Conformal phase transitions at weak and strong coupling

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Introduction

Strongly coupled fermions play an important role in many physical systems:

- QCD
- Technicolor
- Condensed matter

Typically, at weak coupling the dynamics is simple, while at strong coupling one finds many interesting phenomena, such as dynamical symmetry breaking, mass generation and confinement.

The focus of this lecture will be the transition between the two regimes.

Imagine a situation where the dynamics depends on a continuous parameter λ (``the coupling"), such that for $\lambda < \lambda_c$ the order parameter (say a dynamically generated mass) vanishes, while for larger coupling it is non-zero.

The transition at $\lambda = \lambda_c$ may be either first order or continuous. A conformal phase transition is a particular kind of continuous transition, which apparently plays a role in QCD and other strongly coupled systems.

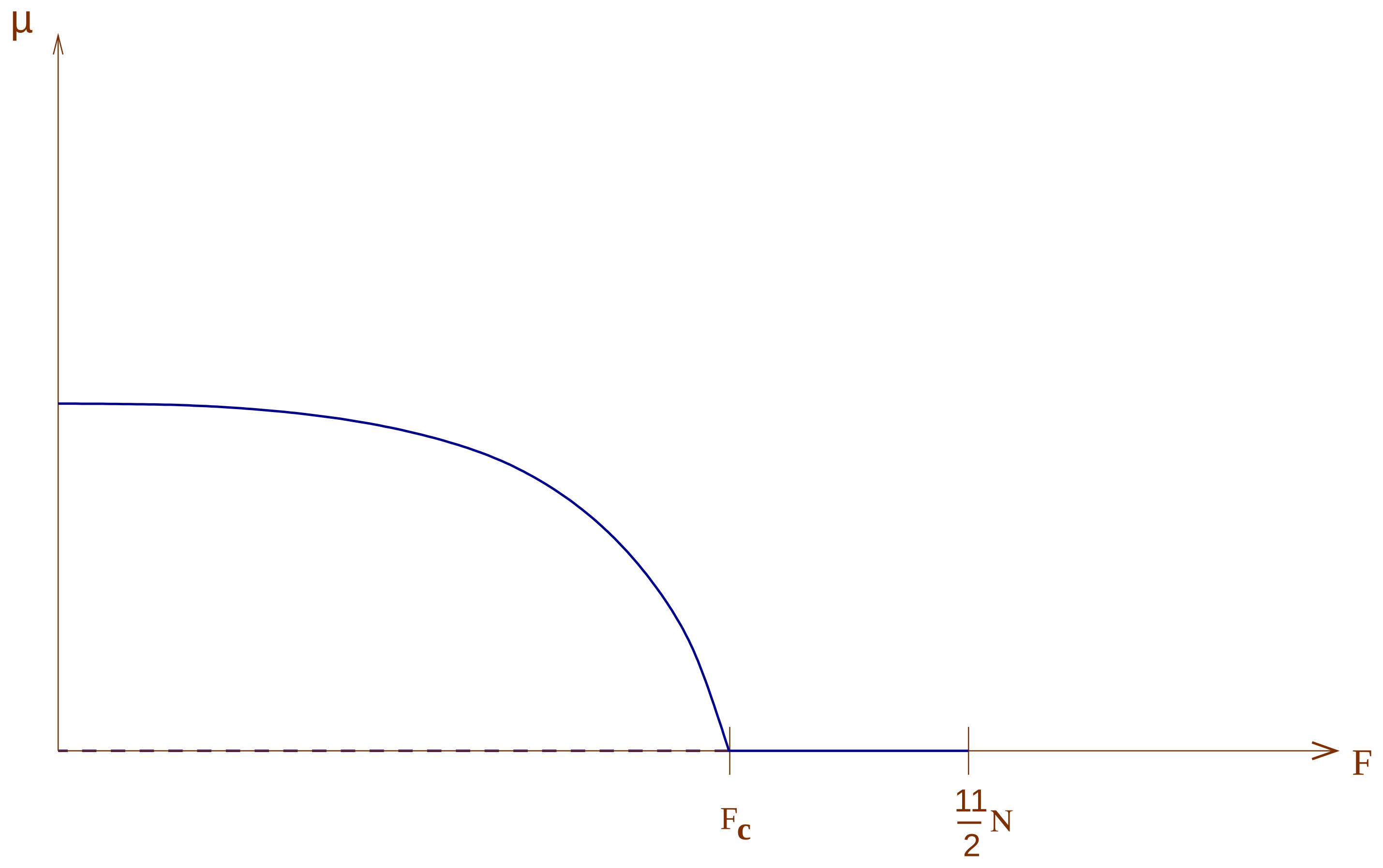
Conformal phase transition in QCD

Consider an SU(N) gauge theory coupled to F flavors of fermions in the fundamental rep. In the large N (Veneziano) limit, $N, F \to \infty$ with x = N / F fixed, the infrared dynamics of this theory depends on x. For $x \le \frac{2}{11}$ the theory is infrared free. For smaller F the IR dynamics is non-trivial.

If x is only slightly larger than 2/11, the gauge coupling runs from zero in the UV to a small non-zero value in the IR. The theory dynamically generates a scale Λ_{QCD} , the crossover scale between the UV and IR . For energies well below this scale the dynamics is governed by a weakly coupled interacting fixed point (Banks, Zaks).

As x increases (or F decreases), the IR theory becomes more strongly interacting. When it exceeds a critical value, x_c, the model undergoes a phase transition to a phase in which conformal symmetry is broken, and the quarks get a non-zero mass μ. The chiral symmetry $SU(F)_L \times SU(F)_R$ is broken to its diagonal subgroup.

This phase transition is believed to be continuous:



Near the transition one has:

$$\mu \simeq \Lambda_{QCD} e^{-\frac{a}{\sqrt{x - x_c}}}$$

Miransky scaling

This phase structure is obtained in various uncontrolled approximations, and it would be nice to understand it better. Among other things this is important for technicolor, the attempt to understand electroweak symmetry breaking as a consequence of strong gauge dynamics. It was recognized long ago that viable models of this sort must live in the vicinity of such a phase transition; they are known as walking technicolor .

In QCD, it is difficult to analyze the dynamics near the transition even in the large N limit, since this involves solving a strongly coupled matrix model. In this lecture we will consider a class of theories which exhibit a similar phase transition but that can be solved at large N.

These models are of interest in their own right, as they involve 2+1 dimensional fermions strongly interacting with 3+1 dimensional gauge fields. This type of dynamics may be experimentally realized in condensed matter systems, such as graphene (Rey).

N=4 SYM coupled to defect fermions

The basic model we will consider can be thought of as the low energy theory on a non-supersymmetric brane system consisting of N D3-branes and F D7-branes, oriented as follows:

$$\begin{array}{ccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ D3 & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & & & & & & \\ D7 & \mathrm{x} & \mathrm{x} & \mathrm{x} & & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \end{array}$$

Low energy dynamics

Spectrum:

- 3-3 strings: 3+1 dimensional N=4 SYM
- 3-7 strings: 2+1 dimensional fermion ψ

Low energy Lagrangian:

$$S = S_{N=4} + \int d^3x (i\bar{\psi}\gamma^\mu D_\mu \psi + g\bar{\psi}\psi\phi^9)$$

This gauge theory describes 2+1 dimensional fermions with a tunable coupling and we will see that it exhibits interesting dynamics when the coupling is cranked up. We will study it at large N, with F of order one, the standard `t Hooft limit.

The classical Lagrangian is conformally invariant. The quantum theory preserves this symmetry for $\lambda < \lambda_c$, and undergoes a continuous phase transition at λ_c .

