

## The Chern-Simons diffusion rate in Improved Holographic QCD

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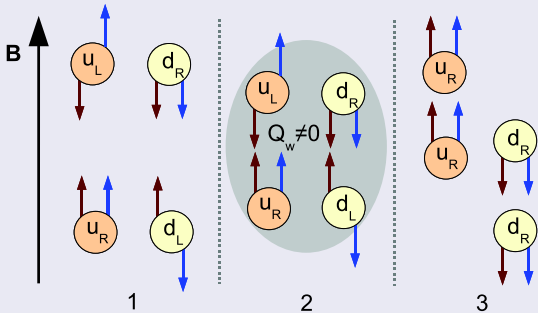
## Outline

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## The chiral magnetic effect

- The Chiral magnetic effect is argued to take place when matter with net chirality is put in a background magnetic field, (Fukushima-Kharzeev-McLerran-Warringa).
- Right-handed particles move along the direction of their spin and left-handed particle in opposite direction than their spin. In case of strong magnetic field the spins of positively charged particles align in the direction of the field. The spin of negative charged particles is opposite.
- Therefore, net chirality creates an electric current parallel to the magnetic field.

## The chiral magnetic effect



**Figure :** The red and blue arrows denote the momentum and spin, respectively. When a strong magnetic field,  $\vec{B}$ , is exerted in a medium of charges the positive charges align towards  $\vec{B}$  and the negative opposite to  $\vec{B}$ . Moreover, right-handed particles move in the same direction with their spin and left-handed in the opposite, (1). When, the medium has net chirality, (2), separation of charges takes place, (3), (Kharzeev-McLerran-Warringa).

- Chern-Simons number,  $N_{CS}$ , characterizes vacua of  $SU(N_c)$  YM which cannot be connected with continuous gauge transformations

$$N_{CS} \equiv \frac{1}{8\pi^2} \int d^3x \epsilon_{ijk} \text{Tr} \left[ A_i \partial_j A_k - \frac{2ig}{3} A_i A_j A_k \right], \quad (1)$$

where  $i, j, k = 1, 2, 3$ .

- The change in  $N_{CS}$  is

$$\Delta N_{CS} = \int d^4x q(x^\mu), \quad q(x^\mu) \equiv \frac{1}{16\pi^2} \text{Tr} [F \wedge F] = \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma}. \quad (2)$$

- Configurations with non-zero  $N_{CS}$  lead to axial anomalies. Consider  $N_f$  massless fermions coupled to the YM. Then, the theory enjoys  $U(N_f)_L \times U(N_f)_R$  global symmetry. The isospin singlet axial current is

$$j_5^\mu = \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu \gamma^5 \psi_i \quad (3)$$

- This current is not conserved due to a quantum anomaly

$$\partial_\mu j_5^\mu = -\frac{N_f}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu} F_{\rho\sigma}. \quad (4)$$

- Taking the expectation value of the above equation in some state and integrating over spacetime

$$\Delta N_5 = N_L - N_R|_{t=\infty} - (N_L - N_R|_{t=-\infty}) = 2N_f \Delta N_{CS}. \quad (5)$$

Thus, changes in  $N_{CS}$  corresponds to a change in chirality of the medium.

- Consider now the transition amplitude,  $A$ , from a state  $|\psi_1(t_1)\rangle$  to  $|\psi_2(t_2)\rangle$  times the change in axial number

$$A \Delta N_5 = 2 \int d^4x \langle \psi_2 | q(x) | \psi_1 \rangle \quad (6)$$

- Taking the square of the above equation, summing over all possible  $|\psi_2(t_2)\rangle$  and using the completeness relation  $\mathbb{I} = \sum |\psi_2\rangle\langle\psi_2|$

$$\langle(\Delta N_5)^2\rangle = 4 \int d^4x d^4y \langle\psi_1|q(x)q(y)|\psi_1\rangle \quad (7)$$

- The Chern-Simons diffusion rate,  $\Gamma_{CS}$ , is the rate of change of  $N_{CS}$  per unit volume and is given by the Wightman two-point function of  $q(x^\mu)$ ,

$$\Gamma_{CS} \equiv \frac{\langle(\Delta N_{CS})^2\rangle}{Vt} = \int d^4x \langle q(x^\mu)q(0)\rangle_w . \quad (8)$$

- In equilibrium states with finite temperature,  $\Gamma_{CS}$  is given by

$$\Gamma_{CS} = - \lim_{\omega \rightarrow 0} \frac{2T}{\omega} \text{Im} G_R(\omega, \vec{k} = 0), \quad (9)$$

where  $G_R(\omega, \vec{k})$  is the Fourier transformation of the retarded Green function of  $q(x^\mu)$ .

