# Holographic Constraints on a Vector boson

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# Introduction-Motivation

Holography is one of the main tools for studying the proeprties of of strongly coupled QFTs. Recently: also proved useful for investigating formal aspects of QFTs

e.g.

- holographic c-theorem and entanglement entropy
- energy-flux constraints and unitarity .

Such results often come from enlarging the set of gravity duals under consideration, e.g. higher derivative gravities.

# Introduction-Motivation

*In this talk*:

**A massive vector boson propagates in the bulk of AdS. We investigate the consistency of its electromagnetic and gravitational couplings.**

**In top-down holographic models such couplings are fixed** *(e.g. consistent truncations, DBI)*. **[talk by C. Rosen]**

**Here we take a phenomenological approach. Assume AdS gravity defines a CFT and check if it consistent.**

**Based on work with R. Rahman.**

# Introduction-Motivation

*Is there reason to expect such constraints from holography?*

- **Assuming duality relates two theories, certain pathologies may be more manifest on one side of the duality.**
- **Explicit example in the past – Gauss-Bonnet gravity**

$$\mathcal{L} = R - \frac{d(d+1)}{2} L^2 + \lambda_{GB} \frac{L^2}{2} \left( R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2 \right)$$

**For arbitrary values of the Gauss-Bonnet coupling, $\lambda_{GB}$, the theory is classically consistent (causal propagation, no ghosts).**

**The dual CFT violates causality unless $\lambda_{GB}$:**
$-\frac{(3d+2)(d-2)}{4(d+2)^2} \leq \lambda_{GB} \leq \frac{(d-2)(d-3)(d^2-d+6)}{4(d^2-3d+6)}$.

[Brigante etal][Buchel etal][deBoer, Parnachev, MK][Camanho etal]

# Outline

- Massive vector field and its interactions.
- AdS/CFT setup: background and fluctuations.
- Fluctuation analysis and WKB.
- Holographic Constraints.
- Summary, conclusions and open questions.

# Massive spin-1: Free case.

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 - \frac{1}{2}m^2 W_\mu W^\mu$$
$$= -\frac{1}{2}(\partial_\mu W_\nu)^2 + \frac{1}{2}(\partial_\mu W^\mu)^2 - \frac{1}{2}m^2 W^\mu W_\mu$$

**The two lines are equivalent since ordinary derivatives commute with each other.**

**EOM and constraint:**

$$\left(\Box - m^2\right) W_\mu = 0, \qquad \partial_\mu W^\mu = 0, \qquad m^2 \neq 0$$

**In $d+1$ dimensions there are $d$ (causally) propagating degrees of freedom.**

# Massive spin-1: Electromagnetic Interactions.

**Lorentz invariance, parity and time-reversal implies**

$$\mathcal{L} = -|\nabla_\mu W_\nu|^2 + |\nabla_\mu W^\mu|^2 - m^2 W_\mu^* W^\mu + iqg F^{\mu\nu} W_\mu^* W_\nu +$$
$$+ \frac{i}{2}\Delta Q_e \left[ W_\mu^* \left( \nabla_\nu W_\rho - \nabla_\rho W_\nu \right) \partial^\mu F^{\nu\rho} - \textbf{h.c.} \right]$$

$$[\nabla_\mu, \nabla_\nu] W_\rho = iq F_{\mu\nu} W_\rho, \qquad \mu_m = \frac{qg}{2m}, \qquad Q_e = q\frac{1-g}{m^2} + \Delta Q_e$$

- $q$ **is the charge**
- $g$ **is the gyromagnetic ratio or $g$-factor.**
- $Q_e$ **the quadrupole moment.**

# Massive spin-1: Electromagnetic Interactions.

**EOM and constraint:**

$$\Box W_\mu - \frac{iq(g-2)}{m^2}\nabla_\mu \left(F^{\rho\nu}\nabla_\rho W_\nu\right) + \cdots = 0$$

$$\nabla_\mu W^\mu - \frac{iq(g-2)}{m^2}F^{\mu\nu}\nabla_\mu W_\nu + \Delta Q_e \left(\cdots\right) = 0$$

$d$ **propagating degrees of freedom in** $d+1$ **dimensions.**

**Causallity requires**

- $g$**-factor arbitrary**
- $\Delta Q_e = 0$ **[Velo-Zwanziger]**

# Massive spin-1:Electromagnetic Interactions

Determine Hyperbolicity and Causality of the EOM by the method of characteristic determinant.

