Generalized Holographic Renormalization

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Motivation

- The number of holographic models describing strongly coupled gauge theories, either as conjectured exact duals or as effective descriptions, has proliferated dramatically over the last few years.
- More often than not one in fact *defines* (at least a sector of) the strongly coupled field theory via some weakly coupled gravity/supergravity theory (e.g. Lifshitz backgrounds, holographic superconductors).
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  - given a generic gravitational theory, is there a holographic dual?
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- What I will discuss instead is *how* these questions can be addressed systematically and in a very broad context.
- In particular, given a gravitational theory, *without* assuming the existence and/or the form of any potential holographic dual, I want to be able to know a priori:
  - Where would any potential dual "live"?
  - What would be the spectrum of gauge-invariant observables?
  - How would correlation functions of such observables look like?
  - What thermodynamic properties any potential dual would possess?
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The problem

Given a gravitational theory...

- I want to formulate the gravitational dynamics as an "integration from infinity" problem. i.e. I want to define the variational problem with boundary conditions at infinity and to reconstruct the bulk dynamics from data specified at infinity. (cf. Fefferman-Graham program for Euclidean AdS gravity.)
- The potential dual field theory is assumed to "reside" on this infinity, where by "reside" I simply mean that (a sector of) the symplectic space of field theory observables is to be identified with the symplectic space of data at infinity, which parameterize the bulk dynamics.
- One can view this construction as an effective field theory (EFT) for gravity in terms of data specified at *infinity*. There are numerous EFT approaches to gravity dynamics in terms of different degrees of freedom, but I find this most natural in the context of holography. (cf. Membrane Paradigm, Blackfolds etc.)
- In the case of asymptotically AdS supergravity it is precisely this EFT description that happens to coincide with the large-N, large 't Hooft coupling of $\mathcal{N} = 4$ super Yang-Mills.

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- Variational problems with boundary conditions set at infinity require some care!
- Besides adding any Gibbons-Hawking boundary term, required to make the variational problem well defined on a space of *finite* volume (i.e. with a cut-off) or equivalently in order to ensure that the theory admits a radial Hamiltonian description, the variational problem can always be made well defined by:
  - the field variations at infinity are restricted to arbitrary variations within the space of asymptotic solutions of the equations of motion,
  - adding a further boundary term.
- The boundary term, S_b, required on a cut-off surface is *universal*:

$$S_b = -\mathcal{S}$$

where $\mathcal{S}$ is a solution of the radial Hamilton-Jacobi equation. (cf. [de Boer, Verlinde, Verlinde '99] for asymptotically AdS gravity.)

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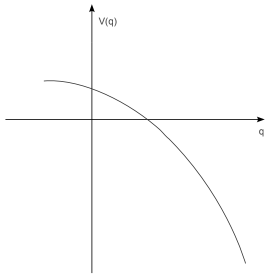
- This result is independent of holography or indeed gravity. We will derive it using a very simple point particle example, but the same argument goes through for any Hamiltonian system that admits a variational problem at infinity.
- Consider a point particle described by the classical action

$$S = \int_0^t dt' \, L = \int_0^t dt' \left(\frac{1}{2}\dot{q}^2 - V(q) \right)$$

- We will take the potential to be unbounded from below as $q \to \infty$



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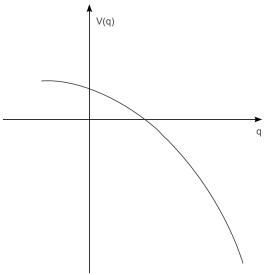

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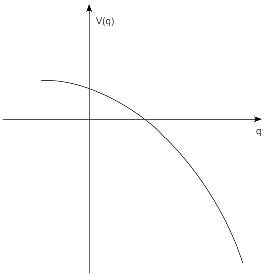

- More precisely, we demand that $V(q) \to -\infty$ as $q \to \infty$ and that the particle reaches $q = \infty$ at infinite time, i.e. the integral

$$t - t_0 = \int_{q_0}^{q} \frac{dq'}{\sqrt{2(E - V(q'))}}$$

diverges as $q \to \infty$.