- The contribution of instanton backgrounds to  $\Gamma_{CS}$  is exponentially suppressed.
- $\Gamma_{CS}$  can be generated by thermal fluctuations in finite temperature states. Those excite shaleron configurations which produce non-zero  $\Gamma_{CS}$  upon decay.
- Finite  $\Gamma_{CS}$  in QCD signals the creation of net chirality bubbles (domains of more left-handed than right-handed quarks or the opposite) because of the anomaly of the axial current.
- When those bubbles are in a background magnetic field, the chiral magnetic effect is created.



## Improved Holographic QCD (IHQCD)

### Fields & the dual operators

- Can we use gauge/gravity duality to calculate  $\Gamma_{CS}$ ?
- **IHQCD** is an effective holographic model for **4d pure Yang-Mills**. It's construction uses intuition from string theory and input from QCD (Gursoy-Kiritsis-Nitti).
- The bulk fields and their dual operators are

Bulk Fields		Dual Operator
Metric	$g_{\mu\nu}$	$\text{Tr} \left[ F_{\mu\rho} F_{\nu}^{\rho} - \frac{\delta_{\mu\nu}}{4} F^2 \right]$
Dilaton	$\phi$	$\text{Tr} [F^2]$
Axion	$\alpha$	$\text{Tr} [F \wedge F]$

## Improved Holographic QCD (IHQCD)

## The action

- The action of IHQCD is

$$S = M_p^3 N_c^2 \int d^5x \sqrt{-g} \left[ R - \frac{4}{3} \frac{(\partial\lambda)^2}{\lambda^2} + V(\lambda) - \frac{Z(\lambda)}{2N_c^2} (\partial\alpha)^2 \right], \quad (10)$$

where  $M_p^3 = 1/(16\pi G_5 N_c^2)$ .

- $\lambda = e^\phi$  is the holographic 't Hooft coupling with potential  $V(\lambda)$ .
- $\alpha(r)$  is the holographic  $\theta$  angle.
- $Z(\lambda)$  encodes the interaction of the dilaton with the axion.
- Requiring IHQCD free energy to follow Stefan-Boltzmann law for a gluon gas at high  $T$ , we fix  $(M_p L)^{-3} = 45\pi^2$ .

## The vacuum

- The zero temperature vacuum solution is of the type

$$ds^2 = e^{2A_0(r)}(dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu), \quad \lambda = \lambda(r), \quad (11)$$

where  $e^{A_0(z)}$  is the metric scale factor corresponding to the energy scale of QCD.

- As  $r \rightarrow 0$ ,  $e^{A_0(r)} \rightarrow L/r + \mathcal{O}(1/\log(r))$  and  $\lambda \rightarrow -1/\log r$ , so as to match the running of the large- $N_c$  YM coupling.
- The axion field contribution to the vacuum energy is of order  $\mathcal{O}(1/N_c^2)$ .
- There are two branches of black hole solutions to the model. **The large black holes are the thermodynamically preferred** solutions. A first order confinement/deconfinement phase transition happens.

## The dilaton potential

- The running of 't Hooft coupling in YM fixes the **UV asymptotics of  $V(\lambda)$**

$$V(\lambda) \simeq \frac{12}{L^2} \left( 1 + v_0 \lambda + v_1 \lambda^2 + \mathcal{O}(\lambda^3) \right). \quad (12)$$

- The criteria for confinement and asymptotic linear Regge trajectories of the glueball spectra fix the **IR behavior of  $V(\lambda)$**

$$V(\lambda) \propto \lambda^{\frac{4}{3}} \sqrt{\log \lambda}. \quad (13)$$

## The action

- The axion part of the action in IHQCD is reminded

$$S_\alpha = -\frac{1}{2} M_p^3 \int d^5x \sqrt{-g} Z(\lambda) (\partial\alpha)^2, \quad (14)$$

(Gursoy-Kiritsis-Nitti).