- Replace the highest derivatives in EOM $i\partial_\mu$ by $n_\mu$
  $n_\mu$ is the normal to the characteristic hypersurface.
- Find the determinant $\Delta(n)$ of the resulting coefficient matrix.

  The system is hyperbolic when:

  $\Delta(n) = 0$ has real solutions for $n_0$ for any $\mathbf{n}$.

  The system is causal when:

  $u_{max} = \frac{|n_0|}{|\mathbf{n}|} \leq 1$. Timelike $n_\mu \Rightarrow$ acausal propagation.

# Massive spin-1: Adding Gravity.

$$\mathcal{L} = -|\nabla_\mu W_\nu|^2 + |\nabla_\mu W^\mu|^2 - m^2 W_\mu^* W^\mu + iqgF^{\mu\nu} W_\mu^* W_\nu - \\ - hR^{\mu\nu} W_\mu^* W_\nu$$

- **A gravitational quadrupole term is present due to the non-commutativity of the covariant derivatives.**
- $h$ **is referred to as the gravimagnetic ratio or** $h$**-factor**
- **The theory is classically consistent for arbitrary values of the** $h$**-factor**

$$\left(\nabla^2 - m^2\right) W_\mu - \nabla_\mu (\nabla \cdot W) + iqgF_{\mu\nu} W^\nu - hR_{\mu\nu} W^\nu = 0$$

# Holography: the setup

*Objective:* To assess the consistency of the couplings $g$, $h$ using holography.

Qubic couplings $\Leftrightarrow$ three-point functions in CFT.

Investigating three-point functions in the dual boundary CFT holographically, involves writing down an action third order in the fluctuating fields.

- Is there a simpler way?

Consider the quadratic action in an asymptotically AdS background with non-trivial profiles for $F_{\mu\nu}$, $R_{\mu\nu}$.

# Holography: the setup

**Simplest choice:**

**The extremal AdS-Reissner-Nordstrom black hole.**

$$ds^2 = \frac{L^2}{r^2}\frac{dr^2}{f(r)} + \frac{r^2}{L^2}\left[-f(r)dt^2 + d\vec{x}^2\right], \quad A_\sigma = \mu\left(1 - \frac{r_0^{d-2}}{r^{d-2}}\right)\delta_\sigma^t$$

$$f(r) = 1 + \frac{d}{d-2}\left(\frac{r_0}{r}\right)^{2d-2} - \frac{2(d-1)}{d-2}\left(\frac{r_0}{r}\right)^d$$

**Chemical Potential and Charge Density**

$$\mu = \sqrt{\frac{d(d-1)}{2(d-2)^2}}\frac{g_F r_0}{L^2} \qquad \rho = \frac{\sqrt{2d(d-1)}}{g_F}\left(\frac{L}{l_p}\right)^{d-1}\left(\frac{r_0}{L^2}\right)^{d-1}$$

# Holography: the setup

**AdS/CFT viewpoint:**

- **massive, spin-1** $W_\mu \Longleftrightarrow$ **non-conserved operator** $\mathcal{O}_\mu$
  $m^2 \neq 0 \to \Delta > d-1$
- **RN-background** $\Longleftrightarrow$ **CFT at finite charge density**
  $\langle J_t \rangle \neq 0$.
- **Quadratic action for** $W^\mu$ **in the bulk** $\Longleftrightarrow$
  **2-point function of** $\mathcal{O}_\mu$

**Consider the eom**

$$\left(\nabla^2 - m^2\right) W_\mu - \nabla_\mu \left(\nabla \cdot W\right) + iqgF_{\mu\nu}W^\nu - hR_{\mu\nu}W^\nu = 0$$

**as eom for fluctuations of** $W_\mu$ **in the bulk.**

# Fluctuation analysis

**Define**

$$\delta g = 2 - g \qquad \delta h = h - 1 \qquad \mathfrak{m}^2 = m^2 - d\delta h$$

