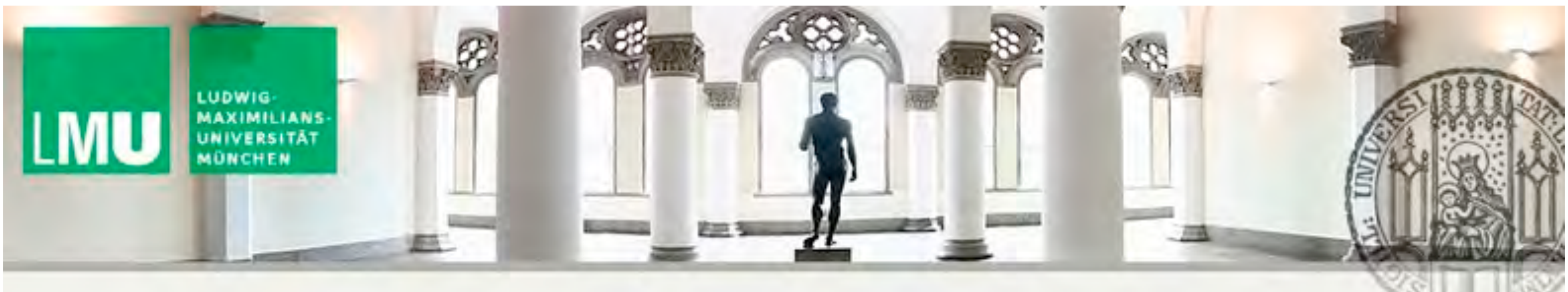
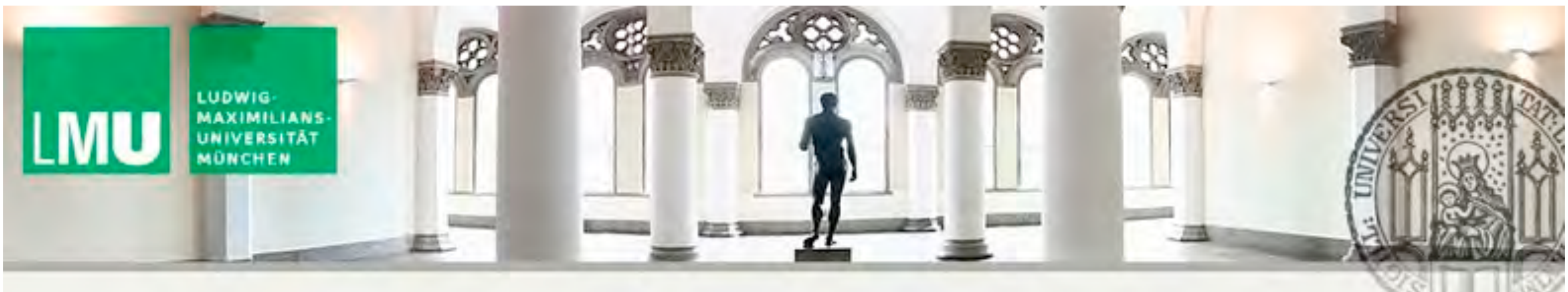


Seventh Crete Regional Meeting in String Theory, κολυμπαρι, June 22nd, 2013

Κλειστές Χορδές, μη γεωμετρικές λύσεις καί μή αντιμεταθετική/ μή προσεταιριστική γεωμετρία

ΝΤΙΕΤΕΡ ΛΟΥΣΤ (ΛΜΥ, ΜΠΙ)



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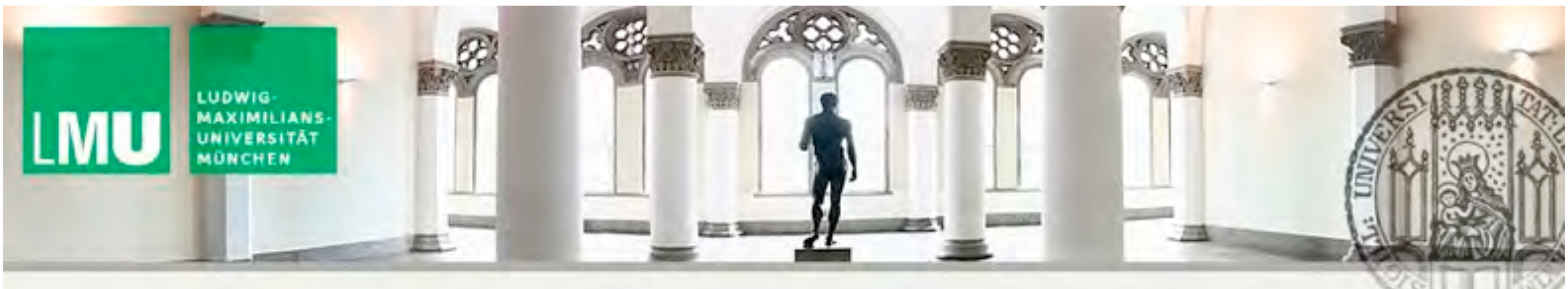
Closed strings, non-geometric backgrounds and non-commutative/ non-associative geometry

DIETER LÜST (LMU, MPI)



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Closed strings, non-geometric backgrounds and non-commutative/ non-associative geometry

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In collaboration with D. Andriot, I. Bakas, R. Blumenhagen, C. Condeescu, A. Deser, I. Florakis, O. Hohm, M. Larfors, C. Kounnas, P. Patalong, E. Plauschinn, F. Rennecke, B. Zwiebach

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I) Non-geometric backgrounds - Introduction

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World sheet point of view:
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effective action from double field theory

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Target space point of view:

effective action from double field theory

V) (Intersecting) Q- and R-branes

6D, supersymmetric, non-geometric flux compactifications & moduli stabilization

I) Non-geometric flux compactifications

Geometry in general depends on, with what kind of objects you test it.

Point particles in classical Einstein gravity „see" continuous Riemannian manifolds.

Strings may see space-time in a different way.

At the string scale L_s :

We expect the emergence of a new kind of stringy geometry.

Recall standard Riemannian geometry:

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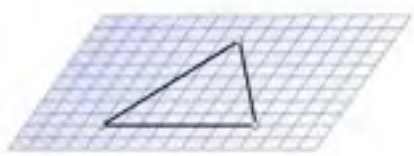
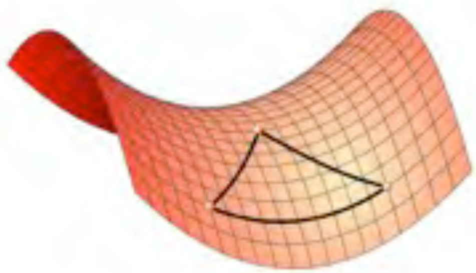
- Flat space: Triangle: $\alpha + \beta + \gamma = \pi$
- Curved space: Triangle: $\alpha + \beta + \gamma > \pi (< \pi)$





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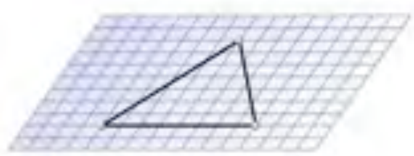

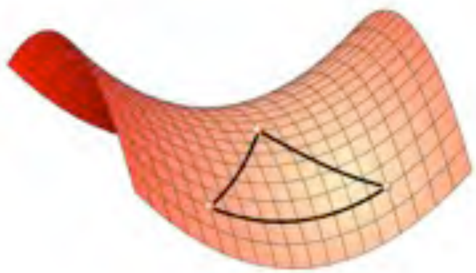

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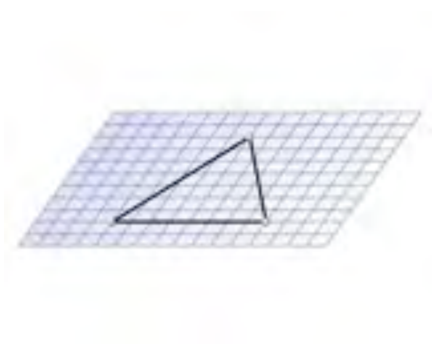

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- $[X^i, X^j] = 0$

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Here new observation for closed strings:

D.L., arXiv:1010.1361;

R. Blumenhagen, E. Plauschinn, arXiv:1010.1263;

R. Blumenhagen, A. Deser, D.Lüst, E. Plauschinn, F. Rennecke, arXiv:1106.0316

- Non-commutative geometry:

$$[X^i(\tau,\sigma),X^j(\tau,\sigma)]\simeq Q_k^{ij}\ \tilde{p}^k\,,\qquad Q_k^{ij}=\partial_k\beta^{ij}$$

- Non-associative geometry:

$$[[X^i(\tau,\sigma),X^j(\tau,\sigma)],X^k(\tau,\sigma)]+\text{perm.}\simeq R^{ijk}\,,\qquad R^{ijk}=3\tilde{D}^{[i}\beta^{jk]}$$