- $\alpha(x, r)$  is dual to  $q(x)$ .  $\alpha(r=0) = \kappa\theta$ , where  $\kappa$  is a constant and  $\theta$  is the YM  $\theta$ -angle.
- The asymptotics of  $Z(\lambda)$  are

$$Z(\lambda) \propto \begin{cases} Z_0 + \mathcal{O}(\lambda), & \lambda \rightarrow 0, \\ \lambda^4 + \mathcal{O}(1/\lambda), & \lambda \rightarrow \infty, \end{cases}. \quad (15)$$

Its IR asymptotics is decided by requiring universal linear glueball Regge trajectories for large excitation number.

## The interaction with dilaton ( $Z(\lambda)$ )

- The simplest form for  $Z(\lambda)$ , which was first considered in IHQCD, is

$$Z(\lambda) = Z_0(1 + c_4\lambda^4), \quad (16)$$

- $Z_0$  is fixed by matching to the large- $N$  YM lattice results for the **topological susceptibility**,  $\chi \equiv -\frac{d^2\mathcal{E}(\theta)}{d\theta^2}$ , where  $\mathcal{E}(\theta)$  is the vacuum energy density.

$$\chi \simeq (191\text{MeV})^4 \Rightarrow \kappa^2 Z_0 = 33.25 \quad (17)$$

(Vicari-Panagopoulos)

## The interaction with dilaton ( $Z(\lambda)$ )

- $c_4$  is fixed by matching to lattice results for the first **axial glueball mass**.

$$m_{0^{-+}}/m_{0^{++}} = 1.50 \Rightarrow c_4 = 0.26. \quad (18)$$

- The next mass of the  $0^{-+}$  tower is very well predicted

$$(m_{0^*{}^{-+}}/m_{0^{++}})_{IHQCD} = 2.10, \quad (m_{0^*{}^{-+}}/m_{0^{++}})_{lat.} = 2.11(6). \quad (19)$$

(Morningstar-Peardon)

## Holographic $\Gamma_{CS}$

- We consider axion fluctuation around the deconfined (large black hole) background of the IHQCD and compute the imaginary part of the retarded Green's function. Then, by definition (33) we find

$$\frac{\Gamma_{CS}}{sT/N_c^2} = \frac{\kappa^2 Z(\lambda_h)}{2\pi}, \quad (20)$$

which has **implicit dependence on  $T$  through  $Z(\lambda_h)$** .  $\lambda_h$  is the value of the dilaton at the black hole horizon.

- Due to the small values of  $\lambda_h$  in the large black hole branch, terms with lower power than  $\lambda^4$  in  $Z(\lambda)$  play an important role in  $\Gamma_{CS}$ .



## Different forms of $Z(\lambda)$

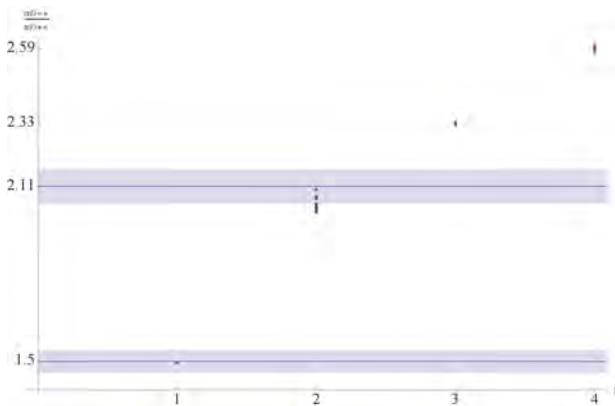
- The largest polynomial correction to  $Z(\lambda)$  for small values of  $\lambda$  is linear,

$$Z(\lambda) = Z_0 \left( 1 + c_1 \lambda + c_4 \lambda^4 \right). \quad (21)$$

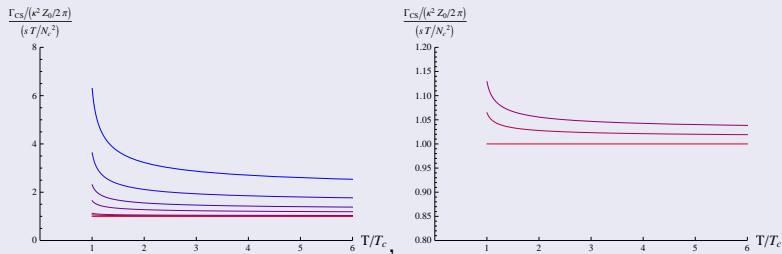
Upon fitting to the first axial glueball mass we find that for any  $c_1 > 0$  there is a  $c_4$  for which the value of the second excited state exhibits at most 3% discrepancy from the large  $N_c$  lattice value.

