Review of constraints on renormalization group flows in QFT

Based on published and unpublished work with
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Plan

1)CFT setup,Flows,The Cardy Conjecture

2)Trace Anomaly Matching for Spontaneously Broken CFT,

the Dilaton Effective Action

3)The “Proof” of the a-Theorem for Spontaneously Broken CFT

4)The Basic Sum Rule and the a-Theorem for Massive Flows

5)Flows with Scale Invariant End Points ?

6) Universal Relation of a-Theorems to d=2 ??

7)Open problems

1)Setup

Consider a Conformal Field Theory at the quantum level i.e. the energy-momentum tensor obeys the operatorial identities:

$$\partial^m T_{mn} = 0$$

and

$$T^m{}_m = 0$$

the beta function being 0.

The Ward identities are put in evidence by coupling to an external metric:

$$g_{mn} = \eta_{mn} + h_{mn}$$

The metric transforms under Weyl transformations as:

$$g_{mn} \to e^{2\sigma(x)} g_{mn}$$

The generating functional W is invariant under Weyl transformations except trace anomalies reflecting violations of tracelessness in certain 3-point functions of the energy momentum tensor:

$$\delta_\sigma W = c \int d^4x \sqrt{g}\, \sigma\, C_{mnpq} C^{mnpq} - a \int d^4x \sqrt{g}\, \sigma\, E_4$$

A generic QFT can flow between two CFTs , the UV and the IR in two ways :

a)Spontaneous breaking of conformal symmetry: a scalar operator gets a v.e.v. The theory becomes massive. Some states remain massless generically producing an interactive IR CFT.In particular there is necessarily a dilaton:the Goldstone boson of the spontaneously broken conformal symmetry.

b)Massive flows: the UV CFT is perturbed by a relevant or marginally relevant operator and flows to the IR .

Generalizing Zamolodchikov’s theorem in d=2,

Cardy proposed that also in d=4 for unitary theories the flows are irreversible :

$$a_{IR} < a_{UV}$$

"a" measures the degrees of freedom with very special weights per degree of freedom:

scalar=1 , massless fermion=11/4 , gauge boson =31

There is a lot of evidence for the validity of the conjecture , especially in SUSY models using the Intriligator+Wecht a-maximization procedure .

2)Trace Anomaly Matching for SB CFT

A CFT can exist in two phases:

a)unbroken:the vacuum is invariant under the full group of SO(2,4) transformations

b)spontaneously broken to the Poincare group if some dimensionful scalar operator(s) get vacuum expectation values:

$$\langle 0|\mathcal{O}|0\rangle = v^{\Delta}$$

This phase has massless and massive states and due to Goldstone's theorem necessarily a massless dilaton with a linear coupling to the energy momentum tensor:

$$\langle 0|T_{mn}|\tau; q\rangle = \frac{1}{3} f q_m q_n$$

Anomaly Matching : the a and c coefficients are the same in the two phases , the dilaton playing an essential role.

Fast argument: a and c are dimensionless and therefore cannot depend on the dimensionful parameter v. However a singular behaviour at the limit is not a priori excluded and this argument still would leave the possibility of dependence on dimensionless ratios when there are “multiple breakings” .

Sketch of the general proof(for the a anomaly):
in the 3-point functions of energy momentum tensor the conservation and tracelessness Ward identities reduce at the symmetric point in phase space to :

$$A(q^2) = 0\,, \qquad A(q^2) - q^2 B(q^2) = 0$$

Since the imaginary part cannot be anomalous , in both phases:

$$B(q^2) = \frac{a}{q^2}$$

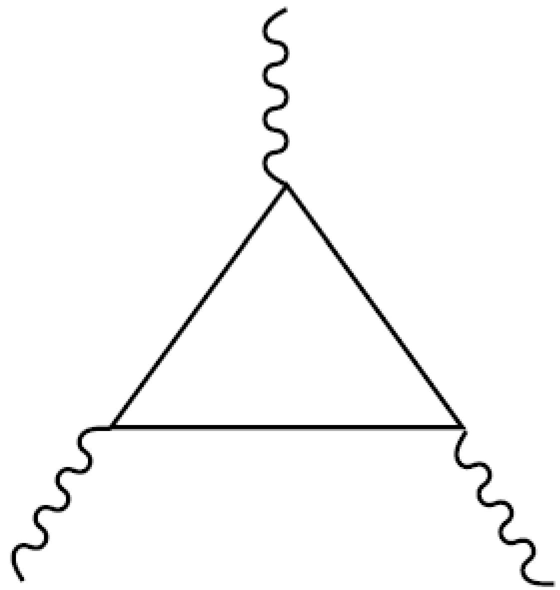
with possibly different coefficients in the two phases.
However in the broken phase for:

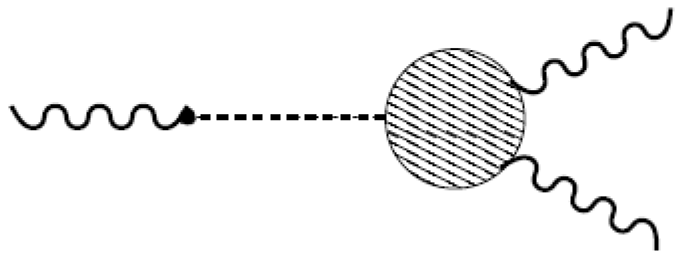
$$q^2 \gg v^2$$

one should recover the amplitudes of the unbroken phase i.e. the coefficient should be the same .
The analytic structure at general momenta of the correlators will be completely different in the two phases(e.g. massive singularities in the broken phase) however at the special kinematical point which "carries" the info about the trace anomaly the amplitudes are the same. Related to that the

$$q^2 = 0$$

pole in the amplitude does not represent a physical particle in the unbroken phase but a collapsed , unfactorizable cut ; in the broken phase it represents the contribution of the dilaton:





The dilaton couplings to two energy momentum tensors are fixed by the anomalies , in particular it should contain an a equal to:

$$a_{UV} - a_{IR}$$

analogous to the pion couplings in ‘t Hooft chiral anomaly matching.

(a_{IR} includes the “ordinary” =loop contribution of the dilaton)

The Dilaton Effective Action

Given the transformations under Weyl of the metric and the dilaton (as a Goldstone boson) :

$$g_{\mu\nu} \longrightarrow e^{2\sigma} g_{\mu\nu} \ , \qquad \tau \longrightarrow \tau + \sigma \ .$$

one constructs an effective action which reproduces the anomalies. One can use the general Wess-Zumino procedure or an iterative approach .The result is :

$$-a \int d^4x \sqrt{-g} \Big(\tau E_4 + 4 \big(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \big) \partial_\mu \tau \partial_\nu \tau$$

$$- 4 (\partial\tau)^2 \Box \, \tau + 2 (\partial\tau)^4 \Big) + c \int d^4x \sqrt{-g} \tau W^2_{\mu\nu\rho\sigma}$$

There is a Wess-Zumino term , i.e. a dilaton self interaction term at flat metric.

The action has ambiguities :

a)The anomalies can get a contribution from a cohomologically trivial term , $\int \sigma \Box R$ which however can be cancelled by a local term $\delta_\sigma \int R^2$ vanishing on flat space .

b)One can add to the action Weyl invariant terms; these can be generically written in terms of curvatures constructed from

$$\widehat{g} = e^{-2\tau} g_{\mu\nu}$$

Expanding in the number of derivatives ,the two derivative term contains the kinetic term for the dilaton :

$$S = f^2 \int d^4x e^{-2\tau} (\partial\tau)^2$$

which in terms of the field $\varphi = 1 - e^{-\tau}$

becomes canonical . At four derivatives one has quadratic polynomials in the curvatures and two total derivatives , $\widehat{\Box}\,\widehat{R}$ and Pontryagin.