*Note:* $\mathfrak{m}$ **is the effective mass in pure AdS.**

**To avoid instabilities $\mathfrak{m}^2 > 0$ (BF bound).**

**Set $d = 4$ and Fourier transform:**

$$W_\mu(r,t,x_3) = \int \frac{d\omega dk}{(2\pi)^2} \widehat{W}_\mu(r)\, e^{i(kx_3 - \omega t)}.$$

**Degrees of freedom are distinguished into longitudinal $\widehat{W}_{\mu=0,3}$ and transverse $\widehat{W}_{\mu=2,3}$. $\widehat{W}_r$ is not a dynamical field; can be completely determined from the longitudinal modes.**

# Fluctuation Analysis

Transverse modes decouple from the rest (no obvious pathology). To study the longitudinal modes make a field redefinition

$$\mathcal{E}_+(r) \equiv k\widehat{W}_t(r) + [\omega - qA_t(r)]\,\widehat{W}_3(r),$$
$$\mathcal{E}_-(r) \equiv [\omega - qA_t(r)]\,\widehat{W}_t(r) + kf(r)\widehat{W}_3(r).$$

Now the boundary operators dual to the new fields are completely decoupled.

EoM still coupled and complicated. Can be simplified in a suitable scaling limit, where frequency $\omega$, momentum $k$ and chemical potential $\mu \equiv q\mu$ are large.

# Fluctuation Analysis

The scaling limit is

$$kz \gg 1 \ \& \ \mu z \gg 1, \ \text{with } u \equiv \frac{\omega}{k} = \text{fixed} \ \& \ v \equiv \frac{q\mu}{k} = \text{fixed}.$$

where the new radial variable is defined as $z = \frac{r}{r_0}$.

The coupled eom for the modes $\mathcal{E}_\pm$ are

$$\mathcal{E}_+'' + ka(z)\mathcal{E}_-' + k^2 b(z)\mathcal{E}_+ = 0,$$
$$\mathcal{E}_-'' + kc(z)\mathcal{E}_+' + k^2 d(z)\mathcal{E}_- = 0,$$

- Functions $a(z)$, $b(z)$, $c(z)$, $d(z)$ have no simple form but are all rational functions of $z$.
- $a(z)$ and $c(z)$ are proportional to $v\delta g$. When either the field is uncharged or $\delta g = 0$ the equations decouple.

# WKB

WKB for coupled differential equations [Yabana-Horiuchi].

Consider the standard ansatz

$$\mathcal{E}_\pm(z) = e^{ikS_\pm(z)}, \qquad S_\pm \equiv S_\pm^{(0)} + k^{-1} S_\pm^{(1)} + \cdots$$

The standard amplitude factor for the WKB solution is hidden in $S_\pm^{(1)}$.

Substitute the ansatz into eom - order by order analysis:

- $S_\pm^{(0)} = S^{(0)}$, *i.e.*, $S^{(0)}$ is independent of the mode.
- Following standard conventions, set

$$p(z) \equiv \frac{dS_0}{dz}$$

# WKB

- Leading order analysis shows that a non-trivial solution requires

$$\mathbb{G} \equiv \begin{vmatrix} -p^2 + b(z) & ia(z)p(z) \\ ic(z)p(z) & -p^2(z) + d(z) \end{vmatrix} = 0.$$

  and thus determines $p(z)$.

- Next order analysis determines $S_\pm^{(1)}$ etc.

The leading order WKB solution is of the usual form

$$\mathcal{E}_\pm \simeq \mathcal{A}_\pm(z) e^{ik \int p_\pm(z)}$$

where $\mathcal{A}_\pm(z)$ is real and inversely proportional to $\sqrt{\frac{\partial \mathbb{G}}{\partial p}}$.