- Simple non-monotonic  $Z(\lambda)$ s with the desired asymptotics, (15), produce glueball masses that deviate from lattice results more than 10%.
- The assumption that  $Z(\lambda)$  is monotonic in  $\lambda$  sets the bound  $Z(\lambda) > Z_0$  for any temperature  $T$ .

## Different forms of $Z(\lambda)$



**Figure :** IHQCD masses of the  $0^{-+}$  glueball states with excitation number  $n$ , normalized to the  $0^{++}$  mass for various  $(c_1, c_4)$ . From the top (red) dots to the bottom (blue) dots the values of  $(c_1, c_4)$  increase,  $(c_1, c_4) = (0, 0.26), (0.5, 0.87), (1, 2.2), (5, 24), (10, 75), (20, 230), (40, 600)$ . The two horizontal blue lines with surrounding blue bands indicate the results and errors, respectively, of the large- $N$  YM lattice masses of the lowest and first excited states,  $n = 1$  and  $n = 2$ , (Morningstar-Peardon).



**Figure :** (a) The numerical result for  $(\Gamma_{CS}/\kappa^2 Z_0/2\pi)/(sT/N_c^2)$ , as functions of  $T/T_c$ , for the  $Z(\lambda)$  in eq. (21), for increasing values of the parameters  $(c_1, c_4)$ , from the bottom (red) curve to the top (blue) curve,  $(c_1, c_4) = (0, 0.26), (0.5, 0.87), (1, 2.2), (5, 24), (10, 75), (20, 230), (40, 600)$ . (b) Close-up of the curves for (from bottom to top)  $(c_1, c_4) = (0, 0.26), (0.5, 0.87), (1, 2.2)$ . In all of these cases, as  $T$  approaches  $T_c$  from above  $\Gamma_{CS}/(sT/N_c^2)$  increases by anywhere from 0.01% up to a factor greater than six. The increase occurs mostly between  $2T_c$  and  $T_c$ .

- A numerical estimate of  $\Gamma_{CS}$  near the confinement-deconfinement phase transition is

$$\Gamma_{CS}(T_c)/T_c^4 = 0.31 \times \frac{\kappa^2 Z(\lambda_c)}{2\pi} > 0.31 \times \frac{\kappa^2 Z_0}{2\pi} = 1.64, \quad (22)$$

where we have used the lattice result for the entropy density,  $s(T_c) = 0.31 N_c^2 T_c^3$ , (Lucini-Teper-Wenger).

- An upper bound can also be set to  $\Gamma_{CS}(T_c)$  by considering a  $Z(\lambda)$  of the form of (21), with  $c_1 = 5$  and  $c_4 = 50$ , which produce the axial glueball spectrum within one sigma with respect to the lattice results. Then,

$$1.64 \leq \Gamma_{CS}(T_c)/T_c^4 \leq 2.8. \quad (23)$$

$\Gamma_{CS}$  in  $\mathcal{N} = 4$  sYM.

- The result in large- $N$  strongly coupled  $\mathcal{N} = 4$  sYM is

$$\Gamma_{CS}^{\mathcal{N}=4} = \frac{\lambda_t^2}{2^8 \pi^3} T^4. \quad (24)$$

(Son-Starinets).

- For some typical values  $\lambda_t = 6\pi$  and  $N = 3$  we have

$$\Gamma_{CS}^{\mathcal{N}=4} / T^4 \simeq 0.045, \quad (25)$$

which is much smaller than the IHQCD result.

