Stochastic Quantization and AdS/CFT

Dileep Jatkar

Harish-Chandra Research Institute, Allahabad

With Jae-Hyuk Oh, (arXiv:1209.2242 and 1305.2008)

Outline

Motivation

Recent progress in holographic Wilsonian renormalisation group (HWRG) has emphasised two aspects:

1. Flow equations have Hamilton-Jacobi form,
2. Complete description of HWRG flow necessarily requires multi-trace operators.Nickel-Son, Heemskerk-Polchinski, Faulkner-Liu-Rangamani

Stochastic Quantization

- There has been a proposal of describing AdS/CFT in terms of Stochastic Quantization (SQ) of a theory in one lower dimension. Lifschytz-Periwal, Polyakov, Akhmedov, Mansi-Mauri-Petkou
- They involve Hamilton-Jacobi set up with Fokker-Planck Hamiltonian.

Motivation Contd. . .

Questions

- Is it possible to relate SQ to HWRG in AdS/CFT correspondence?
- What is the relation between Fokker-Planck Hamiltonian and the HWRG Hamiltonian?
- What is the relation between the stochastic time and the radial direction in AdS?
- Is it possible to accommodate multi-trace operators in the SQ formulation?
- If yes, is it consistent with the SQ-HWRG dictionary?

Relating SQ to HWRG

The Proposal

- (The stochastic time) $t = r$ (the radial coordinate in AdS)
- (Classical action in SQ) $S_c \equiv 2\Gamma(\phi)$ (Classical effective action in AdS/CFT) Mansi-Mauri-Petkou
- (The Fokker-Planck Hamiltonian) $\mathcal{H}_{FP}(t) = \mathcal{H}_{RG}(r)$ (HWRG Hamiltonian in AdS/CFT).

Check

- We check our proposal by studying three examples, massless scalar field theory in AdS_2, $U(1)$ gauge theory on AdS_4 and conformally coupled scalar field in AdS_d.
- In each case, description of double trace operator deformation agrees between SQ and HWRG.

Relating SQ to HWRG Contd...

Relation

1. The boundary action S_B in AdS/CFT is given by

$$S_B = \int_{t_0}^{t} d\tilde{t} \int d^d x \, \mathcal{L}_{FP}(\phi(\tilde{t}, x)), \qquad (1)$$

$\mathcal{L}_{FP}$ is the Fokker-Planck Lagrangian density

2. The Langevin dynamics gives the relation between stochastic 2-point correlation functions and the double trace coupling in AdS/CFT

$$\langle \phi_q(t)\phi_{q'}(t)\rangle_H^{-1} = \langle \phi_q(t)\phi_{q'}(t)\rangle_S^{-1} - \frac{1}{2}\frac{\delta^2 S_c}{\delta\phi_q(t)\delta\phi_{q'}(t)}, \qquad (2)$$

Lightning Review of HWRG

- Consider a bulk action in the Euclidean AdS$_{d+1}$

$$S = \int_{r>\epsilon} dr d^d x \sqrt{g} \mathcal{L}(\phi, \partial\phi) + S_B[\phi, \epsilon], \tag{3}$$

  S_B is the boundary effective action and ϵ is radial cutoff.

- The AdS metric is

$$ds^2 = \frac{dr^2 + \sum_{i=1}^{d} dx_i dx_i}{r^2}. \tag{4}$$

- Canonical momentum is defined with boundary condition

$$\Pi_\phi = \sqrt{g}\frac{\partial \mathcal{L}}{\partial(\partial_r \phi)} = \frac{\delta S_B}{\delta \phi(x)}. \tag{5}$$