Among the polynomials , the Euler combination is also a total derivative and the Weyl combination vanishes in flat space. Therefore one is left with one term :

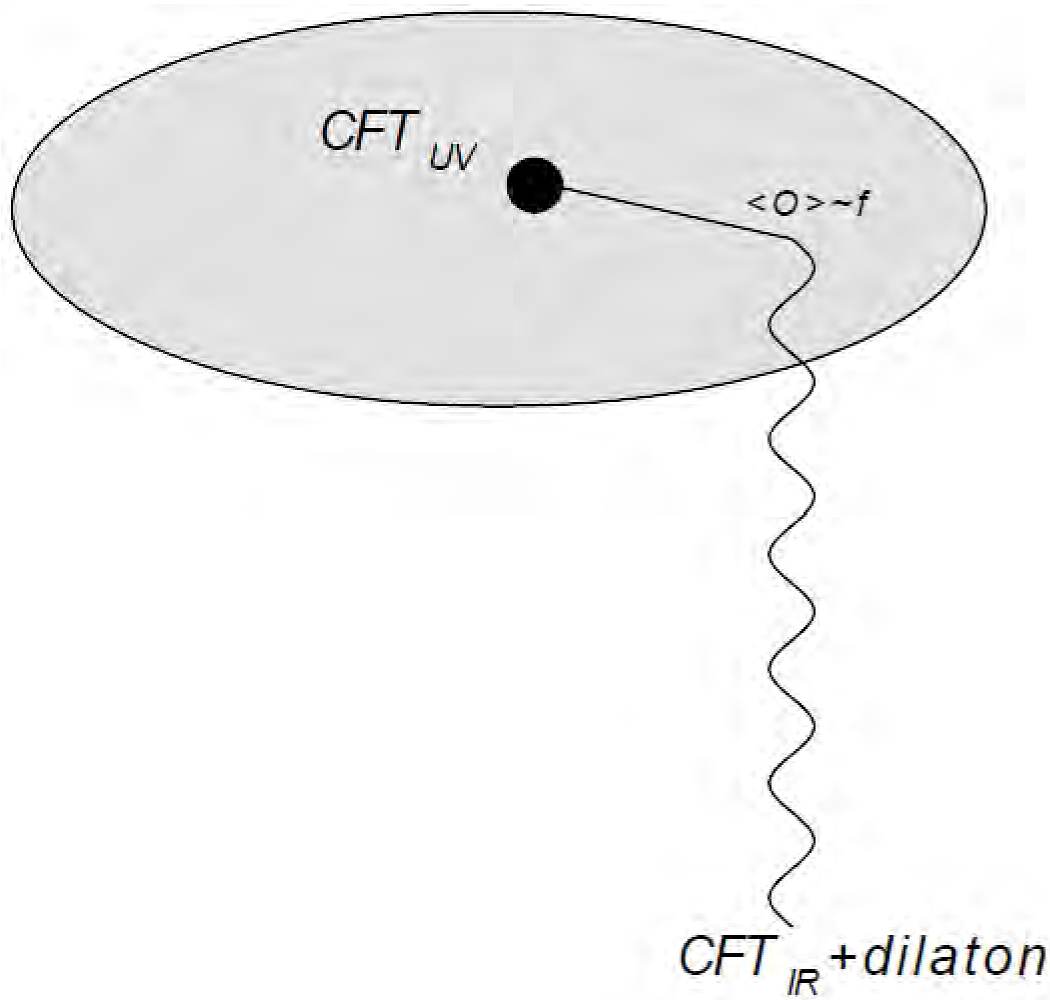
$$\int d^4x \sqrt{-\widehat{g}}\widehat{R}^2 \bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = 36 \int d^4x \left(\Box\,\tau - (\partial\tau)^2\right)^2$$

Combining with the anomalous functional , up to four derivatives one has two terms ,one normalized by the anomaly , the other arbitrary. On the mass shell however the arbitrary term vanishes since it contains explicit $\Box\,\varphi$ and the action becomes:

$$2a \int d^4x (\partial\tau)^4$$

3)The proof of the a-Theorem for SB CFT

Consider a CFT which has a moduli space such that except at the origin conformal symmetry is spontaneously broken :one has massive particles,



the dilaton, and in the infrared generically a CFT . Typical example :Coulomb branch of N=4 SUSY (P.Fayet,1979)
Due to anomaly matching the dilaton has an effective action normalized by $a_{UV} - a_{IR}$

Therefore on mass shell one gets a low energy theorem for the dilaton-dilaton scattering amplitude:

$$\mathcal{A}(s,t) = \frac{a_{UV} - a_{IR}}{f^4}\left(s^2 + t^2 + u^2\right) + \cdots$$

In the forward direction the value given by the low energy theorem is related through a dispersion relation to the positive imaginary part of the scattering amplitude :

$$a_{UV} - a_{IR} = \frac{f^4}{\pi}\int_{s'>0} ds' \frac{Im\mathcal{A}(s')}{s'^3}$$

One can define a monotonic interpolating function between the UV and IR :

$$a(\mu) \equiv a_{UV} - \frac{f^4}{\pi}\int_{s'>\mu} ds' \frac{\sigma(s')}{s'^2}$$

4)The Basic Sum Rule :Massive Flows

Consider a massive unitary flow between two conformal theories :the UV and IR.