# WKB

**The phase factor is**

$$p_\pm^2 = \frac{1}{z^4 f^2(z)} \left[ \left( \sqrt{E_\pm} + \frac{v}{z^2} \right)^2 - V_\pm \right],$$

**where**

$$V_\pm(z) = \frac{f(z)}{\mathfrak{m}^2 z^6 - 8\delta h} \left[ H(z) + \delta g^2 v^2 z^2 \mp 2\sqrt{\delta g^2 v^2 z^2 H(z) + K^2(z)} \right]$$

$$H(z) = \delta g\,(2 + \delta g)\, v^2 z^2 + \mathfrak{m}^2 z^6 - 2\delta h \qquad K(z) = \delta g\, v^2 z^2 + 3\delta h.$$

$V_\pm(z)$ **vanishes at the horizon** $z = 1$ **and tends to unity at the boundary** $z = \infty$**.** $E_\pm$ **is the square of the phase velocity** $u - v$**.**

# WKB

The equation

$$\left.\frac{\partial \mathbb{G}}{\partial p}\right|_{z_t} = 0 \quad \Rightarrow \quad p(z_t) = 0 \quad \textbf{or} \quad p(z_t) = -\tfrac{1}{2}(ac - b - d).$$

determines the turning points where the WKB approximation breaks down.

The first solution $p(z) = 0$ is familiar from the single channel WKB analysis. The second solution is a new feature of the coupled system and corresponds to the point where the phases of the different modes $p_+(z)$ and $p_-(z)$ coalesce.

# WKB

**Study of the turning points reveals:**

- **The second class of turning points do not exist in this case.**
- **The first class exists and the standard approach for matching the solutions can be used.**

**Treating the boundary $z = \infty$ as an infinite wall yields**

$$k \int_{z_t}^{\infty} p(z) + \mathcal{O}(\widetilde{k}^0) = \pi \left( n \pm \tfrac{1}{4} \right), \qquad n = 1, 2, \ldots \ldots .$$

# WKB

The quantization condition yields the group velocity

$$u_g \equiv \frac{d\omega}{dk} - v = \frac{\int u\frac{\partial p}{\partial u} + v\frac{\partial p}{\partial v} - p}{\int \frac{\partial p}{\partial u}} - v$$

The integrals are strongly peaked around the turning point and one can approximate the group velocity with:

$$u_g^2 \simeq E\left[1 - \frac{v}{\sqrt{E}}\left(\frac{\partial u}{\partial v}\right)_p\right]^2_{z=z_t} .$$

At the turning point the phase velocity $E > 1$. For a neutral field $v = 0$ and for $v << 1$ in general, the existence of a turning point will lead to causality violation.

# WKB

- Should appropriately constrain the bulk couplings $g$, $h$ so that a turning point does not exist.

Consider the effective potential ($v > 0$)

$$V_{\pm}^{\text{eff}}(z) \equiv \left(\sqrt{V_{\pm}} - \frac{v}{z^2}\right)^2,$$

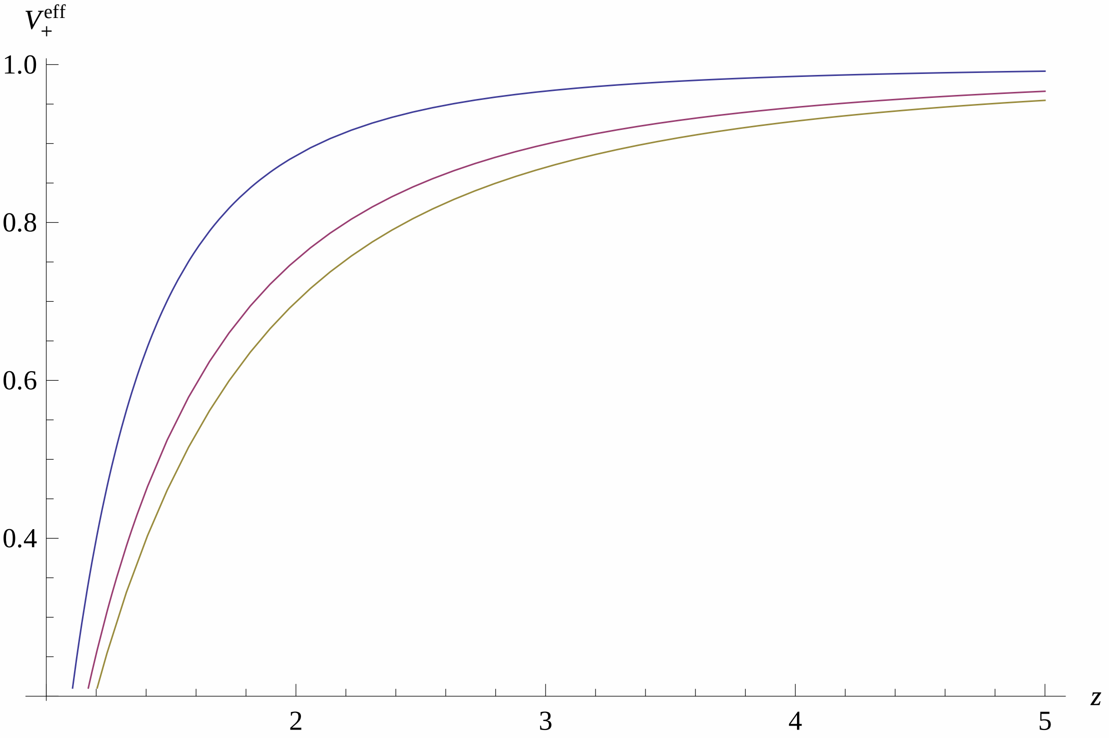
equal to the phase velocity $E$ at the turning point $z_t$.