Lightning Review

- Cutoff independence of the action implies

$$\begin{aligned} \partial_\epsilon S_B &= \int_{r=\epsilon} d^d x \left(\frac{\delta S_B}{\delta \phi} \partial_r \phi - \mathcal{L}(\phi, \partial \phi) \right) \\ &= \int_{r=\epsilon} d^d x \, \mathcal{H}_{RG}\left(\frac{\delta S_B}{\delta \phi}, \phi\right), \end{aligned} \tag{6}$$

  second equality follows from the Legendre transform.
- With a wavefunctional $\psi_H = \exp(-S_B)$, we can write

$$\partial_\epsilon \psi_H = -\int_{r=\epsilon} d^d x \, \mathcal{H}_{RG}\left(-\frac{\delta}{\delta \phi}, \phi\right) \psi_H, \tag{7}$$

  where we have assumed $\left(\frac{\delta S_B}{\delta \phi}\right)^2 >> \frac{\delta^2 S_B}{\delta \phi^2}$.

Stochastic Quantization

- Stochastic quantization is a Hamiltonian description of Euclidean field theory evolving along fictitious stochastic time t. Damgaard-Huffel, Dijkgraaf-Orlando-Reffert
- The probability distribution $P(\phi, t)$ describes evolution of the system and reduces to the Boltzmann measure at late time.
- N-point correlator is given by

$$\langle \phi(x_1)...\phi(x_N) \rangle = \int D\phi \ P(\phi, t) \ \phi(x_1)...\phi(x_N). \tag{8}$$

Basics

The Langevin equation

$P(\phi, t)$ is determined using the Langevin equation

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{1}{2}\frac{\delta S_c}{\delta \phi(x,t)} + \eta(x,t), \tag{9}$$

where $\eta(t)$ is the Gaussian white noise, with following properties

$$\begin{aligned} \langle \eta_{i,q}(t) \rangle &= 0 \\ \langle \eta_{i,q}(t)\eta_{j,q'}(t') \rangle &= \delta_{ij}\delta^d(q-q')\delta(t-t'), \end{aligned} \tag{10}$$

with the Gaussian weight function

$$Z = \int D\eta(x,t) \exp(-\frac{1}{2}\int d^d x dt \, \eta^2(x,t)). \tag{11}$$

Langevin to Fokker-Planck

- The Fokker-Planck description is obtained by eliminating $\eta(x,t)$ in

$$Z = \int D\eta(x,t) \exp(-\frac{1}{2}\int d^d x dt\, \eta^2(x,t)), \qquad (12)$$

using the Langevin equation.

- This gives

$$P(\phi,t) = \exp\left[-\frac{S_c(\phi(t))}{2} - \int_{t_0}^{t} d\tilde{t} \int d^d x \mathcal{L}_{FP}(\phi(\tilde{t},x))\right], \qquad (13)$$

where the Fokker-Planck Lagrangian density is

$$\mathcal{L}_{FP} = \frac{1}{2}\left(\frac{\partial\phi(x)}{\partial t}\right)^2 + \frac{1}{8}\left(\frac{\delta S_c}{\delta\phi(x)}\right)^2 - \frac{1}{4}\frac{\delta^2 S_c}{\delta\phi^2(x)}. \qquad (14)$$

Probability distribution

- Equation satisfied by $P(\phi, t)$ can be put in a suggestive form by defining

$$\psi_S(\phi, t) \equiv P(\phi, t) \exp(\frac{S_c}{2}), \tag{15}$$

and then equation for $P(\phi, t)$ becomes the Schrödinger type equation for $\psi_S(\phi, t)$

$$\partial_t \psi_S(\phi, t) = -\int d^d x \, \mathcal{H}_{FP}(\frac{\delta}{\delta\phi}, \phi) \, \psi_S(\phi, t), \tag{16}$$

where,

$$\mathcal{H}_{FP} = -\frac{1}{2}\frac{\delta^2}{\delta\phi^2(x)} + \frac{1}{8}\left(\frac{\delta S_c}{\delta\phi(x)}\right)^2 - \frac{1}{4}\frac{\delta^2 S_c}{\delta\phi^2(x)}, \tag{17}$$

where $\mathcal{H}_{FP}$ is the Fokker-Planck Hamiltonian.