- A naive extrapolation of the perturbative result, which is valid for  $\alpha_s \ll 1$ ,  $(-\ln \alpha_s)^{-1} \ll 1$ , to  $\alpha_s = 0.5$ ,

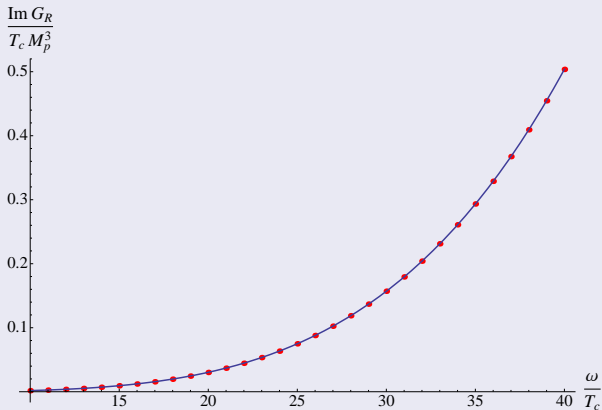
$$\Gamma_{CS}^{(per.)}(T_c) / T_c^4 = 30 \alpha_s^5 \simeq 1 \quad (26)$$

(Moore-Tassler)

## Im $G_R(\omega, \vec{k} = 0)$

- We now calculate the axion spectral function ( $\text{Im}G_R(\omega, \vec{k})$ ) for non-zero  $\omega$  and  $|\vec{k}|$ . We firstly calculate it for  $\vec{k} = 0$
- For small  $\omega$ , the axion fluctuation equation yields  $\text{Im} G_R(\omega, \vec{k} = 0) \propto \omega$ .
- For large  $\omega$ , we expect  $\text{Im} G_R(\omega, \vec{k} = 0) \propto \omega^4$  because the theory is conformally invariant in the UV and  $q(x^\mu)$  is dimension four.

# $\text{Im } G_R(\omega, \vec{k} = 0)$



**Figure :**  $\text{Im } G_R(\omega, \vec{k} = 0)/(T_c M_p^3)$  as a function of  $\omega/T_c$ , at  $T_c$ , for the  $Z(\lambda)$  in eq. (16). The red dots are our numerical results that match the solid blue curve,  $(1.6 \times 10^{-7}) \times (\omega/T_c)^{4.051}$ , which is the expected large  $\omega$  behavior of the spectral function.

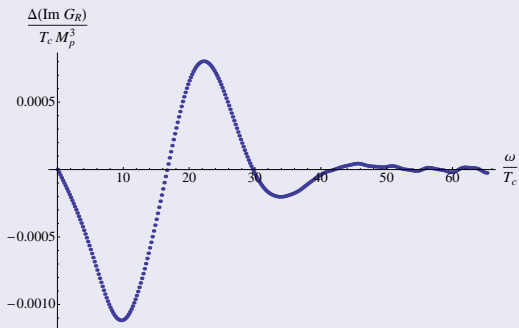
## The renormalized $\text{Im } G_R(\omega, \vec{k} = 0)$

- The  $\omega^4$  scaling of  $\text{Im } G_R(\omega, \vec{k} = 0)$  at large  $\omega$  is a divergence in the coincidence limit of the two-point function, which may overwhelm peaks in  $\text{Im } G_R(\omega, \vec{k})$ , rendering them practically invisible.
- To eliminate the large- $\omega$  divergence we compute  $G_R(\omega, \vec{k})$  at two different temperatures,  $T_1$  and  $T_2$ , and then take the difference,

$$\Delta G_R(\omega, \vec{k}; T_1, T_2) \equiv G_R(\omega, \vec{k}) \Big|_{T_2} - G_R(\omega, \vec{k}) \Big|_{T_1}. \quad (27)$$

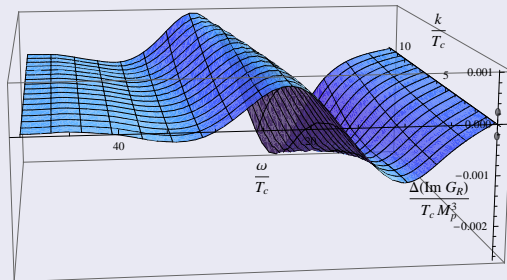


## The renormalized $\text{Im } G_R(\omega, \vec{k} = 0)$



**Figure :** The difference  $\Delta \text{Im } G_R(\omega, \vec{k} = 0; T_c, 2T_c)/(T_c M_p^3)$  as a function of  $\omega/T_c$ , for the  $Z(\lambda)$  in eq. (16). The difference goes to zero as  $\omega/T_c \rightarrow \infty$ . The prominent minimum at  $\omega/T_c \approx 10$  and maximum at  $\omega/T_c \approx 22$  indicate a shift in spectral weight with increasing  $T$ , possibly from the motion of a peak in the spectral function. There is a peak at  $\omega \simeq 20 T_c \simeq 1600 \text{MeV}$  and width  $\sim 10 T_c \sim 1300 \text{MeV}$ .