A good way to probe the phase structure of the model is to calculate the expectation value of the open Wilson line operator

$$OW(x,y) = \bar{\psi}(x)\mathcal{P}\exp\left[ig\int_x^y A\cdot dl\right]\psi(y)$$

In the conformal phase, $\lambda < \lambda_c$, it is determined by conformal symmetry to take the form

$$\langle OW(x,y)\rangle \sim i\gamma^\mu\partial_\mu\frac{1}{(x-y)^{\Delta(\lambda)-1}}$$

The leading deviation from the conformal behavior is associated with the possible addition of a mass term for ψ, and the vev $\langle\bar{\psi}\psi\rangle$, which is non-zero for $\lambda > \lambda_c$. These give

$$\langle OW(x,y)\rangle \sim i\not{\partial}\frac{1}{(x-y)^{\Delta(\lambda)-1}} + \frac{\langle\bar{\psi}\psi\rangle}{(x-y)^{\Delta(\lambda)-\Delta_b}} + \frac{m}{(x-y)^{\Delta(\lambda)-3+\Delta_b}} + \cdots$$

where

$$\Delta_b(\lambda) = \Delta(\bar{\psi}\psi) = 2 + \gamma(\lambda)$$

In momentum space one has

$$\langle OW(p)\rangle \sim p^{\Delta(\lambda)-4}\left(\not{p} + M(p) + \cdots\right),$$

$$M(p) = m\left(\frac{p}{\mu}\right)^{\gamma} + \frac{\langle\overline{\psi}\psi\rangle}{p}\left(\frac{\mu}{p}\right)^{\gamma}$$

which can be summarized as

$$\frac{d}{dp}(p^2 M'(p)) + C(\lambda)M(p) = 0,$$

One can think of M(p) as the order parameter for conformal symmetry breaking.

The function $C(\lambda)$ is related to the anomalous dimension $\gamma(\lambda)$ via the relation

$$\gamma(\lambda) = -\frac{1}{2} + \sqrt{\frac{1}{4} - C(\lambda)}$$

At weak coupling it can be calculated using standard QFT techniques. One finds

$$C(\lambda) = \frac{5\lambda}{12\pi^2} + O(\lambda^2)$$

As the coupling increases, $C(\lambda)$ increases and the anomalous dimension becomes more negative. As long as it remains real, one can show that the order parameter M(p) vanishes. However, if/when $C(\lambda)$ exceeds 1/4, one can show that M becomes non-zero. At large p it behaves like

$$M(p) = A\mu \left(\frac{\mu}{p}\right)^{\frac{1}{2}} \sin\left(\sqrt{\kappa}\ln\frac{p}{\mu} + \phi\right)$$

where

$$\kappa(\lambda) = C(\lambda) - \frac{1}{4}$$

$$\mu = M(0)$$

is the dynamically generated scale. It is related to the UV cutoff by

$$M(\Lambda_{UV}) = 0$$

which is the requirement that the bare mass is zero. This leads to the Miransky-type relation

