambda(\tau)} dx_{\|}^2 + e^{2\nu(\tau)} d\Omega_k^2$$

$D = n + k + 1$

$dx_{\|}^2$ is $n$ dimensional flat metric

$k$ dimensional sphere

# Application to gravitational backgrounds

- In terms of the metric fields and the dilaton

$$S = l_s^{-2}\int d\rho\left([-n(\lambda')^2 - k(\nu')^2 + (\varphi')^2 + ce^{-2\varphi} + (k-1)ke^{-2\nu-2\varphi}]\right.$$
$$\left. -Q_{RR}^2\rho e^{n\lambda - k\nu - \varphi} - Q_{NS}^2 e^{-2k\nu - 2\varphi}\right) \qquad d\tau = -e^{-\varphi}d\rho$$

- The ``null energy" condition which is a Gauss law associated with fixing $g_{\tau\tau} = 1$

- It is identical to the integrand of Derrick's condition

$$\left.\frac{dE(\lambda)}{d\lambda}\right|_{\lambda=1} = \int d\tau\left[n(\partial_\tau\lambda)^2 + k(\partial_\tau\nu)^2 - (\partial_\tau\varphi)^2 + c + (k-1)ke^{-2\nu}\right.$$
$$\left. -Q_{RR}^2 e^{n\lambda - k\nu + \varphi} - Q_{NS}^2 e^{-2k\nu}\right]$$

# Application to gravitational backgrounds

- Consider the following 1+1 dim model with N degrees of freedom

$$\mathcal{L} = \sqrt{g_0(x) + \sum_{i=1}^{N} g_i[(\dot{\phi}_i)^2 - \phi_i')^2] - V(\phi_i)}$$

- The extremum condition reads

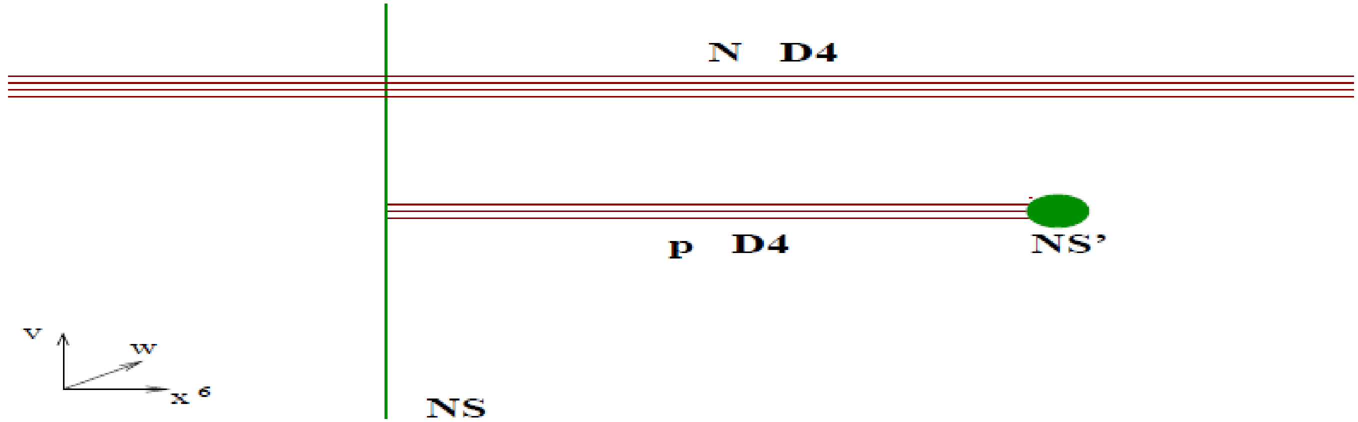
$$\frac{dE(\lambda)}{d\lambda}|_{\lambda=1} = \int dx \left\{ -\sqrt{g_0(\phi_i) - \sum_{i=1}^{N} g_i[(\phi_i')^2] + V(\phi_i)} - \frac{\sum_{i=1}^{N} g_i[(\phi_i')^2]}{\sqrt{g_0(x) - \sum_{i=1}^{N} g_i[(\phi_i')^2]}} \right\}$$

$$= \int dx \left\{ V(\phi_i) - \frac{g_0(\phi_i)}{\sqrt{g_0(x) - \sum_{i=1}^{N} g_i[(\phi_i')^2]}} \right\} = 0$$