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(Hellerman, McGreevy, Williams (2002)
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Transition functions between two coordinate patches are given in terms of O(D,D) T-duality transformations:

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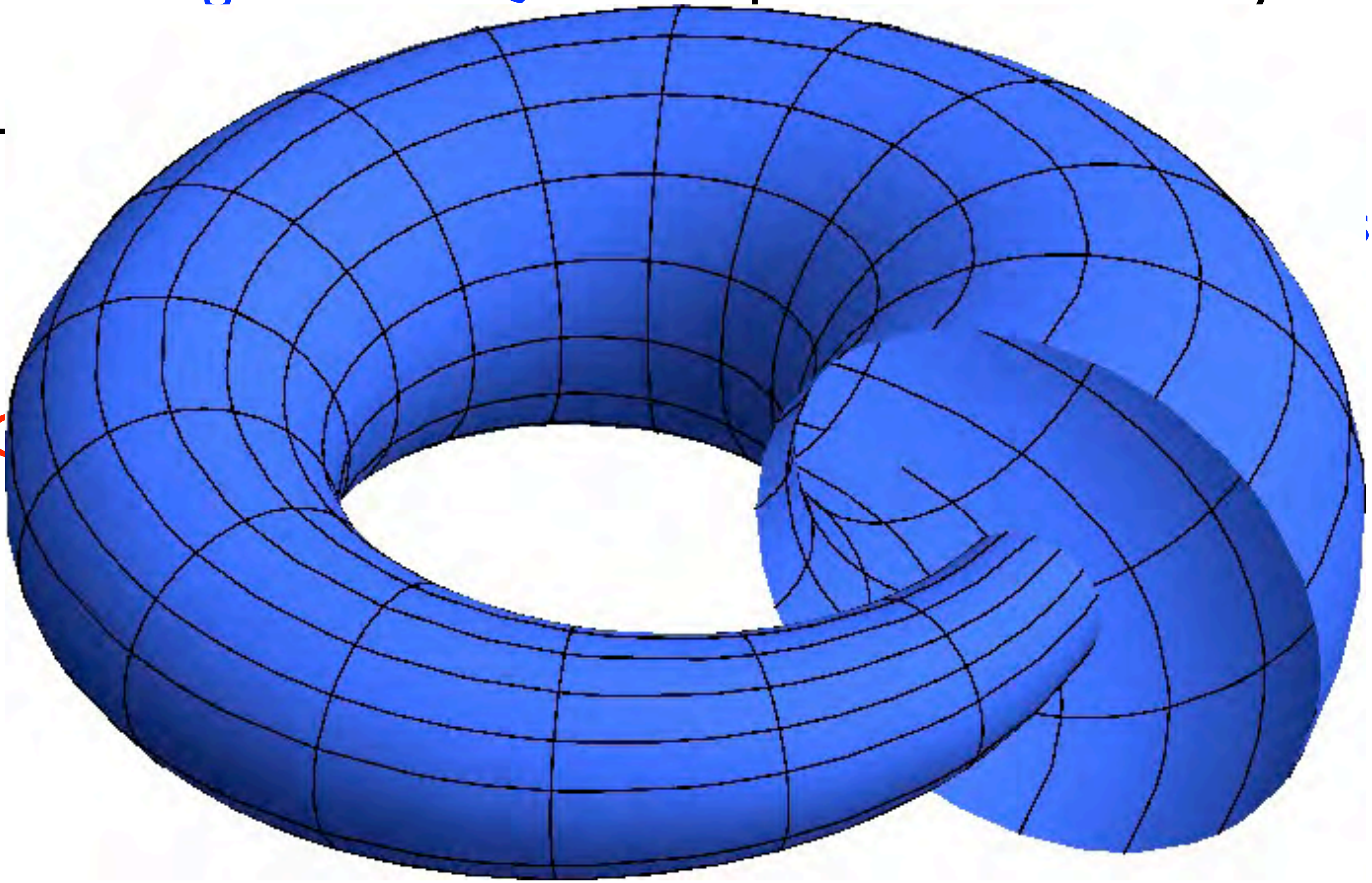

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- Non-geometric R-fluxes: spaces that are even locally not anymore manifolds.

R-space will become non-associative:

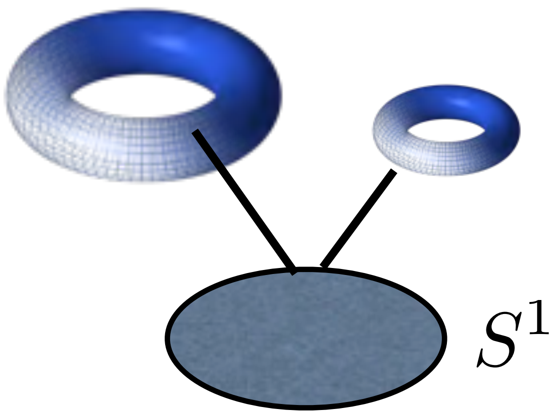
$$\begin{aligned} [X^i, X^j, X^k] \quad &:= \quad [[X^i, X^j], X^k] + \text{cycl. perm.} = \\ &= \quad (X^i \cdot X^j) \cdot X^k - X^i \cdot (X^j \cdot X^k) + \cdots \neq 0 \end{aligned}$$

Example: Three-dimensional flux backgrounds:

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{X^1,X^2} \hookrightarrow M^3 \hookrightarrow S^1_{X^3}$$

The fibration is specified by its monodromy properties.



$$T^2 : \ \mathcal{E}_{ij}(X^3) = G_{ij}(X^3) + B_{ij}(X^3)$$

O(2,2) monodromy: $\mathcal{E}(X^3 + 2\pi) = g_{O(2,2)}\mathcal{E}(X^3)$

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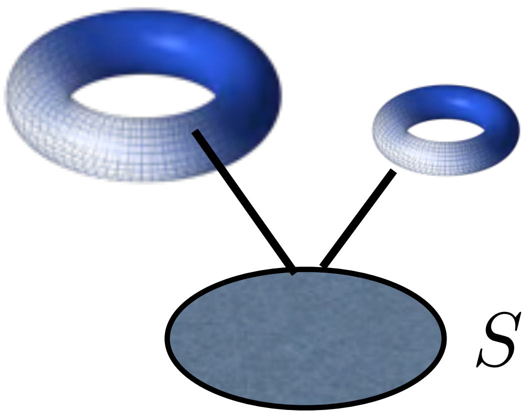

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Complex structure τ of T^2: $\tau(X^3 + 2\pi) = \dfrac{a\tau(X^3) + b}{c\tau(X^3) + d}$

Kähler parameter ρ of T^2 : $\rho(X^3 + 2\pi) = \dfrac{a'\rho(X^3) + b'}{c'\rho(X^3) + d'}$

Chain of four T-dual examples:

(Shelton, Taylor, Wecht, 2005; Dabholkar, Hull, 2005)

(i) Geometric space: 3-dimensional torus with H - flux

$$G_{ij} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix} , \quad B_{12} = H X_H^3 , \ H_{123} = \partial_3 B_{12} = H$$

$$\rho(X_H^3) = i\ R_1 R_2 - H X_H^3$$

$$X_H^3 \rightarrow X_H^3 + 2\pi R_3 \quad \Rightarrow \quad g_{O(2,2)} : \rho(X_H^3 + 2\pi R_3) = \rho(X_H^3) + 2\pi H R_3$$

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T-duality in X^1 :

(ii) Geometric spaces: twisted 3-torus with f - flux $(f \equiv H)$

$$G_{ij} = \begin{pmatrix} \frac{1}{R_1^2} & -\frac{f X_f^3}{R_1^2} & 0 \\ -\frac{f X_f^3}{R_1^2} & R_2^2 + \left(\frac{f X_f^3}{R_1}\right)^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{ij} = 0$$

$$\tau(X_f^3) = i\; R_1 R_2 - f X_f^3$$

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T-duality in X^2 :

(iii) Non-geometric space: T-fold with Q-flux $(Q \equiv f \equiv H)$

$$G_{ij} = \begin{pmatrix} \frac{F}{R_1^2} & 0 & 0 \\ 0 & \frac{F}{R_2^2} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix} \; , \; B_{ij} = F \begin{pmatrix} 0 & -\frac{QX_Q^3}{R_1^2 R_2^2} & 0 \\ \frac{QX_Q^3}{R_1^2 R_2^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; , \; F = \left(1 + \left(\frac{QX_Q^3}{R_1 R_2} \right)^2 \right)^{-1}$$

$$\rho(X_Q^3) = \frac{1}{QX_Q^3 - iR_1 R_2} \Rightarrow g_{O(2,2)} : \rho(X_Q^3 + 2\pi R_3) = \frac{\rho(X_Q^3)}{1 + 2\pi R_3 Q \, \rho(X_Q^3)}$$

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T-duality in X^3 :

(iv) Non-geometric space with R-flux

Now the Buscher rules for T-duality cannot be applied.

There exist no locally defined metric and B-field.