$$\mu \simeq \Lambda_{UV} \exp\left(-\frac{\pi}{\sqrt{\kappa}}\right)$$

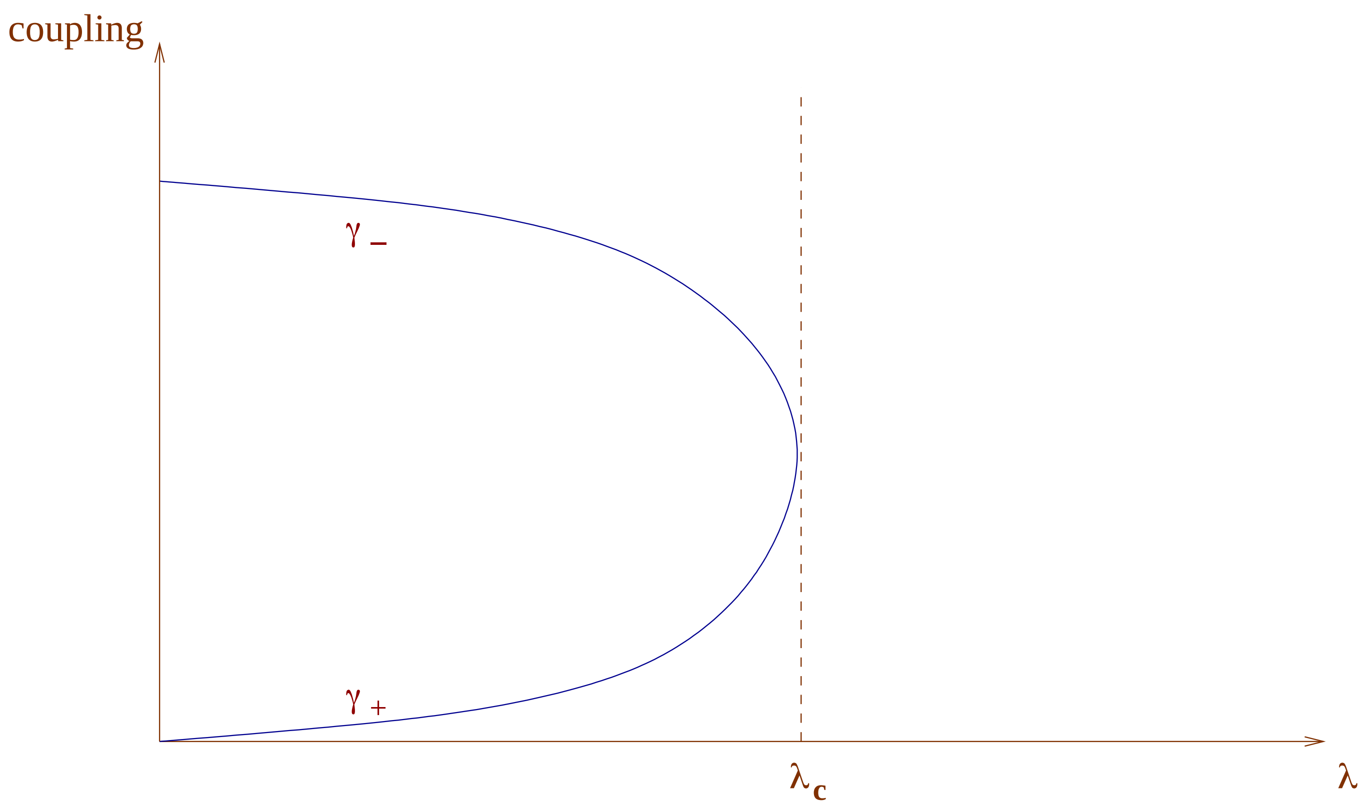
To have a large hierarchy of scales, need $\kappa \ll 1.$

We are led to the following picture:

A conformal phase transition takes place at a value of the coupling satisfying $C(\lambda_c) = \frac{1}{4}$. At that point the dimension of the fermion bilinear takes the value $\Delta(\bar{\psi}\psi) = \frac{3}{2}$ and the two leading terms in M(p) are comparable.

The double trace operator $(\bar{\psi}\psi)^2$ is marginal. This is related to the picture for CPT's proposed by Kaplan, Lee, Son, Stephanov.

The phase diagram is schematically the following:



For $\lambda < \lambda_c$ there are two fixed points which differ in the value of the four-Fermi coupling. The anomalous dimensions take the values $\gamma_{\pm}(\lambda) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - C(\lambda)}.$

As $\lambda \to \lambda_c$, the two fixed points approach each other and above λ_c they move off the real axis.

Comments:

- At $\lambda = 0$ the two fixed points in question are understood from studies of NJL models.

- In the above analysis we assumed that $C(\lambda)$ reaches the value 1/4 at a finite λ . This clearly goes beyond the weak coupling regime. To see what happens we next study this issue at strong coupling.

Strong coupling analysis

At strong coupling, N=4 SYM is described by IIB supergravity on $AdS_5 \times S^5$. The D7-brane can be viewed as a probe propagating in this background.

Metric of $AdS_5 \times S^5$:

$$ds^2 = \left(\frac{r}{L}\right)^2 dx_\mu dx^\mu + \left(\frac{L}{r}\right)^2 \left(d\rho^2 + \rho^2 d\Omega_4^2 + (dx^9)^2\right)$$

The D7-brane wraps $x^0, x^1, x^2, \rho, \Omega_4$. Induced metric on its worlvolume: $AdS_4 \times S^4$

$$ds^2 = \left(\frac{\rho}{L}\right)^2 dx_a dx^a + \left(\frac{L}{\rho}\right)^2 \left(d\rho^2 + \rho^2 d\Omega_4^2\right)$$