Then the correlator of four energy momentum tensors in the massive theory obeys:

$$\int_0^\infty ds \frac{ImS(s,M)}{s^3} = a_{UV} - a_{IR}$$

where

$$S(s,M) =< T[T_i^i(k_1)T_j^j(k_2)T_l^l(-k_1)T_m^m(-k_2)] >$$

and we are in the “forward kinematics “ with

$s = (k_1 + k_2)^2$ and $k_1^2 = k_2^2 = 0$

$M =$ RG invariant mass scale

The a-Theorem follows immediately from the sum rule using the positivity of the integrand .

Analogous sum rules were used for chiral anomalies and by Cappelli et al for an alternative proof of Zamolodchikov’s c-theorem in d=2.

The proof of the Sum Rule(Outline).

Consider the correlator of three energy momentum tensors in the massive theory. Decompose it in invariant amplitudes singling out the dimension -2 amplitude “responsible”

for the a-anomaly in the conformal theory:

$$\tilde{G}^{(3)}_{ij,kl,mn}(k_1, k_2, k_3) = B(s, M) A_{ij,kl}(k_1, k_2) F_{mn}(k_3) + \ldots$$

where $F^{ij}(k) \equiv k^i k^j - k^2 \eta^{ij}$

and

$$
\begin{aligned}
A^{ij,kl}(k_1,k_2) \equiv & -\eta^{ij}\eta^{kl}(k_1\cdot k_2)^2 + k_1^k k_1^l k_2^i k_2^j \\
& -\frac{1}{2}(k_1\cdot k_2)\left(\eta^{ik}k_1^l k_2^j + \eta^{jk}k_1^l k_2^i + \eta^{il}k_1^k k_2^j + \eta^{jl}k_1^k k_2^i\right) \\
& -\frac{1}{2}\left(k_1^k k_1^i k_2^j k_2^l + k_1^k k_1^j k_2^i k_2^l + k_1^l k_1^i k_2^j k_2^k + k_1^l k_1^j k_2^i k_2^k\right) \\
& -\frac{1}{2}(k_1\cdot k_2)\left(\eta^{ik}k_1^j k_2^l + \eta^{jk}k_1^i k_2^l + \eta^{il}k_1^j k_2^k + \eta^{jl}k_1^i k_2^k\right) \\
& +(k_1\cdot k_2)\left(\eta^{ij}k_2^k k_1^l + \eta^{ij}k_2^l k_1^k + \eta^{kl}k_1^i k_2^j + \eta^{kl}k_1^j k_2^i\right) \\
& + k_1^i k_1^j k_2^k k_2^l + \frac{1}{2}(k_1\cdot k_2)^2\left(\eta^{ik}\eta^{jl} + \eta^{jk}\eta^{il}\right)
\end{aligned}
$$

for momenta which obey: $k_1^2 = k_2^2 = 0$

For $s \to \infty$ $B(s,M) \to -\frac{a_{UV}}{s}$

From the dispersion relation:

$$B(s,M) = \int_0^\infty ds' \frac{ImB(s',M)}{s'-s}$$

We obtain: $$\int_0^\infty ds' ImB(s',M) = a_{UV}$$

Since $ImB(s,M)$ contains $a_{IR}\delta(s)$

$$\int_0^\infty ds' \frac{Im[s'^3 B(s',M)]}{s'^3} = a_{UV} - a_{IR}$$

The four point function is related by diffeo invariance WI to the three point function:

$$\tilde{G}^{(4)}_{ij,kl,mn,pq}(k_1, k_2, -k_1, -k_2) =$$

$$= B(s, M)\Big(A_{ij,kl}(k_1, k_2) J_{mn,pq}(k_1, k_2) +$$

$$+ A_{mn,pq}(k_1, k_2) J_{ij,kl}(k_1, k_2) \Big) + ..$$

with :