The Dictionary

SQ ⇔ HWRG

- We can now state the relation between SQ and HWRG.
  1. $t = r$
  2. $\mathcal{H}_{FP}(t) = \mathcal{H}_{RG}(r)$
  3. $\psi_S(\phi, t) \equiv P(\phi, t)\exp(\frac{S_c}{2}) = \psi_H(\phi, t) \equiv \exp(-S_B)$

Implied relation

The second equality also implies a relation between the classical action S_c in SQ and the effective action Γ in AdS/CFT, namely

$$S_c = 2\,\Gamma \tag{18}$$

The Dictionary

2-point functions in SQ

- Langevin dynamics gives

$$\langle \phi_{q_1}(t_1)\phi_{q_2}(t_2)\rangle_S = \int D\phi e^{-S_P(t)}\phi_{q_1}(t_1)\phi_{q_2}(t_2), \tag{19}$$

- $P(\phi, t) \equiv \exp(-S_P(t)) = \exp(-\frac{1}{2}\int \mathcal{K}_q(t)\phi_q(t)\phi_{-q}(t)d^dq)$

  Latter expression is true only for a free theory.
- In a free theory

$$\langle \phi_{q_1}(t_1)\phi_{q_2}(t_2)\rangle_S = \frac{1}{\mathcal{K}_q(t)}\delta^d(q_1+q_2). \tag{20}$$

The Dictionary

2-point functions in AdS/CFT

- According to relation 3, $S_B = S_P - \frac{S_c}{2}$ and

$$\langle \phi_q(r)\phi_{q'}(r)\rangle_H^{-1} = \frac{\delta^2 S_B}{\delta\phi_q(r)\delta\phi_{q'}(r)}. \tag{21}$$

- We thus have a relation

$$\langle \phi_{q_1}(t)\phi_{q_2}(t)\rangle_H^{-1} = \langle \phi_{q_1}(t)\phi_{q_2}(t)\rangle_S^{-1} - \frac{1}{2}\frac{\delta^2 S_c}{\delta\phi_q(t)\delta\phi_{-q}(t)}. \tag{22}$$

AdS/CFT

Scalar field action

- Action for massless scalar field in Euclidean AdS_2

$$S_{bulk} = \frac{1}{2} \int drd\tau \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \tag{23}$$

  where $(g_{\tau\tau}, g_{rr}) = (r^{-2}, r^{-2})$.
- Notice,
  1. The action is invariant under Weyl rescaling of metric,
  2. AdS space is conformally flat.
- Therefore scalar field action can also be written as

$$S_{bulk} = \frac{1}{2} \int_{\mathbb{R}^2_+} drd\tau \partial_\mu \phi \partial_\mu \phi. \tag{24}$$

AdS/CFT

Double trace terms

- $-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4} + 1$ implies massless scalar field admits alternate quantization in AdS_2.
- Assume the form of S_B

$$\begin{aligned} S_B &= \Lambda(\epsilon) + \int \frac{d\omega}{2\pi} \sqrt{\gamma(\epsilon)} \mathcal{J}(\epsilon,\omega)\phi_{-\omega} \\ &\quad -\frac{1}{2}\int \frac{d\omega}{2\pi} \sqrt{\gamma(\epsilon)} \mathcal{F}(\epsilon,\omega)\phi_{\omega}\phi_{-\omega}, \end{aligned} \tag{25}$$

  where $\Lambda(\epsilon)$, $\mathcal{J}(\epsilon,\omega)$ and $\mathcal{F}(\epsilon,\omega)$ are unknown functions of radial cut-off ϵ.
- $\mathcal{F}(\epsilon,\omega)$ is the double trace coupling.