II) World sheet non-commutativity/non-associativity

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Open strings with F-flux:

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Open strings with F-flux:

Coordinates of open string end-points are non-commutative:

2-dimensional D-branes with 2-form F-flux $\Rightarrow$

$$[X^i(\tau), X^j(\tau)] = \epsilon^{ij}\Theta\,, \quad \Theta = -\frac{2\pi i\alpha' F}{1+F^2}$$

(A. Abouelsaood, C. Callan, C. Nappi, S. Yost (1987);
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➢ Non-commutative gauge theories.

(N. Seiberg, E. Witten (1999); J. Madore, S. Schraml, P. Schupp, J. Wess (2000); ....)

$$f_1(x) \star f_2(x) \star \ldots \star f_N(x) :=$$

$$\exp\left[i \textstyle\sum_{m<n} \Theta^{ij}\, \partial_i^{x_m}\, \partial_j^{x_n}\right] f_1(x_1)\, f_2(x_2) \ldots f_N(x_N)\Big|_{x_1=\ldots=x_N=x}$$

$$S \simeq \int d^n x\, \mathrm{Tr} \hat{F}_{ab} \star \hat{F}^{ab}$$

Remark: In the T-dual picture (D1-brane at angle) the coordinates are commutative!

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T-duality: Introduce coordinates and dual coordinates:

$(i = 1, \ldots, D)$

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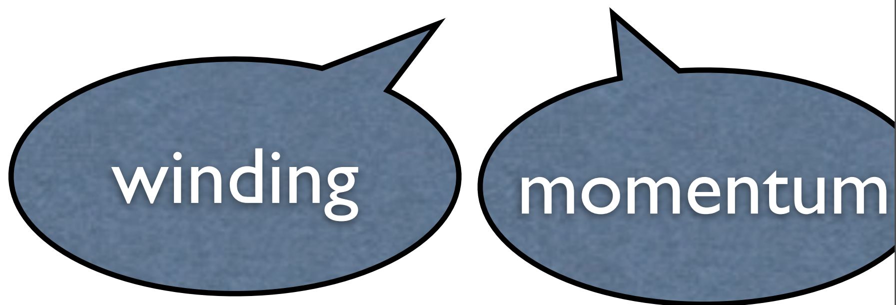

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winding momentum

- O(D,D) transformations: Mix in general X^i with $\tilde{X}_i$.

$$\begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix} , \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in g_{O(D,D)}$$

We considered two different types of models:

(i) (Non-)geometric backgrounds with elliptic monodromy and non-geometric fluxes. D. L., JHEP 1012 (2011) 063, arXiv:1010.1361,

They can be described in terms of (a)symmetric freely acting orbifolds.

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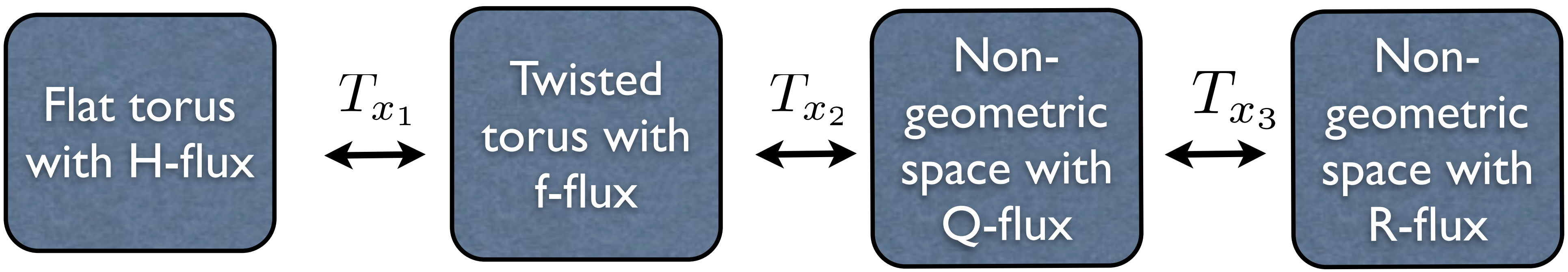

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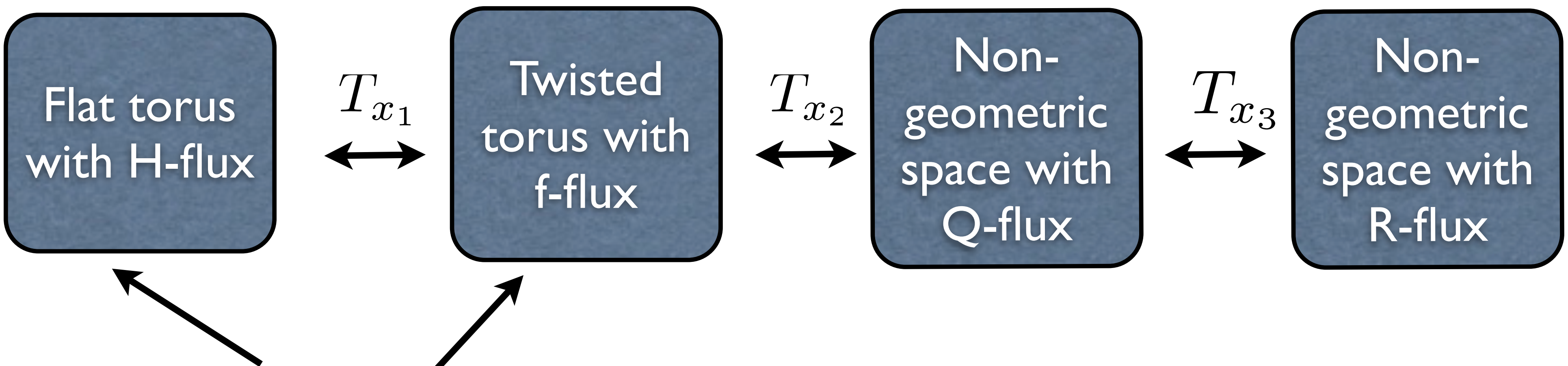

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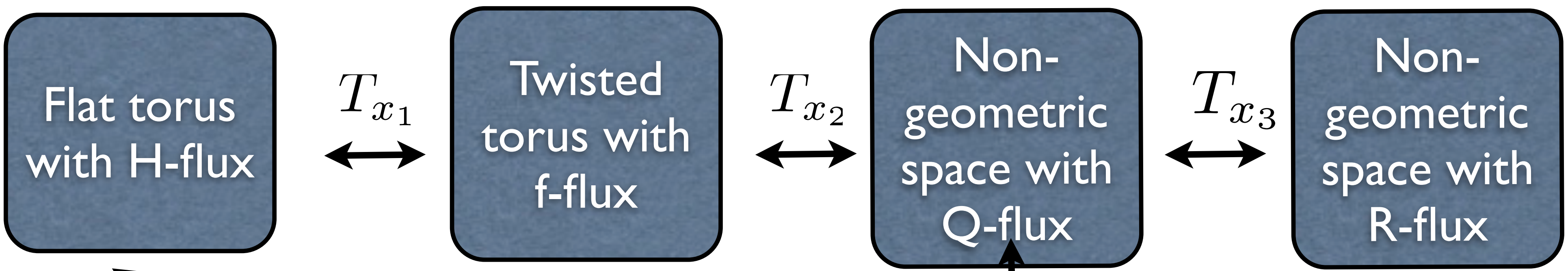

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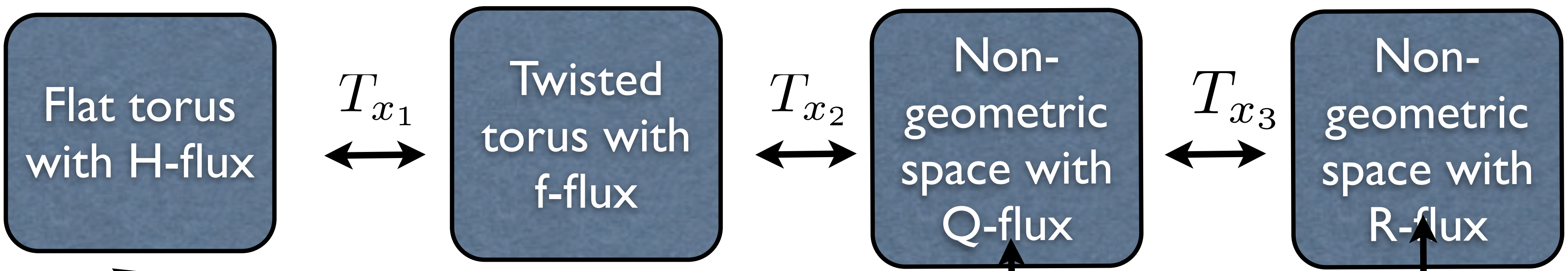

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$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \left(G_{ij}(X)\ \eta^{\alpha\beta} + B_{ij}(X)\ \varepsilon^{\alpha\beta} \right) \partial_\alpha X^i \partial_\beta X^j$$