## The renormalized $\text{Im } G_R(\omega, \vec{k})$



**Figure :**  $\Delta\text{Im } G_R(\omega, \vec{k} = 0; T_c, 2T_c)/(T_c M_p^3)$  in terms of  $\omega/T_c$  and  $|\vec{k}|/T_c$ , for the  $Z(\lambda)$  in eq. (16). As  $|\vec{k}|$  increases up to  $|\vec{k}|/T_c \approx 10$ , the largest peak shifts from  $\omega/T_c \approx 22$  up to  $\omega/T_c \approx 30$ . The width of the peak changes very little.

## Conclusions

- The Chern-Simons diffusion rate was calculated in IHQCD for  $T > T_c$  and found it proportional to  $Z(\lambda_h)$ , which is the value of the axion action normalization factor at  $\lambda = \lambda_h$ .  $\lambda_h$  is the value of the dilaton at the black hole horizon.
- Assuming that  $Z(\lambda)$  is a monotonic function of  $\lambda$ , we argued that it is bounded from below by its value at  $T \rightarrow \infty$ .
- This is argued to be true for a general class of (3+1)-dimensional, confining, strongly-coupled, large- $N_c$  theories with holographic duals.
- A large value of  $\Gamma_{CS}(T_c)/T_c^4$ , compared to the  $\mathcal{N} = 4$  sYM and the extrapolated perturbative results, is favored in IHQCD.
- The presence of a long lived excitation of mass of order of the lightest  $0^{-+}$  glueball at  $T = 0$  was found by studying the axion spectral function.

## Future directions

- Fixing  $Z(\lambda)$  by the fitting to a lattice calculation of the Euclidean 2-point function of  $q(x^\mu)$  is an important future plan.
- Calculate  $\Gamma_{CS}$  in a model which incorporates the flavor degrees of freedom as well (V-QCD). Then, the axion is coupled to the axial vector flavor current and the pseudoscalar phase of the tachyon. Hence,  $\Gamma_{CS}$  is expected to be affected by tachyon dynamics.

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## Dirac equation in a background magnetic field

- The massless Dirac hamiltonan in the presence of a background gauge field,  $\vec{B} = B(x, y)\hat{z}$ , non-zero chemical potential,  $\mu$ , and chiral chemical potential,  $\mu_5$ , is

$$H = -i(\partial_i - ieA_i)\gamma^0\gamma^i + \mu - \mu_5\gamma^5, \quad H\Psi = E\Psi \quad (28)$$

- Decomposing  $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ , the eigenvalue equation becomes,

$$(H_{R/L} - \mu_{R/L})\psi_{R/L} = E\psi_{R/L}, \text{ where } H_{R/L} = \mp i(\partial_i - ieA_i)\sigma^i, \\ H_{R/L}\psi_{R/L} = \epsilon_{R/L}\psi_{R/L} \text{ and } E_{R/L} = \epsilon_{R/L} - \mu_{R/L}.$$

- $H_R = p_3\sigma^3 + H_\perp$ , for eigenstates of momentum in the  $\hat{z}$  direction, where  $H_\perp = -i(\partial_a - ieA_a)$ , ( $a = 1, 2$ ) is the transverse hamiltonian. Since,  $\{\sigma^3, H^\perp\} = 0$ ,  $\sigma_3$  maps eigenstates of  $H^\perp$  to other eigenstates with opposite eigenvalue

$$H_\perp|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow H_\perp\sigma^3|\lambda\rangle = -\lambda\sigma^3|\lambda\rangle \quad (29)$$

## Dirac equation in a background magnetic field

- $[H_R, H_\perp^2] = 0$ , hence for  $\lambda > 0$ ,  $\psi_R = c_1|\lambda\rangle + c_2\sigma^3|\lambda\rangle$ . The eigenvalues of  $H_R$  are  $\epsilon = \pm\sqrt{\lambda^2 + p_3^2}$ .
- $\sigma^3$  maps zero modes of  $H_\perp$  to zero modes.
- Zero modes ( $|\lambda_0\rangle$ ) of  $H_\perp$  are eigenstates of  $H_R$  with  $\epsilon_0 = p_3\sigma^3$ . We consider  $N_\pm$  zero modes of positive/negative spin,

$$N = N_+ - N_- = \text{Index}(H_\perp) = \left[ \frac{e\Phi}{2\pi} \right], \quad \Phi = \int d^2x B(x, y), \quad (30)$$

where  $[x]$  denotes the largest integer below  $x$ .