HWRG

- Holographic Hamilton-Jacobi equation is, Faulkner et al.

$$\partial_\epsilon S_B = -\frac{1}{2}\int_{r=\epsilon} d\omega \left(\left(\frac{\delta S_B}{\delta \phi_\omega}\right)\left(\frac{\delta S_B}{\delta \phi_{-\omega}}\right) - \omega^2 \phi_\omega \phi_{-\omega} \right). \qquad (26)$$

- Substituting S_B into the holographic H-J equation

$$\partial_\epsilon \Lambda(\epsilon) = -\frac{1}{2}\int_\epsilon \frac{d\omega}{(2\pi)^2} J(\epsilon,\omega)J(\epsilon,-\omega), \qquad (27)$$

$$\partial_\epsilon J(\epsilon,-\omega) = \frac{1}{2\pi} J(\epsilon,\omega) f(\epsilon,-\omega), \qquad (28)$$

$$\partial_\epsilon f(\epsilon,\omega) = \frac{1}{2\pi} f(\epsilon,-\omega) f(\epsilon,\omega) - 2\pi\omega^2, \qquad (29)$$

where, $J(\epsilon,\omega) \equiv \sqrt{\gamma(\epsilon)}\mathcal{J}(\epsilon,\omega)$ and $f(\epsilon,\omega) \equiv \sqrt{\gamma(\epsilon)}\mathcal{F}(\epsilon,\omega)$

HWRG

- Solution to these equation is Faulkner et al.

$$f(\epsilon,\omega) = -2\pi\frac{\Pi_\omega(\epsilon)}{\phi_{-\omega}(\epsilon)}, \quad J(\epsilon,\omega) = -\frac{\beta_\omega}{\phi_\omega(\epsilon)}, \tag{30}$$

$$\text{and} \quad \partial_\epsilon\Lambda(\epsilon) = -\frac{1}{2}\int_{r=\epsilon}\frac{d\omega}{(2\pi)^2}\frac{\beta_\omega\beta_{-\omega}}{\phi_\omega(\epsilon)\phi_{-\omega}(\epsilon)},$$

  where Π_ω is momentum conjugate to ϕ_ω and β_ω is independent of ϵ.
- General solution to ϕ equation of motion is

$$\phi_\omega(r) = \phi_\omega^{(0)}\cosh(|\omega|r) + \frac{\phi_\omega^{(1)}}{|\omega|}\sinh(|\omega|r). \tag{31}$$

HWRG

- Consider only double trace coupling term, then

$$S_B(r) = \frac{1}{2}\int d\omega|\omega|\left(\frac{\sinh(|\omega|r) + \tilde{\phi}_\omega \cosh(|\omega|r)}{\cosh(|\omega|r) + \tilde{\phi}_\omega \sinh(|\omega|r)}\right)\phi_\omega\phi_{-\omega}, \tag{32}$$

$$\mathcal{F}(r,\omega) = -2\pi|\omega|r\frac{\sinh(|\omega|r) + \tilde{\phi}_\omega \cosh(|\omega|r)}{\cosh(|\omega|r) + \tilde{\phi}_\omega \sinh(|\omega|r)}. \tag{33}$$

Flows

- As $r \to 0$ (UV), $\mathcal{F}(r,\omega) \to 0$ for $\tilde{\phi} = 0$ and $\mathcal{F}(r,\omega) \to -2\pi$ for $\tilde{\phi} = \infty$. (Two Fixed points in UV)
- As $r \to \infty$ (IR), $\mathcal{F}(r,\omega) \to -\infty$ unless $\tilde{\phi} = -1$, then $\mathcal{F}(r,\omega) \to \infty$. (Two different fixed points in IR)

Stochastic Quantization

Fokker-Planck action

- Recall $S_c = 2\,\Gamma$, and Γ is Legendre transform of on-shell action,

$$S_{cl} = \int_{-\infty}^{\infty} d\omega |\omega| \phi_\omega \phi_{-\omega}. \tag{34}$$

- The Fokker-Planck Lagrangian density is

$$\mathcal{L}_{FP} = \frac{1}{2}\dot{\phi}_\omega \dot{\phi}_{-\omega} + \frac{1}{2}\omega^2 \phi_\omega \phi_{-\omega}, \tag{35}$$

where, dot indicates derivative w.r.to stochastic time t.