This state is conformally invariant (in 2+1 d) but it is unstable to condensation of the scalar field parametrizing the position of the brane in x^9 . Indeed, denoting $x^9 = f(\rho)$ one finds the DBI action

$$S_{D7} = \int d^3x \int_0^\Lambda d\rho \frac{L^2}{\rho^2 + f(\rho)^2} \rho^4 \sqrt{1 + f'(\rho)^2}$$

The linearized eom for f is:

$$\frac{\partial}{\partial\rho}\left(\rho^2 f'(\rho)\right) + 2f(\rho) = 0.$$

This is the KG equation in AdS for a scalar field with mass below the BF bound. Its general solution is

$$f(\rho) = A\,\mu\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \sin\left(\frac{\sqrt{7}}{2}\ln\frac{\rho}{\mu} + \phi\right)$$

One can show that its energy is lower than that of the conformal solution f=0.

The strong coupling analysis mirrors closely the weak coupling one, with

$$M \to f \qquad p \to \rho$$

Thus, we conclude that the strong coupling behavior of the function C is:

$$C_\infty = C(\lambda \to \infty) = 2$$

If C is a continuous function, there must be a phase transition at a finite value of λ .

We conclude that the model undergoes a phase transition at finite `t Hooft coupling, from a conformal phase to one in which a mass is generated dynamically.

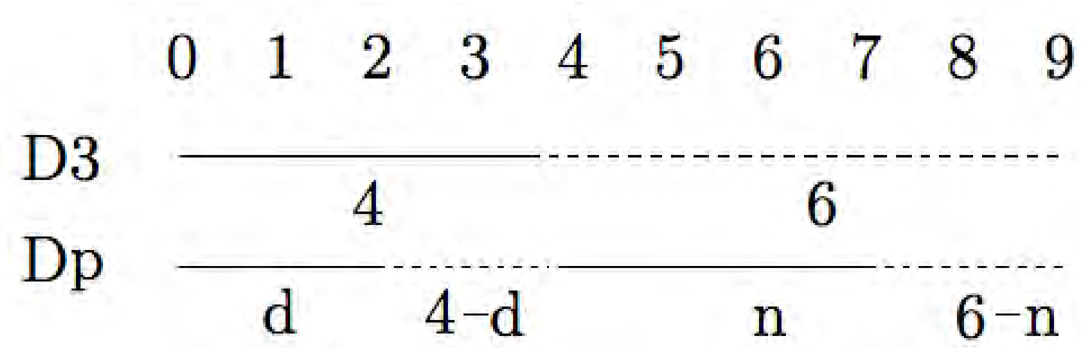
This transition occurs out of the regime of validity of the weak and strong coupling expansions. Therefore, it is hard to study the vicinity of the transition, and even to decide whether the transition is continuous or first order.

We will next discuss approximation schemes which allow one to approach this problem.

Perturbative expansions

The basic idea is to change the parameters of the model so that the phase transition is pushed either to weak coupling, where one can use standard QFT techniques, or to strong coupling, where one can use holography.

To this end we replace the flavor D7-branes by Dp-branes oriented as follows:



- We now have two additional parameters to play with: d, n. The original D3/D7 system corresponds to d=3, n=5, but we can treat d, n as continuous free parameters.

- For n>d one can check that the only massless states of (3,p) strings are fermions, so one can study the issues raised before for general d, n in this range.

- Our basic idea is to vary the parameters so that the phase transition occurs in a region that we can control.

Weak coupling

For general d, n, the perturbative analysis gives:

$$\frac{d}{dp}\left(p^{d-1}M'(p)\right) + C(\lambda)p^{d-3}M(p) = 0$$

$$C(\lambda) = \frac{\lambda}{4d\pi^2}(2(d-1) + (6-n)) + O(\lambda^2)$$

$$\kappa(\lambda) = C(\lambda) - \left(\frac{d-2}{2}\right)^2$$

$$M(p) = m\left(\frac{p}{\mu}\right)^{\gamma} + \frac{\langle\overline{\psi}\psi\rangle}{p^{d-2}}\left(\frac{\mu}{p}\right)^{\gamma}$$

$$\gamma(\lambda) = \gamma_{+}(\lambda) = -\frac{d-2}{2} + \sqrt{\left(\frac{d-2}{2}\right)^{2} - C(\lambda)}$$

The phase transition now occurs when

$$C(\lambda_c) \;=\; (d\,-\,2)^2/4$$

For $d = 2 + \epsilon$, $\lambda_c \sim \epsilon^2$, and we can use perturbation theory to study the transition.