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Quantize at linear order in the flux H or f ,

$$X^i_H(\tau,\sigma) = X^i_{(H0)}(\tau,\sigma) + H\ X^i_{(H1)}(\tau,\sigma)$$

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Quantize at linear order in the flux H or f ,

$$X^i_H(\tau,\sigma) = X^i_{(H0)}(\tau,\sigma) + H\ X^i_{(H1)}(\tau,\sigma)$$

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$$O(2,2)\left\{\begin{array}{rcl} X^1_H(\tau,\sigma+2\pi) & = & X^1_H(\tau,\sigma)\,, \\ X^2_H(\tau,\sigma+2\pi) & = & X^2_H(\tau,\sigma)\,, \\ \tilde{X}_{H1}(\tau,\sigma+2\pi) & = & \tilde{X}_{H1}(\tau,\sigma) - 2\pi\;\tilde{p}^3\;H\;\,X^2_H(\tau,\sigma)\,, \\ \tilde{X}_{H2}(\tau,\sigma+2\pi) & = & \tilde{X}_{H2}(\tau,\sigma) + 2\pi\;\tilde{p}^3\;H\;\,X^1_H(\tau,\sigma)\,. \end{array}\right.$$

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winding number

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winding number

Result:

$$\begin{array}{rcl} [X^i_{H,f}(\tau,\sigma), X^j_{(H,f)}(\tau,\sigma')] & = & 0 \\ [P_i(\tau,\sigma), P_j(\tau,\sigma')] & = & 0 \\ [X^i_{H,f}(\tau,\sigma), P_j(\tau,\sigma')] & = & i\delta^i_j\delta(\sigma-\sigma') \end{array} \qquad (P_i = \frac{\delta\mathcal{L}}{\delta\partial_\tau X^i})$$

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Two consistency requirements:

(i) Canonical T-duality: (E. Alvarez, L. Alvarez-Gaume, Y. Lozano, 1994)

$$\mathrm{T-d.\ along}\ i=2: \quad \begin{aligned} \partial_\tau X^2_Q &= \partial_\sigma X^2_f - Q X^3_f \partial_\sigma X^1_f \\ \partial_\sigma X^2_Q &= \partial_\tau X^2_f - Q X^3_f \partial_\tau X^1_f \end{aligned}$$

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$$X^3_Q(\tau,\sigma+2\pi) = X^3_Q(\tau,\sigma) + 2\pi\ \tilde{p}^3 \quad \Longrightarrow$$

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$(Q \equiv f \equiv H)$

(Similar to DN boundary conditions of open string.)

Mix coordinates with dual coordinates. $\Longleftrightarrow$ Non-geometric background.

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winding number

$$O(2,2)\begin{cases} X^1_Q(\tau,\sigma+2\pi) = X^1_Q(\tau,\sigma) - 2\pi\ \tilde p^3\ Q\ \tilde X_{Q2}(\tau \ldots \\ X^2_Q(\tau,\sigma+2\pi) = X^2_Q(\tau,\sigma) + 2\pi\ \tilde p^3\ Q\ \tilde X_{Q1}(\tau, \ldots \\ \tilde X_{Q1}(\tau,\sigma+2\pi) = \tilde X_{Q1}(\tau,\sigma)\,, \\ \tilde X_{Q2}(\tau,\sigma+2\pi) = \tilde X_{Q2}(\tau,\sigma)\,. \end{cases}$$

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Mix coordinates with dual coordinates. $\Longleftrightarrow$ Non-geometric background.

Then we derive the following result for the commutator of the coordinates:

D. Andriot, M. Larfors, D. L., P. Patalong, arXiv:1211.6437

$$[X_Q^1(\tau,\sigma), X_Q^2(\tau,\sigma')] =$$

$$-\frac{i}{2}Q\,\tilde{p}^3\left(\sum_{n\neq 0}\frac{1}{n^2}e^{-in(\sigma'-\sigma)} - (\sigma'-\sigma)\sum_{n\neq 0}\frac{1}{n}e^{-in(\sigma'-\sigma)} + \frac{i}{2}(\sigma'-\sigma)^2\right)$$

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$\sigma \longrightarrow \sigma'$:

$$[X_Q^1(\tau,\sigma), X_Q^2(\tau,\sigma)] = -i\frac{\pi^2}{6}Q\,\tilde{p}^3$$

The non-commutativity of the torus (fibre) coordinates is determined by the winding in the circle (base) direction.

Corresponding uncertainty relation:

$$(\Delta X_Q^1)^2(\Delta X_Q^2)^2 \geq L_s^6\, Q^2\ \langle\tilde{p}^3\rangle^2$$

The spatial uncertainty in the X_1, X_2 - directions grows with the dual momentum in the third direction: non-local strings with winding in third direction.

R-flux background:

T-duality in x^3-direction $\Rightarrow$ R-flux

$$\tilde{p}^3 \longleftrightarrow p_3\,, \qquad \tilde{X}_{Q,3} \equiv X_R^3$$

$\Rightarrow$ For the case of non-geometric R-fluxes one gets:

$$[X_R^1, X_R^2] = -i\frac{\pi^2}{6} R\, p_3$$

$R \equiv Q$

Use $[X_R^3, p_3] = i \qquad \Longrightarrow$

$$[[X_R^1(\tau,\sigma), X_R^2(\tau,\sigma)], X_R^3(\tau,\sigma)] + \text{perm.} = \frac{\pi^2}{6} R$$

Non-associative algebra!

CFT description: I. Bakas, D.L. to appear.

$$S = -\frac{1}{2\pi}\int_\Sigma d^2\sigma \left(G_{ij}(X)\ \eta^{\alpha\beta} + B_{ij}(X)\ \varepsilon^{\alpha\beta}\right)\partial_\alpha X^i \partial_\beta X^j$$

Introduce „(anti)holomorphic" currents:

$$\begin{aligned} J^i(z,\bar z) \quad &:= \quad \partial \mathcal{X}^i_L = \partial X^i_L + \frac{1}{2}\mathcal{E}_{ij}(X_R)\partial X^j_L\,, \\ \bar J^i(z,\bar z) \quad &:= \quad \bar\partial \mathcal{X}^1_R = \bar\partial X^i_R + \frac{1}{2}\mathcal{E}_{ij}(X_R)\bar\partial X^j_R \end{aligned}$$

$$\mathcal{E}_{ij}(X) = G_{ij}(X) + B_{ij}(X)$$

Action of canonical T-duality on the world-sheet:

$$\begin{aligned} J^1 \quad &\longrightarrow \quad J^1\,, \\ \bar J^1 \quad &\longrightarrow \quad -\bar J^1 \end{aligned}$$

(Automorphism of CFT)

Redefined coordinates (agree with previous first order mode expansion):

$$\mathcal{X}^i(z,\bar{z}) = \mathcal{X}^i_L + \mathcal{X}^i_R$$

$$\text{with} \quad \mathcal{X}^i_L = \int^z J^i(z',\bar{z}')dz' \,, \quad \mathcal{X}^i_R = \int^{\bar{z}} \bar{J}^i(z',\bar{z}')d\bar{z}'$$

T-duality:

$$\mathcal{X}^i = \mathcal{X}^i_L + \mathcal{X}^i_R \;\longrightarrow\; \tilde{\mathcal{X}}_i = \mathcal{X}^i_L - \mathcal{X}^i_R$$

Commutators:

$$\Theta^{ij}_{LL}(\tau,\sigma) := [\mathcal{X}^i_L(\tau,\sigma), \mathcal{X}^j_L(\tau,\sigma')]|_{z\to w} = \lim_{z\to w}\int^{z} J^i(z',\bar{z}')X^j_L(w)dz' \,,$$

$$\Theta^{ij}_{RR}(\tau,\sigma) := [\mathcal{X}^i_R(\tau,\sigma), \mathcal{X}^j_R(\tau,\sigma')]_{z\to w} = \lim_{z\to w}\int^{\bar{z}} \bar{J}^i(z',\bar{z}')X^j_R(\bar{w})d\bar{z}' \,,$$

$$\Theta^{ij}_{LR}(\tau,\sigma) := [\mathcal{X}^i_L(\tau,\sigma), \mathcal{X}^j_R(\tau,\sigma')]_{z\to w} = \lim_{z\to w}\int^{z} J^i(z',\bar{z}')X^j_R(\bar{w})dz' \,,$$

$$\Theta^{ij}_{RL}(\tau,\sigma) := [\mathcal{X}^i_R(\tau,\sigma), \mathcal{X}^j_L(\tau,\sigma')]_{z\to w} = \lim_{z\to w}\int^{\bar{z}} \bar{J}^i(z',\bar{z}')X^j_L(w)d\bar{z}' \,,$$

$$\Theta^{ij} := [\mathcal{X}^i(\tau,\sigma), \mathcal{X}^j(\tau,\sigma)] = \Theta^{ij}_{LL} + \Theta^{ij}_{RR} + \Theta^{ij}_{LR} + \Theta^{ij}_{RL}$$

$$\Theta^{i}_{j} := [\mathcal{X}^i(\tau,\sigma), \tilde{\mathcal{X}}_j(\tau,\sigma)] = \Theta^{ij}_{LL} - \Theta^{ij}_{RR} - \Theta^{ij}_{LR} + \Theta^{ij}_{RL}$$

$$\Theta_{ij} := [\tilde{\mathcal{X}}_i(\tau,\sigma), \tilde{\mathcal{X}}_j(\tau,\sigma)] = \Theta^{ij}_{LL} + \Theta^{ij}_{RR} - \Theta^{ij}_{LR} - \Theta^{ij}_{RL}$$