At the boundary, $V_{\pm}^{\text{eff}}$ is normalized to unity.

A turning point exists when it develops a maximum in the bulk.

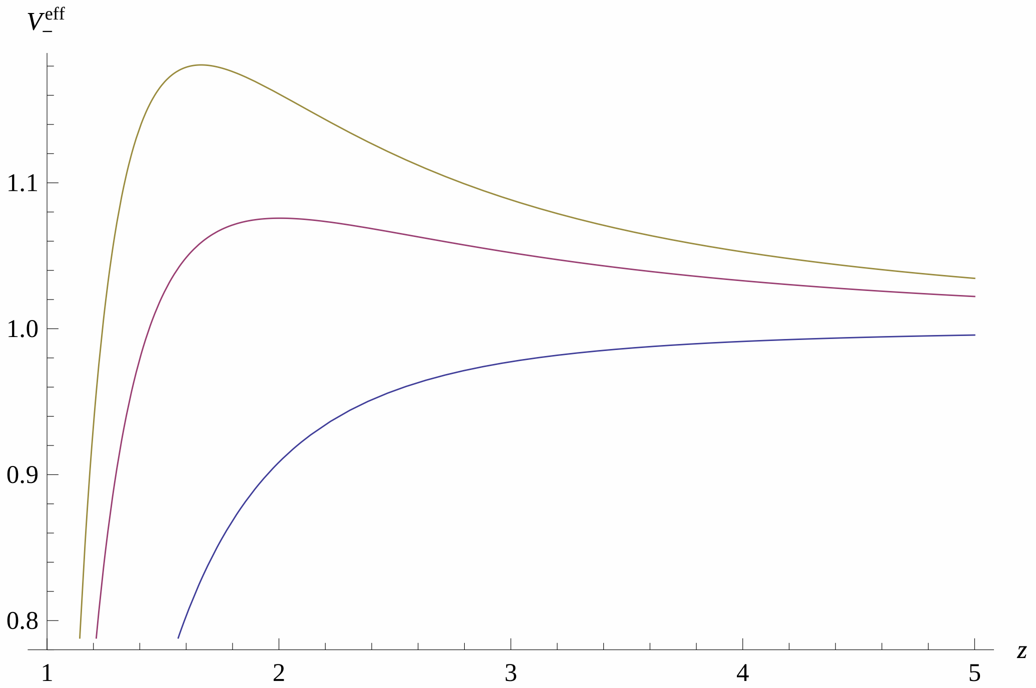
# Holographic Constraints

- **The (+) mode is always consistent.**



# Holographic Constraints

- $V_{eff}^{-}$ **for different values of** $\delta g$.
  **As** $\delta g$ **increases, a turning point develops.**



# Holographic Constraints

**Expand close to the boundary**

$$V_{\pm}^{\text{eff}} = 1 - \frac{2v}{z^2}\left(1 \pm \frac{|\delta g|}{\sqrt{\mathfrak{m}^2}}\right) + \mathcal{O}(z^{-4}),$$