- General solution to eq. of motion derived from $\mathcal{L}_{FP}$ is

$$\phi_\omega(t) = a_{1,\omega} \cosh(|\omega| t) + a_{2,\omega} \sinh(|\omega| t). \tag{36}$$

Stochastic Quantization

Fokker-Planck action

- Consider a boundary condition that at time t we want the solution $\phi_\omega(\tilde{t} = t) = \phi_\omega(t)$ then

$$\phi_\omega(\tilde{t}) = \phi_\omega(t) \frac{\cosh(|\omega|\tilde{t}) + a_\omega \sinh(|\omega|\tilde{t})}{\cosh(|\omega|t) + a_\omega \sinh(|\omega|t)}. \qquad (37)$$

- Substituting this in the Fokker-Planck action gives

$$S_{FP} = \frac{1}{2} \int d\omega |\omega| \phi_\omega(t) \phi_{-\omega}(t) \left(\frac{\sinh(|\omega|t) + a_\omega \cosh(|\omega|t)}{\cosh(|\omega|t) + a_\omega \sinh(|\omega|t)} \right). \qquad (38)$$

- This action is identical to $S_B(r)$, with t replaced by r.
- It therefore has same fixed point structure.

U(1) gauge theory on AdS$_4$

U(1) gauge theory action on AdS$_4$

- The gauge field action in the Euclidean AdS$_4$ is

$$S_{bulk}[A] = \frac{1}{4}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}, \tag{39}$$

  where, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the background metric is

$$ds^2 = \frac{dr^2 + \delta_{ij}dx^i dx^j}{r^2}. \tag{40}$$

- This action also has Weyl invariance. By rescaling the metric $ds^2 \to r^2 ds^2$, this action can be mapped to that on half of $\mathbb{R}^4$. This example works out in a similar fashion as massless scalar field on AdS$_2$.

Conformally coupled scalar field

- Consider conformally coupled scalar field in AdS_{d+1},

$$S = \int_{r>\epsilon} dr d^d x \sqrt{g} \mathcal{L}(\phi, \partial\phi) + S_B,$$

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^{\frac{2(d+1)}{d-1}}, \qquad (41)$$

  where $m^2 = -(d^2 - 1)/4$, which is dictated by conformal coupling of scalar to gravity.
- We will ignore interaction term from now on, *i.e.*, $\lambda = 0$.
- This value of mass falls within the BF window $-d^2/4 \leq m^2 \leq -d^2/4 + 1$.
- It allows alternate quantization of the scalar.

HWRG flow equation

The flow equation for conformally coupled scalar takes the form

$$\partial_\epsilon S_B = -\int_{r=\epsilon} d^d p \left[\frac{1}{2\sqrt{g} g^{rr}} \left(\frac{\delta S_B}{\delta \phi_p} \right) \left(\frac{\delta S_B}{\delta \phi_{-p}} \right) - \frac{1}{2} \sqrt{g} g^{ij} p_i p_j \phi_p \phi_{-p} + \frac{d^2 - 1}{8} \sqrt{g} \phi_p \phi_{-p} \right], \quad (42)$$

To solve this we again make an ansatz

$$S_B = \Lambda(\epsilon) + \int \frac{d^d p}{(2\pi)^d} \sqrt{\gamma} \mathcal{J}(\epsilon, p) \phi_{-p} - \int \frac{d^d p}{2(2\pi)^d} \sqrt{\gamma} \mathcal{F}(\epsilon, p) \phi_p \phi_{-p}. \quad (43)$$

Equation for Λ, $\mathcal{J}$ and $\mathcal{F}$ take similar form and the solution to double trace coupling is

$$\begin{aligned} \sqrt{\gamma}\mathcal{F}(\epsilon, p) &= -(2\pi)^d \frac{\Pi_\phi}{\phi}, \\ \Pi_\phi &= \sqrt{g} g^{rr} \partial_r \phi = \frac{\delta S_B}{\delta \phi} \end{aligned}$$