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- A generic variation of the action takes the form

$$\delta S = -\int_0^t dt' \left(\ddot{q} + V'(q)\right) \delta q + L\delta t + p\delta q$$

- The usual Dirichlet BVP is set up by keeping the location of the boundary fixed, i.e. $t = t_o$ fixed, and requiring

$$\delta q|_{t_o} = 0$$

- However, if we want to set up the BVP at $t = \infty$, setting $\delta t = 0$ at $t = \infty$ does not make sense.
- So, unless $L \to 0$ as $t \to \infty$, the variational problem does not imply the equations of motion *whatever time-independent boundary conditions are imposed on* (p, q).
- Hence, the only admissible boundary conditions are time-dependent: $\delta q \propto \delta t$

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- In order to impose time-dependent asymptotic boundary conditions consistent with the equations of motion we must restrict the space of paths $q(t)$ to the space of asymptotic, as $t \to \infty$, solutions of the equation of motion.
- However, allowing for generic time-dependent boundary conditions within the space of asymptotic solutions does not automatically ensure that $L\delta t + p\delta q \to 0$ as $t \to \infty$ unless a boundary, $S_b(q)$, term is added to the original action.

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- Adding a boundary term $S_b(q)$ a generic variation of the action takes the form:

$$\delta(S+S_b) = -\int_0^t dt' \, (\ddot{q} + V'(q)) \, \delta q + \left(L + \dot{S}_b\right) \delta t + (p + S_b'(q))\delta q$$

- The variational problem is then well defined provided there exists a boundary term $S_b(q)$ such that when q asymptotically approaches generic solutions of the equations of motion

$$\frac{d}{dt}(S+S_b) \xrightarrow{q\to\infty} 0$$

- Since q is asymptotically evaluated on-shell, we can replace the on-shell action S in this expression with Hamilton's principal function, $\mathcal{S}$, i.e. a solution of the Hamilton-Jacobi equation which is a function of q on the boundary

$$\frac{\partial \mathcal{S}}{\partial t} + H\left(\frac{\partial \mathcal{S}}{\partial q}, q, t\right) = 0$$

- The required boundary term, S_b, therefore is *in general* given by

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- The space of asymptotic solutions considered must support a well defined symplectic form, i.e. both non-normalizable and normalizable modes must be considered.
- The solution $\mathcal{S}$ of the radial Hamilton-Jacobi equation is not unique. There are generically discrete equivalent classes of solutions, with solutions within each equivalence class being related continuously. The appropriate solution that must be added as a boundary term corresponds to a specific equivalence class, but any representative within the equivalence class serves equally well as a boundary term.
- In general S_b will be non-local in the transverse space, i.e. non-polynomial in transverse derivatives.
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- The space of asymptotic solutions considered must support a well defined symplectic form, i.e. both non-normalizable and normalizable modes must be considered.
- The solution $\mathcal{S}$ of the radial Hamilton-Jacobi equation is not unique. There are generically discrete equivalent classes of solutions, with solutions within each equivalence class being related continuously. The appropriate solution that must be added as a boundary term corresponds to a specific equivalence class, but any representative within the equivalence class serves equally well as a boundary term.
- In general S_b will be non-local in the transverse space, i.e. non-polynomial in transverse derivatives.
- If the Hamilton-Jacobi equation can be solved exactly, so much the better! However this is usually practically impossible and indeed unnecessary.

Algorithm for solving the Hamilton-Jacobi equation asymptotically...

- Indeed, it suffices to solve the Hamilton-Jacobi equation in an asymptotic sense and only up to certain order.
- So, here is the algorithm:
  1. Write down the radial Hamilton-Jacobi equation for the theory at hand.
  2. Specify the *leading* asymptotic form of the general solutions of the gravity equations - i.e. the non-normalizable modes.
  3. Is this asymptotic form derivable from a Hamilton-Jacobi *potential*, i.e.*without* transverse derivatives?
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The addition of this boundary term...

- makes the variational problem at infinity well defined,
- the on-shell action on arbitrary solutions of the equations of motion is finite,
- induced fields are proportional to non-normalizable modes and the conjugate canonical momenta are proportional to *normalizable* modes – i.e. we have a well defined symplectic space of boundary data at infinity. This space can be identified with the symplectic space of observables in a potential holographically dual theory.
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Generic dilaton-axion system

- Let us now apply this general algorithm to a generic dilaton-axion system of the form

$$S=-\frac{1}{2\kappa^2}\int_{\mathcal{M}}d^{d+1}x\sqrt{g}\left(R[g]-\partial_\mu\varphi\partial^\mu\varphi-Z(\varphi)\partial_\mu\chi\partial^\mu\chi+V(\varphi)\right)+GH$$

- This action includes a large number of physically interesting examples, both asymptotically locally AdS [Mateos & Trancanelli '11], and non asymptotically locally AdS, e.g. Improved Holographic QCD [Gursoy, Kiritsis '07] and non conformal branes [Wiseman & Withers '08], [Kanitscheider, Skenderis, & Taylor '08].