E.g. parabolic H,f,Q,R-field background:

$$
\begin{aligned}
J^i(z,\bar{z}) &= \partial \mathcal{X}^i_L(z,\bar{z}) = \partial X^i_L(z) - \tfrac{1}{2} H_{ijk}\, \partial X^j_L(z)\, X^k_R(\bar{z}) \;, \\
\bar{J}^i(z,\bar{z}) &= \partial \mathcal{X}^i_R(z,\bar{z}) = \partial X^i(\bar{z}) - \tfrac{1}{2} H_{ijk}\, X^j_L(z)\, \bar{\partial} X^k_R(\bar{z})
\end{aligned}
$$

(R. Blumenhagen, A. Deser, D.L. , E. Plauschinn, F. Rennecke, 2011)

Commutators:

$$
\begin{aligned}
\Theta^{ij}_{LL}(\tau,\sigma) &= -\lim_{z\to w} H_{ikl} \int^z \partial X^k_L(z') \overbrace{X^j_L(w) X^l_R(\bar{z}')}\, dz' \\
&= -H_{ijk} p^k_R \lim_{z\to w} \int^z dz' \frac{\log \bar{z}'}{z'-w} = -H_{ijk} p^k_R \lim_{z\to w} Li_2\left(\frac{z}{w}\right) + \ldots \\
&= -\frac{\pi^2}{6} H_{ijk} p^k_R
\end{aligned}
$$

$$
\begin{aligned}
C^{ij}_{RR} &= \frac{\pi^2}{6} H_{ijk}\; p^k_L \,, \qquad C^{ij}_{LR} = -\frac{\pi^2}{6} H_{ijk}\; p^k_L \,, \\
C^{ij}_{RL} &= \frac{\pi^2}{6} H_{ijk}\; p^k_R
\end{aligned}
$$

Then we derive for the four different cases:

H-flux $$[\mathcal{X}^i_H(\tau,\sigma),\mathcal{X}^j_H(\tau,\sigma)]=0$$

f-flux $$[\mathcal{X}^i_f(\tau,\sigma),\mathcal{X}^j_f(\tau,\sigma)]=0$$

Q-flux $$[\mathcal{X}^i_Q(\tau,\sigma),\mathcal{X}^j_Q(\tau,\sigma)]=-\frac{\pi^2}{3}Q^{ij}_k\ \tilde{p}^k$$

R-flux $$[\mathcal{X}^i_R(\tau,\sigma),\mathcal{X}^j_R(\tau,\sigma)]=-\frac{\pi^2}{3}R^{ijk}\ p_k$$

III) Phase space of double geometry

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Non-commutative and non-associative Poisson structure of flux background:

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Start with 2D -dimensional (x^i, p_i) phase space.

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$\star_p$ - product: (D. Mylonas, R. Szabo, P. Schupp, arXiv:1207.0926)

$$f(x,p) \star_p g(x,p) := e^{i\Theta^{IJ}\partial_I \partial_J} f(x,p) g(x,p), \quad \Theta^{IJ} = \begin{pmatrix} R^{ijk}p_k & \delta^i_j \\ -\delta^j_i & 0 \end{pmatrix}$$

$$f(x) \star_p g(x) = e^{iR^{ijk}p_k \partial_i \partial_j} f(x) g(x)$$

$$\Rightarrow \quad [x^i, x^j]_{\star_p} = iR^{ijk} p_k$$

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$$\Rightarrow \quad [x^i, x^j]_{\star_p} = iR^{ijk}p_k$$

This product is non-associative:

$$\big(f(x) \star_p g(x)\big) \star_p h(x) = \exp\left[R^{ijk}\,\partial_i\,\partial_j \partial_k\right]\, f(x) g(x) h(x)$$

$$f(x) \star_p \big(g(x) \star_p h(x)\big) = \exp\left[-R^{ijk}\,\partial_i\,\partial_j \partial_k\right]\, f(x) g(x) h(x)$$

$$\Rightarrow \quad [f(x), g(x), h(x)]_{\star_p} = i \sinh\left[R^{ijk}\,\partial_i\,\partial_j \partial_k\right]\, f(x) g(x) h(x)$$

Generalization:

(Work in progress with I. Bakas)

Consider 4D-dimensional phase space of doubled geometry:

D + D coordinates $X^i\,,\,\tilde{X}_i$ D + D momenta $p_i\,,\,\tilde{p}^i$.

$\Rightarrow$ derive all commutators:

$$\begin{aligned}
\Theta^{ij} &= [X^i(\tau,\sigma), X^j(\tau,\sigma)]\,, & \Theta^{i\,\prime}_{j} &= [X^i(\tau,\sigma), P_j(\tau,\sigma)]\,, & \Theta''_{ij} &= [P_i(\tau,\sigma), P_j(\tau,\sigma)]\,,\\
\Theta^{i}_{j} &= [X^i(\tau,\sigma), \tilde{X}_j(\tau,\sigma)]\,, & \Theta^{ij\,\prime}_{j} &= [X^i(\tau,\sigma), \tilde{P}^j(\tau,\sigma)]\,, & \Theta^{j\,\prime\prime}_{i} &= [P_i(\tau,\sigma), \tilde{P}^j(\tau,\sigma)]\,,\\
\Theta_{ij} &= [\tilde{X}_i(\tau,\sigma), \tilde{X}_j(\tau,\sigma)]\,. & \Theta'_{ij} &= [\tilde{X}_i(\tau,\sigma), P_j(\tau,\sigma)]\,. & \Theta^{ij\,\prime\prime} &= [\tilde{P}^i(\tau,\sigma), \tilde{P}^j(\tau,\sigma)]\,.
\end{aligned}$$

O(D,D) covariant star product:

$$f_1(x,\tilde{x},p,\tilde{p}) \star_{p,\tilde{p}} f_2(x,\tilde{x},p,\tilde{p}) \stackrel{\text{def}}{=} \exp\Big(\Theta^{IJ}(x,p,\tilde{x},\tilde{p})\,\partial_I\,\partial_J\Big)\, f_1(x,\tilde{x},p,\tilde{p})\, f_2(x,\tilde{x},p,\tilde{p})$$

4D-dimensional Poisson tensor:

$$\Theta^{IJ}(x,p,\tilde{x},\tilde{p}) : \qquad \Theta^{IJ} = \begin{pmatrix} \Theta^{ij} & \Theta^{i}_{j} & \Theta^{i\,\prime}_{j} & \Theta^{ij\,\prime} \\ \Theta^{j}_{i} & \Theta_{ij} & \Theta'_{ij} & \Theta^{j\,\prime}_{i} \\ \Theta^{j\,\prime}_{i} & \Theta'_{ji} & \Theta''_{ij} & \Theta^{j\,\prime\prime}_{i} \\ \Theta^{ji\,\prime} & \Theta^{i\,\prime}_{j} & \Theta^{i\,\prime\prime}_{j} & \Theta^{ij\,\prime\prime} \end{pmatrix}$$

(i) Parabolic backgrounds:

Chain of three T-dual backgrounds: $(H \equiv f \equiv Q \equiv R)$

Flux	Commutators	Three-brackets
H-flux	$[\tilde{X}_1, \tilde{X}_2] \simeq H\ \tilde{p}^3$	$[\tilde{X}_1, \tilde{X}_2, \tilde{X}_3] \simeq H$
f-flux	$[X^1, \tilde{X}_2] \simeq f\ \tilde{p}^3$	$[X^1, \tilde{X}_2, \tilde{X}^3] \simeq f$
Q-flux	$[X^1, X^2] \simeq Q\ \tilde{p}^3$	$[X^1, X^2, \tilde{X}_3] \simeq Q$
R-flux	$[X^1, X^2] \simeq R\ p_3$	$[X^1, X^2, X^3] \simeq R$

$$\Theta^{IJ} = \begin{pmatrix} R^{ijk}p_k & 0 & \delta^i_j & 0 \\ 0 & 0 & 0 & \delta^j_i \\ -\delta^j_i & 0 & 0 & 0 \\ 0 & -\delta^i_j & 0 & 0 \end{pmatrix}$$

(ii) Elliptic backgrounds:

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Pair of T-dual geometric/non-geometric spaces:

$$\text{Geometric space (S)}: \; \tau(y) = \frac{\tau_0 \cos(fy) + \sin(fy)}{\cos(fy) - \tau_0 \sin(fy)}\,, \quad f \in \frac{1}{4} + \mathbb{Z}\,,$$
$$\rho = \rho_0$$

$$\tau' = \tau(y + 2\pi) = -1/\tau(y)$$

$$\text{Non}-\text{geometric space (A)}: \; \rho(y) = \frac{\rho_0 \cos(Qy) + \sin(Qy)}{\cos(Qy) - \rho_0 \sin(Qy)}\,, \quad Q \in \frac{1}{4} + \mathbb{Z}\,,$$
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$$\tau = \tau_0$$

$$\rho' = \rho(y + 2\pi) = -1/\rho(y)$$

They can be described in terms of freely acting symmetric ⇔ asymmetric Z4 orbifold CFTs:

$$X_L' = e^{-\frac{i\pi}{2}} X_L ,$$
$$X_R^{2\prime} = e^{\mp\frac{i\pi}{2}} X_R$$

Commutators and 3-brackets:

Flux	Commutators	Three-brackets
f-flux	$[X^1, \tilde{X}_2] \simeq i\ \Theta(\tilde{p}^3)$	$[X^1, \tilde{X}_2, \tilde{X}^3] \simeq \Theta'(\tilde{p}^3)$
Q-flux	$[X^1, X^2] \simeq i\ \Theta(\tilde{p}^3)$	$[X^1, X^2, \tilde{X}_3] \simeq \Theta'(\tilde{p}^3)$
R-flux	$[X^1, X^2] \simeq i\ \Theta(p_3)$	$[X^1, X^2, X^3] \simeq \Theta'(p_3)$

$$\Theta(p_3) = -\frac{\pi}{2}\cot(\pi p_3 R)\,, \quad \Theta'(p_3) = \frac{\partial \Theta(p_3)}{\partial p_3} = \frac{\pi^2 R}{2\sin^2(\pi \tilde{p}_3 R)}$$

D. L., JHEP 1012 (2011) 063, arXiv:1010.1361

Poisson structure:

$$\Theta^{IJ} = \begin{pmatrix} -2i\Theta^{ij} & 0 & \delta^i_j & 0 \\ 0 & -2i\Theta_{ij} & 0 & \delta_i^j \\ -\delta_i^j & 0 & 0 & 0 \\ 0 & -\delta^i_j & 0 & 0 \end{pmatrix} \qquad \Theta^{ij} = \begin{pmatrix} \epsilon^{ij}\Theta & 0 \\ 0 & 0 \end{pmatrix}$$

Non-geometric spaces that are NOT T-dual to a geometric space:

C. Condeescu, I. Florakis, D. L., JHEP 1204 (2012), 121, arXiv:1202.6366
C. Condeescu, I. Florakis, C. Kounnas, D.L., work in progress.

$$\tau(y) = \frac{\tau_0 \cos(fy) + \sin(fy)}{\cos(fy) - \tau_0 \sin(fy)}, \quad f \in \frac{1}{8} + \mathbb{Z},$$

$$\rho(y) = \frac{\rho_0 \cos(Qy) + \sin(Qy)}{\cos(Qy) - \rho_0 \sin(Qy)}, \quad Q \in \frac{1}{8} + \mathbb{Z}$$

$$\tau' = \tau(y + 2\pi) = \frac{1 + \tau(y)}{1 - \tau(y)}, \quad \rho' = \rho(y + 2\pi) = \frac{1 + \rho(y)}{1 - \rho(y)}$$

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$$\rho(y) = \frac{\rho_0 \cos(Qy) + \sin(Qy)}{\cos(Qy) - \rho_0 \sin(Qy)}, \quad Q \in \frac{1}{8} + \mathbb{Z}$$

$$\tau' = \tau(y + 2\pi) = \frac{1 + \tau(y)}{1 - \tau(y)}, \quad \rho' = \rho(y + 2\pi) = \frac{1 + \rho(y)}{1 - \rho(y)}$$

It can be described in terms of freely acting asymmetric Z4 orbifold CFT:

$$X'_L = e^{-\frac{i\pi}{2}} X_L ,$$
$$X_R^{2\prime} = X_R$$

Commutators and 3-brackets:

Flux	Commutators	Three-brackets
Q-flux	$[X^1, X^2] \simeq i\ \Theta(\tilde{p}^3)$	$[X^1, X^2, \tilde{X}_3] \simeq \Theta'(\tilde{p}^3)$
R-flux	$[X^1, X^2] \simeq i\ \Theta(p_3)$	$[X^1, X^2, X^3] \simeq \Theta'(p_3)$

$$\Theta(p_3) = -\frac{\pi}{2}\cot(\pi p_3 R)\,, \quad \Theta'(p_3) = \frac{\partial\Theta(p_3)}{\partial p_3} = \frac{\pi^2 R}{2\sin^2(\pi\tilde{p}_3 R)}$$

Poisson structure:

$$\Theta^{IJ} = \begin{pmatrix} i\Theta^{ij} & i\Theta^i_j & \delta^i_j & 0 \\ i\Theta_i^j & i\Theta_{ij} & 0 & \delta_i^j \\ -\delta_i^j & 0 & 0 & 0 \\ 0 & -\delta^i_j & 0 & 0 \end{pmatrix}$$

IV) Target space effective action

D. Andriot, M. Larfors, D.L., P. Patalong, arXiv:1106.4015
D. Andriot, O. Hohm, M. Larfors, D.L., P. Patalong, arXiv:1202.3060, 1204.1979
F. Hassler, O. Hohm, D.L., B. Zwiebach, work in progress

Consider the standard 10-dimensional effective action for the NS background fields $G_{ij}\,,\ B_{ij}\,,\ \phi:\ \ (H_{ijk}=\partial_{[i}B_{jk]})$

$$S=\int \mathrm{d}^{10}x\, e^{-2\phi}\sqrt{|G|}\left(\mathcal{R}+4(\partial\phi)^2-\frac{1}{12}H_{ijk}H^{ijk}\right)$$

This action is in general not well-defined for non-geometric backgrounds.

However a well-defined (10D) effective action for non-geometric backgrounds can be constructed.

Mathematical framework: Double field theory.

O(D,D) background:

$$\mathcal{H}^{MN} = \begin{pmatrix} G_{ij} - B_{ik}G^{kl}B_{lj} & B_{ik}G^{kj} \\ -G^{ik}B_{kj} & G^{ij} \end{pmatrix}$$

O(D,D) invariant action:

$$S_{\rm DFT} = \int dX d\tilde{X}\, e^{-2d} \Big(\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK} - 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d \Big) .$$

Strong constraint (level matching):

$$\eta^{MN}\partial_M\partial_N = 0\,, \quad \eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

e.g. set all $\tilde{\partial}^i = 0$

$\Longrightarrow$ Standard NS action.

Now we perform the following field redefinition (O(D,D) transformation):

bi-vector

$$\mathcal{E} = G + B \quad \rightarrow \quad \tilde{\mathcal{E}}^{-1} = \tilde{\mathcal{E}} = \tilde{G}^{-1} + \beta$$

O(D,D) background:

$$\mathcal{H}^{MN} = \begin{pmatrix} \tilde{G}_{ij} & -\tilde{G}_{ik}\beta^{kj} \\ \beta^{ik}\tilde{G}_{kj} & \tilde{G}^{ij} - \beta^{ik}\tilde{G}_{kl}\beta^{lj} \end{pmatrix}$$

(B-field gauge transformations $\longrightarrow$ β - transformations)

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O(D,D) background:

$$\mathcal{H}^{MN} = \begin{pmatrix} \tilde{G}_{ij} & -\tilde{G}_{ik}\beta^{kj} \\ \beta^{ik}\tilde{G}_{kj} & \tilde{G}^{ij} - \beta^{ik}\tilde{G}_{kl}\beta^{lj} \end{pmatrix}$$

(B-field gauge transformations $\longrightarrow$ β - transformations)

Introduce the following objects (non-geometric fluxes):

Q-flux: $$Q_k{}^{ij} = \partial_k \beta^{ij}$$

(not a tensor but a connection)

R-flux: (tensor):

$$R^{ijk} = 3\tilde{D}^{[i}\beta^{jk]}\,, \quad \tilde{D}^i \equiv \tilde{\partial}^i - \beta^{ij}\partial_j$$

R-flux needs dual coordinates!