$$J^{ij,kl}(k_1, k_2) \equiv \frac{1}{2}\left(\eta^{kl} k_2^i k_2^j + \eta^{ij} k_1^k k_1^l\right) - \frac{1}{4}\eta^{ij}\eta^{kl} k_1 \cdot k_2$$

$$- \frac{1}{4}\left(\eta^{jl} k_1^i k_2^k + \eta^{ik} k_1^j k_2^l + \eta^{jk} k_1^i k_2^l + \eta^{il} k_1^j k_2^k\right)$$

$$+ \frac{1}{4}\left(\eta^{ij} k_1^k k_2^l + \eta^{kl} k_1^i k_2^j + \eta^{ij} k_1^l k_2^k + \eta^{kl} k_1^j k_2^i\right)$$

$$-\frac{1}{4}\Big(\eta^{ik}k_2^j k_2^l + \eta^{jk}k_2^i k_2^l + \eta^{il}k_2^j k_2^k + \eta^{jl}k_2^i k_2^k +$$

$$+ \eta^{jl}k_1^i k_1^k + \eta^{il}k_1^j k_1^k + \eta^{jk}k_1^i k_1^l + \eta^{ik}k_1^j k_1^l\Big)$$

$$-\frac{1}{8}\Big(\eta^{ik}k_1^l k_2^j + \eta^{jk}k_1^l k_2^i + \eta^{il}k_1^k k_2^j + \eta^{jl}k_1^k k_2^i\Big)$$

$$+\frac{3}{8}\left(\eta^{ik}\eta^{jl} + \eta^{il}\eta^{jk}\right)k_1\cdot k_2$$

Taking the trace on all the indices we get:

$$\tilde{G}^{(4)}_{ij,kl,mn,pq}\eta^{ij}\eta^{kl}\eta^{mn}\eta^{pq} = s^3 B(s,M) + \ldots$$

One should consider the contributions to the sum rule of other invariant amplitudes .We distinguish between a)amplitudes present in the CFT and b)amplitudes present only in the massive theory ,being “killed” by the conformal Ward identities.

For a): besides the amplitude B related to the “a” anomaly there is an amplitude related to the “c” anomaly and one which is non-anomalous. The last two amplitudes vanish when more than one trace is taken and cannot contribute to the sum rule.

For b) we start with amplitudes $A(s, M)$ with positive dimensions $2r = 0, 2, 4$

Since the amplitudes cannot contribute in a CFT their imaginary parts should be subdominant in the UV and the IR:

$ImA(s, M) \rightarrow s^{r-\epsilon} M^{2\epsilon}$ for $s \rightarrow \infty$

$ImA(s, M) \rightarrow s^{r+\epsilon} M^{-2\epsilon}$ for $s \rightarrow 0$

It follows that we can write an unsubtracted dispersion relation for $\frac{\partial^r A}{\partial s^r}$

From the vanishing of the derivative in the IR :

$$\frac{\partial^r A}{\partial s^r} = 0 \qquad \text{at} \qquad s = 0$$

and the dispersion relation we obtain :

$$\int_0^\infty ds \frac{ImA(s,M)}{s^{r+1}} = 0$$

which is the condition that the amplitude does not contribute (through its descendent) to the sum rule .

For negative dimension $(= -2)$ amplitudes one repeats the argument used for the B-amplitude . Since there is no $1/s$ in the UV or IR one obtains the sum rule for B with 0 r.h.s. i.e. these amplitudes do not contribute either .

The suppressions of the Im parts do not need to be powerlike :a logarithmic one (like in QCD) allows the previous arguments to go through as well.

The possible additional contributions to the sum rule involve either four point functions which are descendents through the diffeo Ward identities from three point functions or contributions which “start” with four point functions .There are no trace anomalies which start with the four point functions . These type of contributions are either amplitudes which survive the conformal limit and therefore cannot contribute to traces or amplitudes which are suppressed in the conformal limit .For the second type an analysis completely analogous to the one performed for the three point amplitudes shows that the suppression "kills" the contribution to the sum rule. In conclusion the sum rule gets its only contribution from the B amplitude .

The vanishing of the other contributions is after being integrated :locally the sum of all contributions is positive .