**The (-) potential develops a maximum when**

$$\delta g^2 > m^2$$

**While for $v \equiv \frac{q\mu}{k} << 1$, the group velocity behaves like**

$$u_g^2 = 1 + v + \mathcal{O}(v^2) > 1$$

**Consistency of the dual boundary CFT requires $\delta g^2 < m^2$.**

# Holographic Constraints

We can separately analyze the neutral case $v = 0$. The potential is

$$V_{i=\pm} = \left(1 - \frac{3}{z^4} + \frac{2}{z^6}\right)\left[1 + \delta_i^- \frac{12\delta h}{\mathfrak{m}^2 z^6 - 8\delta h}\right],$$

If the potential increases as we move inwards from the boundary, where $V_i(z = \infty) = 1$, there will be a turning point with $E = V_i > 1$. Group velocity will then be greater than unity.

Causality combined with stability requirements then yield

$$-\tfrac{1}{4}\mathfrak{m}^2 \;\leq\; \delta h \;<\; \tfrac{1}{8}\mathfrak{m}^2.$$

# Summary

- **Holography constraints both $\delta g$ and $\delta h$.**

$$\delta g^2 < \mathfrak{m}^2 \quad \textbf{and} \quad -\tfrac{1}{4}\mathfrak{m}^2 \;\leq\; \delta h \;<\; \tfrac{1}{8}\mathfrak{m}^2.$$

- **Constraints depend on the mass $\mathfrak{m}$ of the bulk field. In the dual CFT this translates to a dependance on the conformal dimension $\Delta$ of the spin-1 operator.**

- **For a field of arbitrary mass consistency requires**

$$g = 2, \qquad h = 1$$

# Discussion-Open Questions

- Constraints obtained in simlar manner to the case of Gauss-Bonnet gravity.

  In Gauss-Bonnet theory $\lambda_{GB}$ is related via holography to the parameters which determine the two- and three- point functions of the stress energy tensor.

  Holographically obtained constraints on $\lambda_{GB}$ turn out to be in "1-1" correspondence to the positivity of energy flux constraints.

  Example: $d+1=5$

$$-\frac{7}{36} \leq \lambda_{GB} = \frac{(a-5c)(a-c)}{4(a-3c)^2} \leq \frac{9}{100} \quad \Rightarrow -\frac{1}{2} \leq \frac{c-a}{c} \leq \frac{1}{2}$$

# Discussion: Energy Flux one-point function

*Definition*: The energy flux operator $\mathcal{E}(\widehat{n})$ per unit angle measured through a very large sphere of radius $r$ is

$$\mathcal{E}(\widehat{n}) = \lim_{r\to\infty} r^{d-2} \int dt\, \widehat{n}^i\, T^0_i(t, r\widehat{n}^i)$$

$n^i$ is a unit vector specifying the position on $S^{d-2}$ where energy measurements may take place. Integrating over all angles yields the total energy flux at large distances.

Focus on the energy flux one-point function on states created by the stress–energy tensor operator

$$\mathcal{O}_q = \epsilon_{ij} T_{ij}(q)$$

with $\epsilon_{ij}$ a symmetric, traceless polarization tensor.

# Discussion: Energy Flux one-point function

- Rotational symmetry fixes the form of the energy flux one–point function up to two independent parameters.

$$\langle \mathcal{E}(\widehat{n}) \rangle_{T_{ij}} = \frac{\langle \epsilon^*_{ik} T_{ik} \mathcal{E}(\widehat{n}) \epsilon_{lj} T_{lj} \rangle}{\langle \epsilon^*_{ik} T_{ik} \epsilon_{lj} T_{lj} \rangle} =$$

$$= \frac{E}{\Omega_{d-2}} \left[ 1 + t_2 \left( \frac{\epsilon^*_{il} \epsilon_{lj} n_i n_j}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{1}{d-1} \right) + t_4 \left( \frac{|\epsilon_{ij} n_i n_j|^2}{\epsilon^*_{ij} \epsilon_{ij}} - \frac{2}{d^2-1} \right) \right]$$

Here $t_2$, $t_4$ are arbitrary constants. By construction, they can be related to $a$, $c$.