- Rewrite DFT action

$$S_{\rm DFT}(\tilde G,\beta,\tilde\phi) = \int dX d\tilde X \sqrt{|\tilde G|}\, e^{-2\tilde\phi}\Big[\mathcal{R}(\tilde G,\partial)+\mathcal{R}(\tilde G^{-1},\tilde\partial) - \frac{1}{4}Q^2 - \frac{1}{12}R^{ijk}R_{ijk} + 4\Big((\partial\tilde\phi)^2+(\tilde\partial\tilde\phi)^2\Big) + \dots$$

- Final action („supergravity limit“): $\tilde\partial = 0$

$$e^{2d}\mathcal{L}_{\rm final}(\tilde G,\beta,d)(x) = \mathcal{R}(\tilde G) + 4(\partial\tilde\phi)^2 - \frac{1}{12}R^{ijk}R_{ijk} - \frac{1}{4}\tilde G_{ik}\tilde G_{jl}\tilde G^{rs}\,Q_r{}^{kl}\,Q_s{}^{ij} + \dots$$

This action is indeed well-defined for non-geometric fluxes!

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$$- \frac{1}{4}Q^2 - \frac{1}{12}R^{ijk}R_{ijk} + 4\Big((\partial\tilde\phi)^2+(\tilde\partial\tilde\phi)^2\Big) + \ldots$$

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This action is indeed well-defined for non-geometric fluxes!

T-fold with Q-flux: $\beta^{12} = QX^3_Q\,,\ Q_3^{12} = \partial_3\beta^{12} = Q$

Monodromy: $\beta(X^3_Q + 2\pi) = \beta(X^3_Q) + 2\pi Q$

- Rewrite DFT action

$$S_{\text{DFT}}(\tilde{G},\beta,\tilde{\phi}) = \int dX d\tilde{X}\, \sqrt{|\tilde{G}|}\, e^{-2\tilde{\phi}} \Big[\mathcal{R}(\tilde{G},\partial) + \mathcal{R}(\tilde{G}^{-1},\tilde{\partial}) - \frac{1}{4}Q^2 - \frac{1}{12}R^{ijk}R_{ijk} + 4\Big((\partial\tilde{\phi})^2 + (\tilde{\partial}\tilde{\phi})^2\Big) + \ldots$$

- Final action („supergravity limit"): $\tilde{\partial} = 0$

$$e^{2d}\mathcal{L}_{\text{final}}(\tilde{G},\beta,d)(x) = \mathcal{R}(\tilde{G}) + 4(\partial\tilde{\phi})^2 - \frac{1}{12}R^{ijk}R_{ijk} - \frac{1}{4}\tilde{G}_{ik}\tilde{G}_{jl}\tilde{G}^{rs}\, Q_r{}^{kl}\, Q_s{}^{ij} + \ldots$$

This action is indeed well-defined for non-geometric fluxes!

T-fold with Q-flux: $\beta^{12} = QX^3_Q\,,\ Q^{12}_3 = \partial_3\beta^{12} = Q$

Monodromy: $\beta(X^3_Q + 2\pi) = \beta(X^3_Q) + 2\pi Q$

T-fold with R-flux: $\beta^{12} = R\tilde{X}_{3R}\,,\ R^{123} = \tilde{\partial}^3\beta^{12} = R$

Relation to world sheet quantities:

(i) Non-commutativity (commutator):

$$[X_Q^i(\tau,\sigma), X_Q^j(\tau,\sigma)] = \oint_{C_k} Q_k{}^{ij}(X)$$

$$Q_k{}^{ij} = \partial_k \beta^{ij}$$

Wilson-line operator: non-local effect due to string winding.

(ii) Non-associativity (3-bracket):

$$[X_R^i(\tau,\sigma), X_R^j(\tau,\sigma), X_R^k(\tau,\sigma)]$$

$$:= [[X_R^i(\tau,\sigma), X_R^j(\tau,\sigma)], X_R^k(\tau,\sigma)] + \text{perm.} = R^{ijk}$$

$$R^{ijk} = 3\tilde{D}^{[i}\beta^{jk]}\,, \quad \tilde{D}^i \equiv \tilde{\partial}^i - \beta^{ij}\partial_j$$

V) (Intersecting) Q- and R-branes

(F. Haßler, D.L., arXiv:1303.1413)

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Standard NS action:

$$S = \int \mathrm{d}^{10}x \, e^{-2\phi} \sqrt{|G|} \left(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} \right)$$

(i) NS 5-brane solution - source of H-flux:

	x^0	x^1	x^2	x^3	y^1	y^2	y^3	y^4	y^5	y^6
NS5	$\otimes$	$\otimes$	$\otimes$					$\otimes$	$\otimes$	$\otimes$

$$ds^2_{NS5} = \sum_i (dx^i_{||})^2 + h(r) \sum_k (dx^k_{\perp})^2$$

$$H_{mnp} = \epsilon_{mnpq} \partial_q h(r)$$

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$$\begin{aligned} ds^2_{NS5} &= \sum_i (dx^i_{\|})^2 + h(r)\sum_k (dx^k_{\perp})^2 \\ H_{mnp} &= \epsilon_{mnpq}\partial_q h(r) \end{aligned}$$

(ii) T-duality in y^1-direction $\rightarrow$ KK-monopole - source of f-flux:

	x^0	x^1	x^2	x^3	y	y^2	y^3	y^4	y^5	y^6
KK	$\otimes$	$\otimes$	$\otimes$		$\bullet$			$\otimes$	$\otimes$	$\otimes$

$$ds^2_{KK} = \sum_{\mu=0,1,2}(dx^\mu)^2 + \sum_{i=4,5,6}(dy^i)^2 + \frac{1}{h(r)}\left(dy + \sum_{i=2,3} A_i dy^i\right)^2 + h(r)\left((dx^3)^2 + \sum_{i=2,3}(dy^i)^2\right)$$

(iii) T-duality in y^2-direction $\longrightarrow$ Q-brane - source of Q-flux:

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	x^0	x^1	x^2	x^3	y	y'	y^3	y^4	y^5	y^6
Q	$\otimes$	$\otimes$	$\otimes$		$\bullet$	$\bullet$		$\otimes$	$\otimes$	$\otimes$

$$ds_Q^2 = \sum_{\mu=0,1,2} (dx^\mu)^2 + \sum_{i=4,5,6} (dy^i)^2 + \frac{h(r)}{h(r)^2 + A_2^2}(dy^2 + dy'^2) + h(r)\Big((dx^3)^2 + (dy^3)^2\Big)$$

$$B_{y,y'} = -\frac{A_2}{h(r)^2 + A_2^2}$$

See also: Bergshoeff, Ortin, Riccioni (2011); de Boer, Shigemori (2010,2012)

This is a non-geometric configuration and hence it is globally ill defined with respect to the standard NS action.

(iii) T-duality in y^2-direction $\longrightarrow$ Q-brane - source of Q-flux:

	x^0	x^1	x^2	x^3	y	y'	y^3	y^4	y^5	y^6
Q	$\otimes$	$\otimes$	$\otimes$		$\bullet$	$\bullet$		$\otimes$	$\otimes$	$\otimes$

$$ds_Q^2 = \sum_{\mu=0,1,2}(dx^\mu)^2 + \sum_{i=4,5,6}(dy^i)^2 + \frac{h(r)}{h(r)^2+A_2^2}(dy^2+dy'^2) + h(r)\Big((dx^3)^2+(dy^3)^2\Big)$$

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See also: Bergshoeff, Ortin, Riccioni (2011); de Boer, Shigemori (2010,2012)

This is a non-geometric configuration and hence it is globally ill defined with respect to the standard NS action.

However the Q-brane is a well-defined solution of the redefined NS action:

$$\tilde{S} = \int \mathrm{d}^{10}x\,\sqrt{|\tilde{G}|}e^{-2\tilde{\phi}}\left(\tilde{\mathcal{R}} + 4(\partial\tilde{\phi})^2 - \frac{1}{4}Q^2\right)$$

$$d\tilde{s}_Q^2 = \sum_{\mu=0,1,2}(dx^\mu)^2 + \frac{1}{h(r)}\Big(dy^2+dy'^2\Big) + h(r)\Big((dx^3)^2+(dy^3)^2\Big) + \sum_{i=4,5,6}(dy^i)^2\,,$$

$$\beta_Q^{y,y'} = -A_2$$

Q-flux: $Q_3^{y,y'} = \partial_{y^3}\beta_Q^{y,y'} = -Q$

(iv) T-duality in y^3-direction $\longrightarrow$ R-brane - source of R-flux:

	x^0	x^1	x^2	x^3	y	y'	y''	y^4	y^5	y^6
R	$\otimes$	$\otimes$	$\otimes$		$\bullet$	$\bullet$	$\bullet$	$\otimes$	$\otimes$	$\otimes$

However there does not exist a local metric in terms of the original coordinates.

One has to use a dual coordinate:

$$\begin{aligned} d\tilde{s}^2_R &= \sum_{\mu=0,1,2} (dx^\mu)^2 + \frac{1}{h(r)}\left(dy^2 + dy'^2 + d\tilde{y}''^2\right) + h(r)(dx^3)^2 + \sum_{i=4,5,6} (dy^i)^2 \,, \\ \beta_R^{y,y'} &= -R\tilde{y}'' \,, \end{aligned}$$

R-flux: $R^{y,y',y''} = \partial_{\tilde{y}''}\beta_R^{y,y'} = -R$

The intersection of NS 5-branes, KK-monopoles, Q-branes and R-branes leads to supersymmetric type IIA/B backgrounds of the form:

$$M_{10} = AdS_4 \times M_6^{H,f,Q,R}$$

See also: Kounnas, D.L., Petropoulos, Tsimpis (2007)

$M_6^{H,f,Q,R}$ are compact, 6-dimensional flux backgrounds with SU(3) x SU(3) group structure.