The convergence of the sum rule is guaranteed by the way the CFT in the UV and IR are approached from the massive theory:

In the UV:

for $s \to \infty$ $$ImS(s, M) \to s^{2-\epsilon} M^{\epsilon}$$

In the IR :

for $s \to 0$ $$ImS(s, M) \to s^{2+\epsilon} M^{-\epsilon}$$

The limits can be approached logarithmically e.g. in the first line one could have :

$$\frac{s^2}{(log(s/M^2))^{\epsilon}}$$

5)Flows with Scale Invariant End Points?

In principle massless theories can exist without having the full SO(d,2) group as a symmetry but just a conserved dilation current.Such theories are “SFT”.

Wess(1960) proved that from locality it follows that even though the energy momentum tensor is not anymore traceless but its trace T obeys:

$T = \partial_i V^i$ where V^i is the “virial current”

It is convenient to couple the virial current to a “virial gauge field” C_i

such that the Weyl invariance implying the Ward identity is extended to the transformations:

$$g_{ij} \to e^{2\sigma} g_{ij} \,, \qquad C_i \to C_i + \partial_i \sigma$$

The analysis of the possible anomalies is extended now to functionals depending on the metric and the virial gauge fields.New anomalies appear but the “old” anomalies are still present in particular the a-anomaly.

Luty,Polchinski,Rattazzi (1204.5221) presented an argument which in the language of the sum rule says that the basic sum rule is still formally valid , however the l.h.s. is not anymore convergent since there are contributions from amplitudes which do not vanish in a scale-only invariant theory .

Since the r.h.s. is the difference of the a-anomalis there is a contradiction unless the l.h.s. vanishes in the IR limit . As a consequence the matrix element

$$\langle X|T(p_1)T(p_2)|0\rangle$$

involving the "on shell " traces should vanish for an arbitrary state in a unitary theory which is "almost" the statement that the trace vanishes as an operator i.e. that the IR theory is actually conformally invariant.

6)Universal Relation of a-Theorems to d=2 ??

In d=2 Zamolodchikov's proof of the c-theorem involving the correlator of two energy momentum tensors when translated into the "dilaton language" is a statement about the positivity of the dilaton kinetic term.

Solodukhin(1304.4411) proposes that a 2d kinetic term for the dilaton normalized to the a-anomaly can appear in the dilaton action also in higher dimension.

The basic assumption(motivated by EE results) is that trace anomaly matching holds also for SINGULAR background metrics . In particular a 4d manifold with a conical singularity such that the near the singularity the manifold has the structure $C_{2,\alpha} \times \Sigma$

where $C_{2,\alpha}$ is a cone with deficit angle $\delta = 2\pi(1-\alpha)$

The various curvatures have singularities , in particular the Ricci tensor is:

$$R^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} + 2\pi(1-\alpha)(n^{\mu}n_{\nu})\delta_{\Sigma} ,$$

Then the $4(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\partial_{\mu}\tau\partial_{\nu}\tau$

term in the dilaton action produces:

$$\Delta a \frac{\pi}{8} \int_{\Sigma} \sqrt{\gamma}\,(\tau R_{\Sigma}(\gamma) + (\partial_{\Sigma}\tau)^2)$$

i.e. a 2d term in the action normalized to the 4d anomaly difference!

From the positivity of the 2d term the a-Theorem in 4d would follow.However the 2d kinetic term is not a leading one and its positivity is not obvious.Moreover it is very difficult to see how the information contained in the action on singular manifolds is related to the properties of energy momentum correlators on flat space.

Nevertheless the proposal that the a-Theorems in even dimensions can be all related to the one in d=2 is a very interesting and intriguing one.In Holography there are universal relations between the type A anomalies in all even dimensions for a given polynomial of curvatures giving the holographic gravitational action.

7)Open Problems

"Physics " problems:

- flows in odd dimensions
- the "gradient " nature of the flow
- existence of "isolated" unitary SFT
- relation between a-Theorems and EE (Casini-Huerta)
- limitations on the values of "a "(Hoffmann-Maldacena-Zhiboedov)"

" Technical" problems:

- the "non abelian" meaning of the WZ term
- the relation to the holographic proofs of the a- Theorem
- proving that the end point in the LPR flow is CFT
- …….