# Discussion: Energy Flux and Constraints

**Demand positivity of the energy flux one point function,** *i.e.*, $\langle \mathcal{E}(\widehat{n}) \rangle \geq 0$.

**Positivity of the energy flux yields constraints on** $t_2, t_4$:

$$C_G(\mathcal{A}, \mathcal{B}, \mathcal{C}) \equiv 1 - \frac{1}{d-1} t_2 - \frac{2}{d^2-1} t_4 \geq 0$$

$$C_V(\mathcal{A}, \mathcal{B}, \mathcal{C}) \equiv 1 - \frac{1}{d-1} t_2 - \frac{2}{d^2-1} t_4 + \frac{t_2}{2} \geq 0$$

$$C_S(\mathcal{A}, \mathcal{B}, \mathcal{C}) \equiv 1 - \frac{1}{d-1} t_2 - \frac{2}{d^2-1} t_4 + \frac{d-2}{d-1}(t_2 + t_4) \geq 0$$

# Discussion: Energy Flux and Holography

- **Positivity of the energy flux constraints on states created by $T_{\mu\nu}$ obtained holography from the study of the two-point function of $T_{\mu\nu}$ in AdS-black hole.**
  - **Compare this calculation with ours:**

Comparison with Calculation of Energy Flux Constraints

| | |
|---|---|
| $\langle T_{\mu\nu} T_{\rho\sigma} \rangle_{\Delta=4}$ | $\langle \mathcal{O}_\mu \mathcal{O}_\nu \rangle_{\Delta(g)}$ |
| $\langle T_{00} \rangle \neq 0$ | $\langle T_{00} \rangle \neq 0$ **&** $\langle J_t \rangle \neq 0$ |
| **BH &** $T \neq 0$ | **RN-BH &** $\mu \neq 0$ |

# Discussion-Open Questions

- What is the meaning of these constraints? What do the $g$, $h$-factors correspond to in field theory?

**Possible interpretation:**

$$h\text{-factor} \quad \longleftrightarrow \quad \langle \mathcal{O}_\mu \mathcal{E} \mathcal{O}_\nu \rangle$$

$$\mathcal{E} = \lim_{r\to\infty} r^{d-2} \int dt\, \hat{n}^i\, T^0_i(t, r\hat{n}^i)$$

$$g\text{-factor} \quad \longleftrightarrow \quad \left\langle \mathcal{O}^\dagger_\mu \mathcal{Q} \mathcal{O}_\nu \right\rangle$$

$$\mathcal{Q} \equiv \lim_{r\to\infty} r^{d-2} \int dt\, \hat{n}^i\, j_i(t, r\hat{n}^i)$$

# Discussion-OpenQuestions

- **The following field redefinition**

$$g_{\mu\nu} \to g_{\mu\nu}\left(1 + a_1|W|^2\right) + a_2 W_\mu^\star W_\nu$$

**absorbs the $h$-coupling and introduces quartic terms** *e.g.* $W_\mu^\star W_\nu W_\rho^\star W_\sigma$**.**

  - **$h$-factor constraints must then be related to four- and higher- point functions of $\mathcal{O}_\mu$.**

- **Constraints result from requiring $\langle \mathcal{E} \rangle \geq 0$.**

**Is there any physical reason for $\langle \mathcal{Q} \rangle \geq 0$ ?**

**Constraints related to higher-point functions?**

# Discussion-Open Questions

More questions ...

- The meaning of the scaling limit:

  Violation of causality observed for $q\mu$ large compared to other couplings but small compared to the momentum $k$.

- Constraints related to the conformal dimension of the operator - here we implicitly work in the strong coupling regime.

# Discussion-Open Questions

More things to do...

- Holographic computation of three point functions.
- CFT analysis of the three- and four- point functions - constraints?
- Similar analysis for fields of other spin?
- Holographic bounds for generic values of $q$.

# Discussion-Open Questions

- Gravity side - a different perspective:
  - What is known for the action of a massive spin-1 field in flat space?
    * Tree level unitarity requires $g = 2$ [Cornwall, Levin, Tiktopoulos].
    * Standard model predicts $g = 2$ at tree level.
    * No known constraint for $h$ unless susy - then $h = 1$ [Giannakis, Liu, Porrati].
  - What is the holographic analysis telling us? Pathologies in classically (seemingly) consistent theories show up in the dual boundary CFT.

*Thank you!*