The spaces M_6 are torus fibrations of T^4 over T^2 .

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The spaces M_6 are torus fibrations of T^4 over T^2.

Corresponding closed string boundary conditions:

$$\begin{aligned}
Y^m(\tau,\sigma+2\pi) &= Y^m(\tau,\sigma) + 2\pi\ \tilde{p}^m \\
Y^i(\tau,\sigma+2\pi) &= Y^i(\tau,\sigma) + f^i_{jm}\,\tilde{p}^m\ Y^j(\tau,\sigma) + Q^{ij}_m\,\tilde{p}^m\ \tilde{Y}_j(\tau,\sigma)\,, \\
\tilde{Y}_i(\tau,\sigma+2\pi) &= \tilde{Y}_i(\tau,\sigma) - f^j_{im}\,\tilde{p}^m\ \tilde{Y}_j(\tau,\sigma) + H^{ijm}\,\tilde{p}^m\ Y^j(\tau,\sigma)\,.
\end{aligned}$$

This leads to the following closed string commutators:

$$[Y^i(\tau,\sigma), Y^j(\tau,\sigma)] \quad \simeq \quad Q^{ij}_m\ \tilde{p}^m$$

(i) IIA with four H-fluxes:

	x^0	x^1	x^2	x^3	y^1	y^2	y^3	y^4	y^5	y^6
NS5	$\otimes$	$\otimes$	$\otimes$		$\otimes$		$\otimes$		$\otimes$	
NS5′	$\otimes$	$\otimes$	$\otimes$		$\otimes$			$\otimes$		$\otimes$
NS5″	$\otimes$	$\otimes$	$\otimes$			$\otimes$		$\otimes$	$\otimes$	
NS5‴	$\otimes$	$\otimes$	$\otimes$			$\otimes$	$\otimes$			$\otimes$

Effective 4D flux superpotential:

$$W_H^{IIA} = H_{2,4,6}S + H_{2,3,5}U_1 + H_{1,4,5}U_2 + H_{1,3,6}U_3$$

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	x^0	x^1	x^2	x^3	y^1	y^2	y^3	y^4	y^5	y^6
NS5	$\otimes$	$\otimes$	$\otimes$		$\otimes$		$\otimes$		$\otimes$	
NS5′	$\otimes$	$\otimes$	$\otimes$		$\otimes$			$\otimes$		$\otimes$
NS5″	$\otimes$	$\otimes$	$\otimes$			$\otimes$		$\otimes$	$\otimes$	
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$$W_H^{IIA} = H_{2,4,6}S + H_{2,3,5}U_1 + H_{1,4,5}U_2 + H_{1,3,6}U_3$$

(ii) IIA with four f-fluxes (Iwasawa manifold):

	y^1	y^2	y^3	y^4	y^5	y^6
KK	$\otimes$	$\bullet$	$\otimes$		$\otimes$	
KK′	$\otimes$	$\bullet$		$\otimes$		$\otimes$
KK″	$\bullet$	$\otimes$		$\otimes$	$\otimes$	
KK‴	$\bullet$	$\otimes$	$\otimes$			$\otimes$

$$\begin{aligned} W_f^{IIA} &= f^2_{4,6}ST_1 + f^2_{3,5}T_1U_1 \\ &+ f^1_{4,5}T_1U_2 + f^1_{3,6}T_1U_3 \end{aligned}$$

(iii) IIA with four Q-fluxes:

	y^1	y^2	y^3	y^4	y^5	y^6
Q	$\otimes$	$\bullet$	$\otimes$	$\bullet$	$\otimes$	
Q′	$\otimes$	$\bullet$	$\bullet$	$\otimes$		$\otimes$
Q″	$\bullet$	$\otimes$	$\bullet$	$\otimes$	$\otimes$	
Q‴	$\bullet$	$\otimes$	$\otimes$	$\bullet$		$\otimes$

$$\begin{aligned} W_Q^{IIA} &= Q_6^{2,4} S T_1 T_2 + Q_5^{2,3} T_1 T_2 U_1 \\ &+ Q_5^{1,4} T_1 T_2 U_2 + Q_6^{1,3} T_1 T_2 U_3 \end{aligned}$$

(iii) IIA with four Q-fluxes:

	y^1	y^2	y^3	y^4	y^5	y^6
Q	$\otimes$	$\bullet$	$\otimes$	$\bullet$	$\otimes$	
Q′	$\otimes$	$\bullet$	$\bullet$	$\otimes$		$\otimes$
Q″	$\bullet$	$\otimes$	$\bullet$	$\otimes$	$\otimes$	
Q‴	$\bullet$	$\otimes$	$\otimes$	$\bullet$		$\otimes$

$$W_Q^{IIA} = Q_6^{2,4} S T_1 T_2 + Q_5^{2,3} T_1 T_2 U_1 + Q_5^{1,4} T_1 T_2 U_2 + Q_6^{1,3} T_1 T_2 U_3$$

(iv) IIBA with four R-fluxes:

	y^1	y^2	y^3	y^4	y^5	y^6
R	$\otimes$	$\bullet$	$\otimes$	$\bullet$	$\otimes$	$\bullet$
R′	$\otimes$	$\bullet$	$\bullet$	$\otimes$	$\bullet$	$\otimes$
R″	$\bullet$	$\otimes$	$\bullet$	$\otimes$	$\otimes$	$\bullet$
R‴	$\bullet$	$\otimes$	$\otimes$	$\bullet$	$\bullet$	$\otimes$

$$W_R^{IIA} = R^{2,4,6} S T_1 T_2 T_3 + R^{2,3,5} T_1 T_2 T_3 U_1 + R^{1,4,5} T_1 T_2 T_3 U_2 + R^{1,3,6} T_1 T_2 T_3 U_3$$

(v) IIB with two H-fluxes and two f-fluxes:

	y^1	y^2	y^3	y^4	y^5	y^6
NS5	$\otimes$		$\otimes$		$\otimes$	
NS5$'$	$\otimes$			$\otimes$		$\otimes$
KK$''$	$\bullet$	$\otimes$		$\otimes$	$\otimes$	
KK$'''$	$\bullet$	$\otimes$	$\otimes$			$\otimes$

(vi) IIB with two f-fluxes and two Q-fluxes:

	y^1	y^2	y^3	y^4	y^5	y^6
KK	$\otimes$	$\bullet$	$\otimes$		$\otimes$	
Q$'$	$\otimes$	$\bullet$	$\bullet$	$\otimes$		$\otimes$
Q$''$	$\bullet$	$\otimes$	$\bullet$	$\otimes$	$\otimes$	
KK$'''$	$\bullet$	$\otimes$	$\otimes$			$\otimes$

(vii) IIB with two Q-fluxes and two f-fluxes:

	y^1	y^2	y^3	y^4	y^5	y^6
Q	$\otimes$	$\bullet$	$\otimes$	$\bullet$	$\otimes$	
R$'$	$\otimes$	$\bullet$	$\bullet$	$\otimes$	$\bullet$	$\otimes$
Q$''$	$\bullet$	$\otimes$	$\bullet$	$\otimes$	$\otimes$	
R$'''$	$\bullet$	$\otimes$	$\otimes$	$\bullet$	$\bullet$	$\otimes$

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- Mixed closed string bound conditions (in analogy to mixed D-N boundary conditions for D-branes) lead to closed string non-commutativity.

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  Unified description of H,f,Q,R-fluxes in double geometry needs a 4D-dimensional phase space.

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- Mixed closed string bound conditions (in analogy to mixed D-N boundary conditions for D-branes) lead to closed string non-commutativity.

- Investigation of the phase space of doubled geometry

  Unified description of H,f,Q,R-fluxes in double geometry needs a 4D-dimensional phase space.

- Is there are non-commutative (non-associative) theory of gravity? (Non-commutative geometry & gravity: P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess (2005 L. Alvarez-Gaume, F. Meyer, M. Vazquez-Mozo (2006))

V) Outlook & open questions

- Mixed closed string bound conditions (in analogy to mixed D-N boundary conditions for D-branes) lead to closed string non-commutativity.

- Investigation of the phase space of doubled geometry

  Unified description of H,f,Q,R-fluxes in double geometry needs a 4D-dimensional phase space.

- Is there are non-commutative (non-associative) theory of gravity? (Non-commutative geometry & gravity: P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess (2005
  L. Alvarez-Gaume, F. Meyer, M. Vazquez-Mozo (2006))

- What is the generalization of quantum mechanics for this non-associative geometry?
  How to represent this algebra?