- The integrand is just the energy of a 0+1 dim. where x is taken to be the time. **Thus the vanishing of the integrand is identical to the ``null energy condition**"

# Flavor branes in MQCD

- The type IIA brane configuration [Aharony,Kutasov,Lunin,Yankielowicz]




- Can be uplifted to M theory background

$$ds^2 = H^{-1/3}\left(dx_\mu^2 + dx_6^2 + dx_{11}^2\right) + H^{2/3}\left(|dv|^2 + |dw|^2 + dx_7^2\right)$$

$$C_6 = H^{-1} d^4x \wedge dx_6 \wedge dx_{11}, \qquad H = 1 + \frac{\pi \lambda_N l_s^2}{|\vec{r} - \vec{r}_0|^3},$$

# Flavor branes in MQCD

- The shape of the curved five brane

$$v = u(x_6) e^{i\phi(x_{11})} \sin\alpha(x_6), \qquad w = u(x_6) e^{-i\phi(x_{11})} \cos\alpha(x_6)$$

- The induced metric is

$$ds^2_{ind} = H^{-1/3} \left\{ dx_\mu^2 + \left[1 + H\left((u\alpha')^2 + (u')^2\right)\right] dx_6^2 + \left(1 + H(u\dot{\phi})^2\right) dx_{11}^2 \right\}$$

- The Lagrangian density

$$L = H^{-1}\sqrt{1 + H(u\dot{\phi})^2}\sqrt{1 + H((u\alpha')^2 + (u')^2)} - H^{-1}$$

# Flavor branes in MQCD

- The Neother charges associated with the shifts of x6 and $\alpha$ are

$$J = \frac{u^2\alpha'\sqrt{1+Hu^2/\lambda_p^2}}{\sqrt{1+H\left[(u\alpha')^2+(u')^2\right]}},$$

$$E = H^{-1} - \frac{H^{-1}\sqrt{1+Hu^2/\lambda_p^2}}{\sqrt{1+H\left[(u\alpha')^2+(u')^2\right]}}.$$

- Applying Derrick's condition yields

$$\frac{dE(\lambda)}{d\lambda}|_{\lambda=1} = \int dx\; H^{-1}\left\{1 - \frac{\sqrt{1+Hu^2/\lambda_p^2}}{\sqrt{1+H\left[(u\alpha')^2+(u')^2\right]}}\right\} = 0$$

- The integrand is identical to the Noether charge E thus the condition translates to `` **null energy condition**"

# Application to spatially modulated models

- Spatial modulation (S.M) was identified in YM+CS theory on an AdS5 black-hole

$$\mathcal{L} = \frac{\sqrt{-g}}{\alpha^2}\left[-\frac{1}{4}\tilde{F}_{IJ}\tilde{F}^{IJ} + \frac{1}{3!\sqrt{-g}}\epsilon^{IJKLM}\tilde{A}_I\tilde{F}_{JK}\tilde{F}_{LM}\right]$$

- The background metric is given by

$$ds^2 = -H(r)dt^2 + H(r)^{-1}dr^2 + r^2 d\vec{x}^2 \qquad \text{with} \qquad \vec{x} = (x_2, x_3, x_4)$$

- With the warp factor

$$H(r) = r^2\left[1 - \left(\frac{r_+}{r}\right)^4\right]$$

# Application to spatially modulated models

- The background electric field is given by

$$\lim_{r\to\infty} \tilde{F}_{0r} = \frac{E}{r^3} = -\frac{2r_+^3}{\pi\tau r^3}$$