Solution to the equation of motion is

$$\phi_p = r^{\frac{d-1}{2}} \left[\phi_0(p) \cosh(|p|r) + \phi_1(p) \sinh(|p|r)\right], \qquad (44)$$

using this $\mathcal{F}(r, p)$ and S_B evaluates to

$$\begin{aligned} \mathcal{F}(r, p) &= -(2\pi)^d \left[\frac{d-1}{2} + |p| r \frac{\sinh(|p|r) + \tilde{\phi}(p) \cosh(|p|r)}{\cosh(|p|r) + \tilde{\phi}(p) \sinh(|p|r)}\right] \\ S_B^{DTD} &= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{\mathcal{F}(r, p)}{r^d} \phi_p \phi_{-p}. \end{aligned} \qquad (45)$$

Stochastic Picture

- We are now faced with a problem of reproducing this result using stochastic quantization.
- Notice we have explicit time dependence in the Langevin equation (for AdS_{d+1}, $\Omega(t) = t^{(d-1)/2}$)

$$\frac{1}{\Omega(t)}\frac{d\phi_p(t)}{dt} = -\frac{1}{\Omega(t)}\left(|p| - \frac{\partial_t \Omega(t)}{\Omega(t)}\right)\phi_p(t) + \eta(t,p). \quad (46)$$

- The general solution to this equation is

$$\phi_p(t) = \Omega(t)\int_{t_0}^{t} dt' e^{-|p|(t-t')}\eta(t',p)\ . \quad (47)$$

- The two point correlation function is given by

$$\langle \phi_p(t)\phi_{p'}(t)\rangle_S = \Omega^2(t)\frac{\delta^d(p-p')}{2|p|}(1-e^{2|p|(t_0-t)}) \, . \qquad (48)$$

  Using this one can read off the double trace coupling.
- In spite of having explicit time dependence in the Langvin equation, Fokker-Planck formulation can be derived, provided $\Omega(t)$ satisfies

$$\frac{d^2\Omega^{-1}(t)}{dt^2} = (\frac{d^2-1}{4})\Omega^{-\frac{d+3}{d-1}} \qquad (49)$$

- $\Omega(t) = t^{(d-1)/2}$ is a solution to this equation!

Field redefinition

- We can simplify the problem by doing the following field redefinition

$$\phi(t,p) = \Omega(t) f_p(t). \tag{50}$$

- The Langevin equation then becomes

$$\frac{df_p(t)}{dt} = -|p| f_p(t) + \eta(t,p), \tag{51}$$

which means in the new variables the classical action is

$$S_c = \int d^d p |p| f_p f_{-p}. \tag{52}$$

- We can carry out Stochastic quantization in new variables in the same manner as in the Weyl invariant case. This result matches with that obtained without doing the field redefinition.

Conclusion: SQ $\Leftrightarrow$ HWRG

The Proposal

- (The stochastic time) $t = r$ (the radial coordinate in AdS)
- (Classical action in SQ) $S_c \equiv 2\Gamma(\phi)$ (Classical effective action in AdS/CFT)
- (The Fokker-Planck Hamiltonian) $\mathcal{H}_{FP}(t) = \mathcal{H}_{RG}(r)$ (HWRG Hamiltonian in AdS/CFT).

Examples

- We have checked our proposal for Weyl invariant actions, e.g., free massless scalar field in AdS_2 and $U(1)$ gauge theory on AdS_4 and conformally coupled scalar field in AdS_d.
- We have studied free theories in the bulk, but they do contain information about multi-trace operators of boundary theory.

Outlook

- It will be interesting to extend the stochastic quantization method to interacting Weyl invariant theories.
- It will also be interesting to see if this method can applied to more general geometries(*e.g.*, Lifshitz metric, hyperscaling violating metric, etc.). It may be possible to obtain different geometries by replacing white noise by correlated noise.
- It will be useful to find out how to accommodate finite temperature AdS/CFT systems in this prescription.

THANK YOU!