## Vector and chiral currents

The current densities are

$$j^3(x) = -e\psi_L^\dagger(x)\sigma^3\psi_L(x) + e\psi_R^\dagger(x)\sigma^3\psi_R(x), \quad j_5^3(x) = \psi_L^\dagger(x)\sigma^3\psi_L(x) + \psi_R^\dagger(x)\sigma^3\psi_R(x)$$

- The vev of the above currents in finite temperature and chemical potential receive contribution only from the transverse zero modes

$$\begin{aligned} \langle \psi_R^\dagger \sigma^3 \psi_R \rangle &= \sum_{\lambda_0} \int \frac{dp_3}{2\pi} n(p_3 \sigma^3 - \mu_R) \langle \lambda_0 | \sigma^3 | \lambda_0 \rangle = N \frac{\mu_R}{2\pi}, \\ \langle \psi_L^\dagger \sigma^3 \psi_L \rangle &= \sum_{\lambda_0} \int \frac{dp_3}{2\pi} n(-p_3 \sigma^3 - \mu_L) \langle \lambda_0 | \sigma^3 | \lambda_0 \rangle = N \frac{\mu_L}{2\pi}, \end{aligned} \quad (31)$$

where  $n(E - \mu) = \frac{\text{sgn}(E)}{e^{\text{sgn}(E)\beta(E - \mu)} + 1}$  is the Fermi-Dirac distribution, (Metliski-Zitnitsky).

- Hence,

$$J^3 = \frac{e^2 \Phi}{2\pi^2} \mu_5, \quad J_5^3 = \frac{e\Phi}{2\pi^2} \mu. \quad (32)$$



- In equilibrium states with finite temperature,  $\Gamma_{CS}$  is given by

$$\Gamma_{CS} = - \lim_{\omega \rightarrow 0} \frac{2T}{\omega} \text{Im } G_R(\omega, \vec{k} = 0), \quad (33)$$

where  $G_R(\omega, \vec{k})$  is the Fourier transformation of the retarded Green function of  $q(x^\mu)$ .

- $\Gamma_{CS}$  can be related to  $\Delta N_5$  by a fluctuation-dissipation relation of the form

$$\frac{d\Delta N_5}{dt} = -\Delta N_5 \frac{6N_f}{N_c} \frac{\Gamma_{CS}}{T^4}. \quad (34)$$

- Hence non-vanishing  $\Delta N_5$  induces topological fluctuations which balance  $\Delta N_5$  and relax it back to zero.

- The axion equation of motion in a background of the form, (11), is

$$\partial_r \left( Z(\lambda(r) e^{3A_0}) \partial_r \alpha(r) \right) = 0. \quad (35)$$

- The solution reads

$$\alpha(r) = \theta + 2\pi k + C \int_0^r \frac{dr}{L} \frac{1}{e^{3A_0} Z(\lambda)}, \quad k \in \mathbb{Z}. \quad (36)$$

- The on-shell action is the vacuum energy per volume

$$E(\theta) = \frac{M^3}{2L} C \alpha(r) \Big|_{r=0}^{r=r_0}. \quad (37)$$

Requiring to have contribution only from the UV boundary we find

$$C = (\theta + 2k\pi) \left( \int_0^{r_0} \frac{dr}{L} \frac{1}{e^{3A_0} Z(\lambda)} \right)^{-1}$$

- The vacuum energy reads

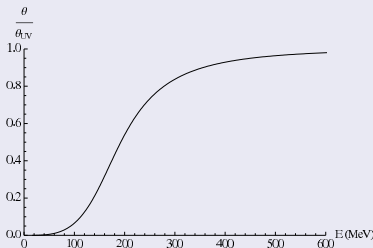
$$E(\theta) = -\frac{M^3}{2L} \frac{(\theta + 2k\pi)^2}{\int_0^{r_0} \frac{dr}{L} \frac{1}{e^{3A_0} Z(\lambda)}}. \quad (38)$$