- The spatially modulated solution

$$\tilde{A}_0 = f(r), \quad \tilde{A}_3 + i\tilde{A}_4 = h(r)e^{ikx_2}$$

- The equations of motion

$$\partial_r \left(r^3 f' + 2kh^2\right) = 0$$

$$\partial_r \left(rHh'\right) - \frac{k^2}{r}h + 4f'kh = 0$$

# Application to spatially modulated models

- Integrating the first equation we end up with

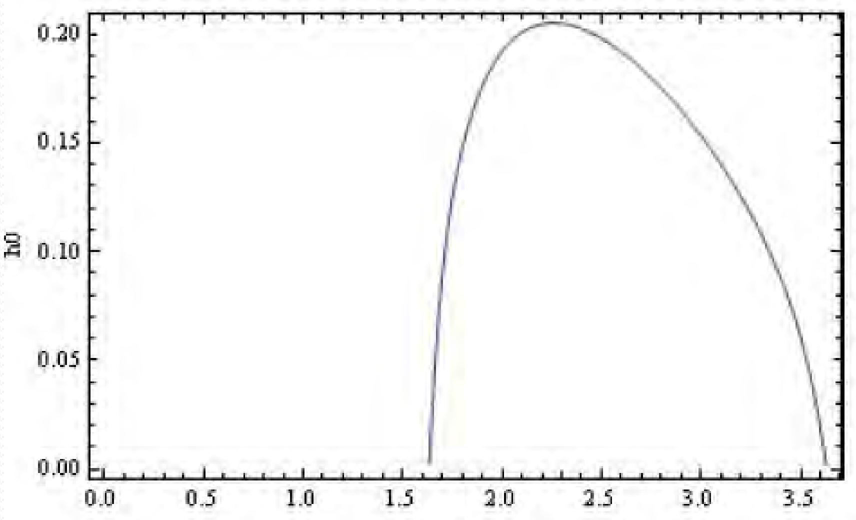
$$r^3\partial_r\left(rHh'\right) - r^2k^2h - 4kh\left(\tilde{E} + 2kh^2\right) = 0$$

$$r^3 f' + 2kh^2 = -\tilde{E}$$

- This equation admits solution with amplitude

$$h(r_+) = h_0 \text{ nonzero}$$

- The relation between ho and k

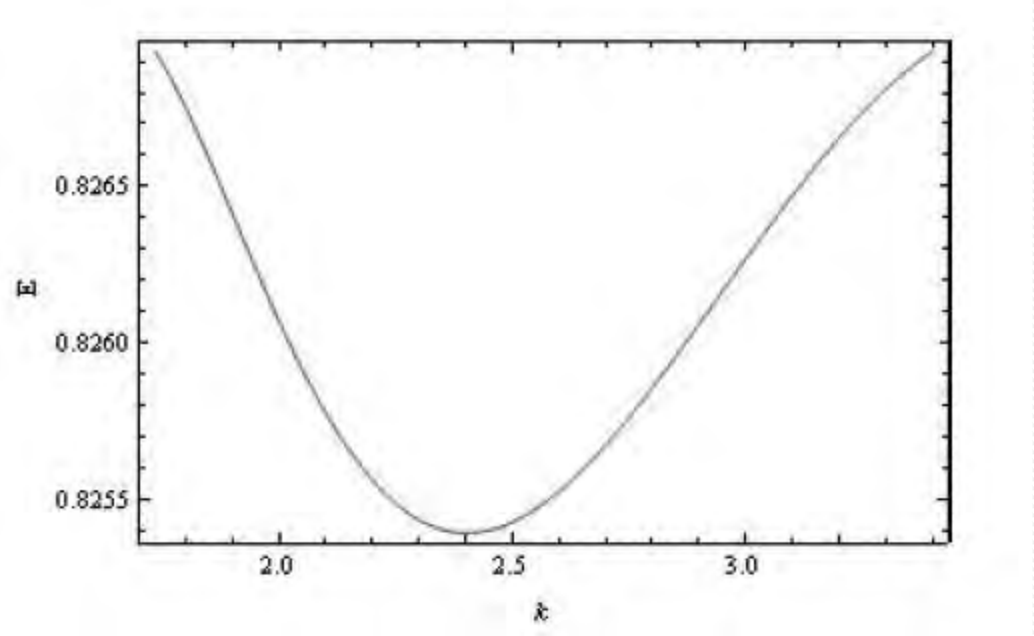


# Application to spatially modulated models

- The energy density of the boundary field theory is

$$\langle \mathcal{E} \rangle = \int_{r_+}^{\infty} dr \left[ \frac{1}{2r^3} \left( \tilde{E} + 2kh^2 \right)^2 + \frac{k^2 h^2}{2r} + \frac{r H h'^2}{2} \right]$$

- It is minimized at $kr_+ = 2.38$



# The stiffness tensor

- The energy density is given by

$$\mathcal{E} = \int_{r_+}^{\infty} dr\, \sqrt{-g}\left[\frac{1}{2}|g^{00}|g^{ij}F_{0i}F_{0j} + \frac{1}{2}|g^{00}|g^{rr}F_{0r}F_{0r} + \frac{1}{2}g^{rr}g^{ij}F_{ri}F_{rj} + \frac{1}{4}g^{ik}g^{jl}F_{ij}F_{kl}\right]$$