Strong coupling

For general d, n, it is convenient to write the metric as

$$ds^2 = \left(\frac{r}{L}\right)^2 dx_\mu dx^\mu + \left(\frac{L}{r}\right)^2 (d\rho^2 + \rho^2 d\Omega_{n-1}^2 + df^2 + f^2 d\Omega_{5-n}^2)$$

The Dp-brane wraps d of the x^μ, as well as S^{n-1}, and forms a curve $f(\rho)$. The DBI action for f is

$$S_{Dp} = \int d^d x \int d\rho \left(\frac{\rho^2 + f^2}{L^2}\right)^{\frac{d-n}{2}} \rho^{n-1} \sqrt{1 + f'^2}.$$

The linearized eom for f is:

$$\frac{\partial}{\partial \rho}\left(\rho^{d-1} f'\right) + (n-d)\rho^{d-3} f = 0.$$

As before, we can use it to read off

$$C_\infty = \lim_{\lambda \to \infty} C(\lambda) = n - d$$

and

$$\kappa_\infty = \lim_{\lambda \to \infty} \kappa(\lambda) = n - d - \left(\frac{d-2}{2}\right)^2 .$$

If $\kappa_\infty < 0,$ the system remains in the conformal phase for all λ . On the other hand, for $\kappa_\infty > 0,$ i.e. for

$$n > n_c = d + \left(\frac{d-2}{2}\right)^2 ,$$

the system undergoes a CPT at a finite coupling. We can explore the transition in gravity by taking

$$n = n_c + \delta$$

with $\kappa_\infty = \delta \ll 1$.

This leads to a kind of gravitational epsilon-expansion. To leading order in δ, we need to solve the DBI eom for f. One finds the by now familiar large ρ behavior

$$f(\rho) = A\mu \left(\frac{\mu}{\rho}\right)^{\frac{d-2}{2}} \sin\left(\sqrt{\kappa_\infty} \ln \frac{\rho}{\mu} + \phi\right).$$

The dynamically generated scale is

$$f(0) = \mu \simeq \Lambda \exp\left(-\frac{\pi}{\sqrt{\kappa_\infty}}\right)$$

Mesons

To study mesons in the massive phase we need to expand about the background solution and study small excitations. Find $(\bar{\mu} \sim \mu/\sqrt{\lambda})$:

σ - mesons: $m^2/\overline{\mu}^2 \approx 0.44, 9.65, 26.63, 51.35, 84, \cdots$

vector mesons: $m^2/\bar{\mu}^2 \approx 3.08, 15.12, 34.87, 62.32, 97.46, \cdots$

Note the anomalously light scalar meson; it can be thought of as an analog of the techni-dilaton.

In technicolor there is a long-standing debate about the fate of the dilaton near the CPT in QCD. There are two schools of thought:

(1) $m_{\rm td}/m_{\rm meson} \to 0$

(2) $m_{\rm td}/m_{\rm meson} \to {\rm const}$

as $\kappa \to 0\,.$

We find that (2) is correct.

Intriguingly, in QCD it was argued (by M. Hashimoto and K. Yamawaki) that the mass of the techni-dilaton is smaller than that of the lightest vector meson by a factor of about 2.8. In our system this ratio is about 2.6...

Comments

- One can also discuss the system at finite temperature and chemical potential. Find a line of first order phase transitions separating the broken phase from the unbroken one.

- Our results confirm the ideas of Kaplan et al, that CPT's are stable because of their topological nature - they arise when two RG fixed points approach each other and ``annihilate."

- Understanding the phase transition in QCD requires more work. From our perspective this has to do with generalizing the discussion from open strings (DBI) to closed strings (gravity).
- It would also be interesting to see if one can realize this kind of transition experimentally in systems of 2 dimensional electrons interacting with 3 dimensional fields.