- The topological susceptibility is

$$\chi \equiv \frac{d^2 \mathcal{E}(\theta)}{d\theta^2} = \int d^4 x \langle q(x^\mu) q(0) \rangle_E = \frac{\kappa^2 M_p^3}{\int_0^\infty \frac{dr}{e^{3A_0(r)} Z(\lambda(r))}}. \quad (39)$$

## The axion profile

- Typical behavior of the axion in a confining background solution ( $T \leq T_c$ ) with IR normalizable boundary conditions. The axion is interpreted as the **running  $\theta$ -angle** of the dual field theory.



- In the deconfined background ( $T > T_c$ ), the only non-singular solution is  $\alpha(r) = \kappa\theta = \text{const.}$  and the topological susceptibility is zero.

## The fluctuation equation

- The linear fluctuation equation of the axion is considered in the black hole background,

$$\frac{1}{Z(\lambda(r))\sqrt{-g}} \partial_r \left[ Z(\lambda(r))\sqrt{-g} g^{rr} \partial_r \delta\alpha(r, k^\mu) \right] - g^{\mu\nu} k_\mu k_\nu \delta\alpha(r, k^\mu) = 0, \quad (40)$$

where we have considered solutions of the form

$\delta\alpha(r, x^\mu) = \int \frac{d^4 k}{(2\pi)^4} e^{ikx} \delta\alpha(r, k^\mu) a(k^\mu)$ . The UV normalization is  $\lim_{r \rightarrow 0} \delta\alpha(r, k^\mu) = 1$ , and in-going wave boundary condition is set on the horizon.

- The on shell action of the fluctuation is

$$S_\alpha^{\text{on-shell}} = \int \frac{d^4 k}{(2\pi)^4} a(-k^\mu) \mathcal{F}(r, k^\mu) a(k^\mu) \Big|_0^{r_h}, \quad (41)$$

$$\mathcal{F}(r, k^\mu) \equiv -\frac{M_p^3}{2} \delta\alpha(r, -k^\mu) Z(\lambda(r)) \sqrt{-g} g^{rr} \partial_r \delta\alpha(r, k^\mu).$$

AdS/CFT yields that the retarded Green's function is

$$\hat{G}_R(\omega, \vec{k}) = -2 \lim_{r \rightarrow 0} \mathcal{F}(r, k^\mu). \quad (42)$$

Solving the fluctuation equation for  $\omega = 0$  and  $\vec{k} = 0$  we find

$$\delta\alpha = C_1 + C_2 \int_0^r \frac{dr'}{Z(\lambda(r')) e^{3A(r')} f(r')}, \quad (43)$$

UV normalization fixes  $C_1 = 1$ . For non-zero but small  $\omega$ , we expect  $C_2 \propto \omega$ . Expanding (43) close to the horizon we have

$$\delta\alpha = 1 + \frac{C_2}{Z(\lambda_h) e^{3A(r_h)} f'(r_h)} \log(r_h - r) + \mathcal{O}(r_h - r). \quad (44)$$

Then, we find

$$\lim_{r \rightarrow 0} \mathcal{F}(r, k^\mu) = -\frac{M_p^3}{2} C_1 C_2. \quad (\omega \ll T, \vec{k} = 0). \quad (45)$$

Going back and solving the fluctuation equation at  $r \rightarrow r_h$  for finite  $\omega$  and setting the outgoing solution to zero, we have

$$\delta\alpha = C_+(r_h - r)^{\frac{i\omega}{4\pi T}} + C_-(r_h - r)^{-\frac{i\omega}{4\pi T}} = C_- - i\frac{\omega}{4\pi T} C_- \log(r_h - r) + \mathcal{O}(\omega^2/T^2). \quad (46)$$

Matching the two solutions, we find that in the limit  $\omega \rightarrow 0$ ,  $C_- = 1$  and  $C_2 = -i\omega Z(\lambda_h) e^{3A(r_h)}$ . We finally have

$$\frac{\Gamma_{CS}}{sT/N_c^2} = \frac{\kappa^2 Z(\lambda_h)}{2\pi}, \quad (47)$$