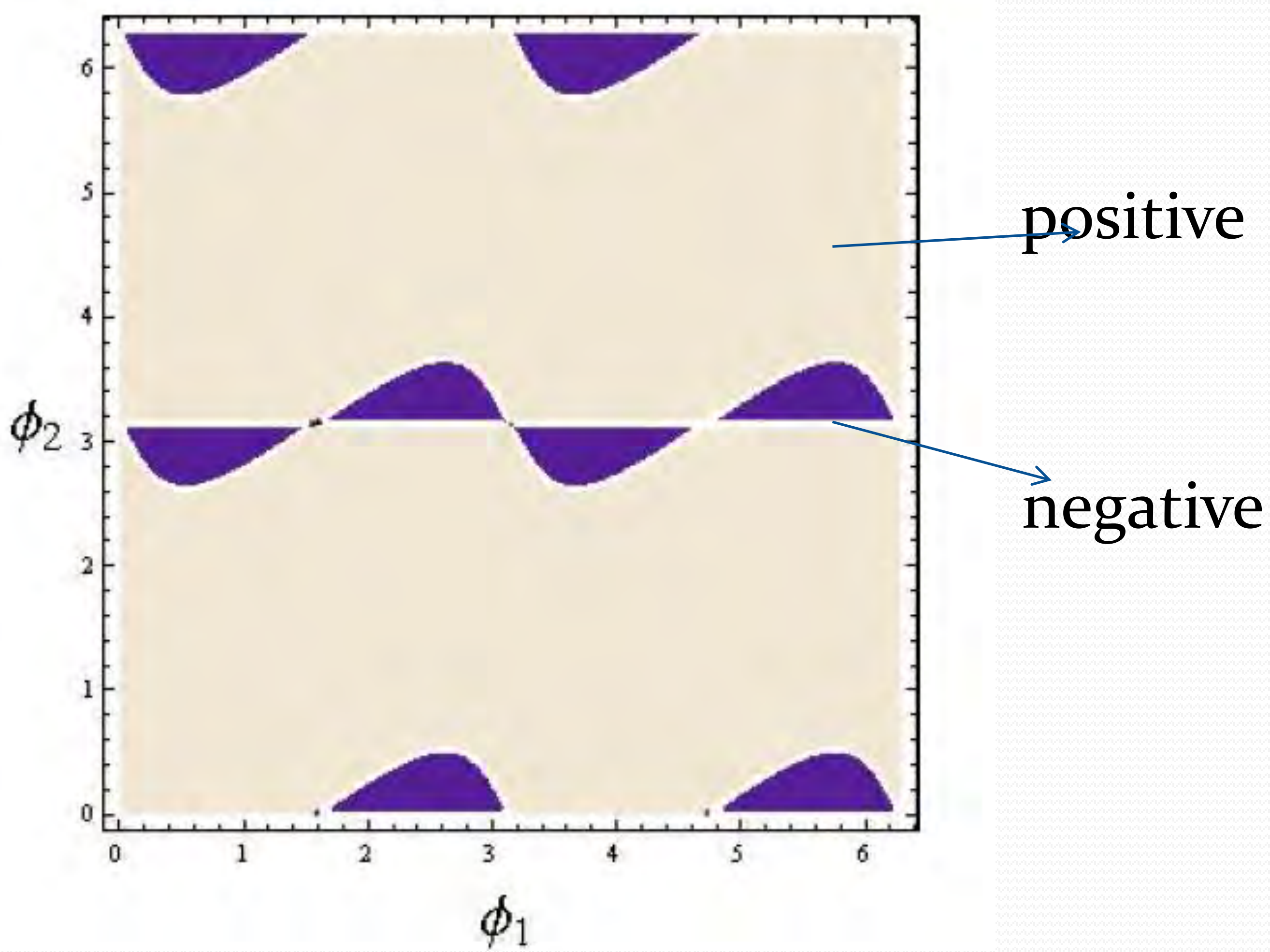
- The expression for the stiffness tensor is complicated
- For the unit vectors

$$\hat{a} = (0, \cos\varphi_1, \sin\varphi_1), \quad \hat{b} = (0, \cos\varphi_2, \sin\varphi_2)$$

It is given by

$$C^{il}_{jm}\hat{a}_i\hat{a}_l\hat{b}^j\hat{b}^m = \int_{r_+}^{\infty} dr\, \frac{k^2h^2}{r}\left[1 + \frac{1}{2}\cos(2(\varphi_1 - \varphi_2)) - \frac{1}{2}\cos(2(kx_2 - \varphi_1)) - \cos(2(kx_2 - \varphi_2))\right]$$

# The stiffness tensor



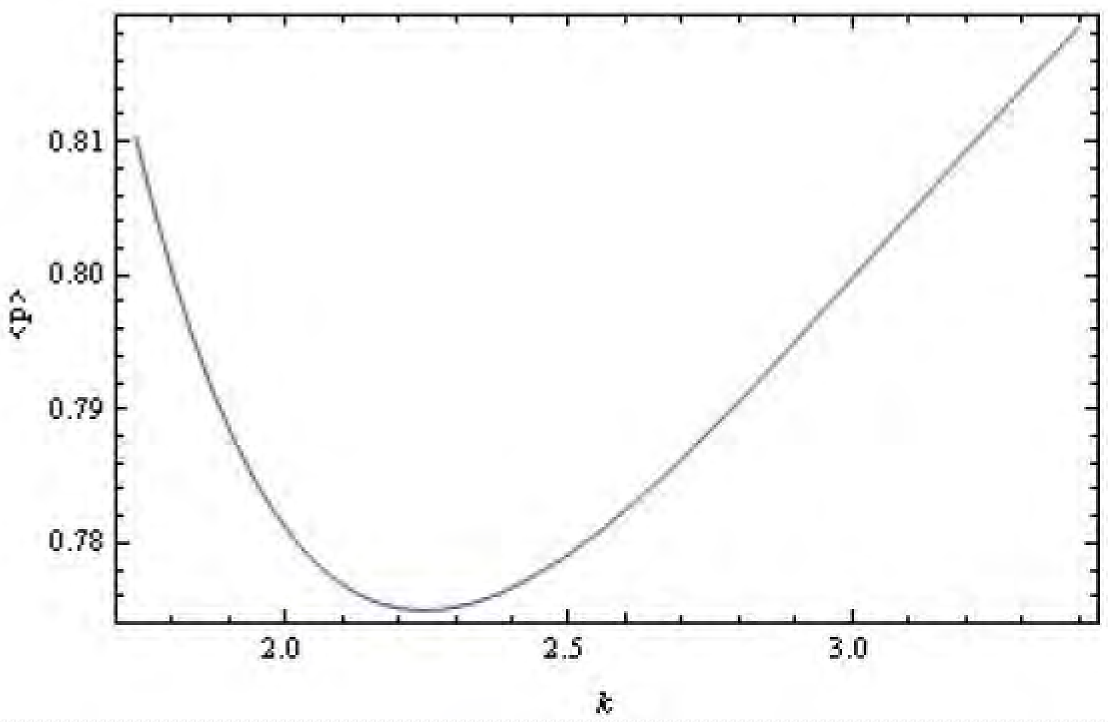

Thus there are regions which indeed correspond to minima but other ( blue ones) correspond to maxima

# Stress foreces

- We check now for the stability against deformation in the x2 direction

$$p_2 = T^2{}_2 = -\frac{1}{2r^3}\left(\tilde{E} + 2kh^2\right)^2 - \frac{k^2h^2}{2r} + \frac{rHh'^2}{2}$$

- The pressure is negative for all k and has a maximum for $kr_+ \simeq 2.25$
- In the region $\partial_k \langle p \rangle < 0$ the system is not restored
- The minimum of the free energy at $kr_+ \simeq 2.4.$ is in the instability region


0.81
0.80
0.79
0.78
<P>
2.0
2.5
3.0
k

# Summary and open questions

- We unified and generalized **Derrick's and Manton constraints on solitons.**
- We have applied the condtions to sytems of soltions with global currents
- Sigma model and higher derivative actions
- DBI electromagnetism
- Dbranes  including the DBI and WZ terms
- The method can be applied to many more `` modern solitons"
- In particular we are investigating the stability of the spatially modulated  brane and bulk solutions.