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Radial Hamiltonian formulation

- In order to derive the radial Hamilton-Jacobi equation we start by the standard ADM decomposition of the metric

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j$$

- Substituting this metric into the above action leads to a Lagrangian for the induced fields $\gamma_{ij}, N, N^i, \varphi, \chi$ on the radial slices Σ_r, with Hamiltonian

$$H = \int_{\Sigma_r} d^d x \left(N\mathcal{H} + N_i \mathcal{H}^i \right)$$

- N and N^i are Lagrange multipliers leading to the Hamiltonian and momentum constraints

$$\begin{aligned} 0 = \mathcal{H} &= 2\kappa^2 \gamma^{-\frac{1}{2}} \left(\pi^i_j \pi^j_i - \frac{1}{d-1}\pi^2 + \frac{1}{4}\pi^2_\varphi + \frac{1}{4} Z^{-1}(\varphi)\pi^2_\chi \right) \\ &\quad + \frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi - Z(\varphi)\partial_i\chi\partial^i\chi + V(\varphi) \right), \\ 0 = \mathcal{H}^i &= -2D_j\pi^{ij} + \pi_\varphi \partial^i\varphi + \pi_\chi\partial^i\chi. \end{aligned}$$

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- Gauge-fixing the Lagrange multipliers to $N=1$, $N^i=0$, the canonical momenta are related to the first radial derivatives of the induced fields as

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{\gamma}_{ij}} = -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right), \quad K_{ij} = \frac{1}{2}\dot{\gamma}_{ij},$$
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- The Hamilton-Jacobi formulation of the dynamics amounts to writing the canonical momenta as gradients

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- Now that we have the Hamilton-Jacobi equation, it is important to pause for a moment and take note of a crucial observation:

  - Every solution of the Hamilton-Jacobi equations defines a *first order flow* in field space (cf. BPS equations, fake supergravity, Ricci flows):

$$\begin{aligned}
\dot{\gamma}_{ij} &= 4\kappa^2\left(\gamma_{ik}\gamma_{jl} - \frac{1}{d-1}\gamma_{kl}\gamma_{ij}\right)\frac{1}{\sqrt{\gamma}}\frac{\delta \mathcal{S}}{\delta\gamma_{kl}}, \\
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  - The general solution of the Hamilton-Jacobi equation contains as many integration functions as dynamical fields. Finding such a general solution corresponds to integrating the second order equations once.

- These flow equations are the key ingredient in *deriving* the generalized Fefferman-Graham expansions. Given an asymptotic solution of the Hamilton-Jacobi equation up to the order where the integration functions appear, one can immediately write down the corresponding generalized Fefferman-Graham expansions by making use of the above flow equations.

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Just to make it clear...

- What the Hamilton-Jacobi formalism provides us with, therefore, is a machine that is fed non-normalizable modes and spits out normalizable modes, or more precisely the canonically conjugate variables in the symplectic space of general boundary conditions.
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So, how do we actually solve the Hamilton-Jacobi equation?

- To begin with, having identified the correct non-normalizable modes, the Hamitlon-Jacobi equation admits a solution of the form

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} U(\varphi, \chi)$$

where $U(\varphi, \chi)$ satisfies

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Asymptotic solution of the Hamilton-Jacobi equation

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$$\mathcal{S}_r = \int_{\Sigma_r} d^d x \mathcal{L}(\gamma, \varphi, \chi)$$

and expand

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(2)} + \mathcal{S}_{(4)} + \cdots$$

in eigenfunctions of the operator

$$\delta_\gamma = \int d^d x 2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}}$$

- Inserting this expansion into the Hamilton-Jacobi equation leads to *linear* equations for $\mathcal{S}_{(2n)}$, $n > 0$.
- In particular, applying the identity identity

$$\pi^{ij}\delta\gamma_{ij} + \pi_\varphi \delta\varphi + \pi_\chi \delta\chi = \delta\mathcal{L} + \partial_i v^i(\delta\gamma, \delta\varphi, \delta\chi)$$

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$$U'(\varphi)\frac{\delta}{\delta\varphi}\int d^dx\mathcal{L}_{(2n)} - \left(\frac{d-2n}{d-1}\right)U(\varphi)\mathcal{L}_{(2n)} = \mathcal{R}_{(2n)}, \quad n > 0,$$

where the source terms are given by

$$\mathcal{R}_{(2)} = -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi - Z(\varphi)\partial_i\chi\partial^i\chi\right),$$

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Schematic diagram of the recursion

$$
\begin{array}{ccccc}
\mathcal{R}_{(2n)} & \xrightarrow{\int} & \mathcal{L}_{(2n)} & \xrightarrow{\delta} & \{\pi_{(2n)}\} \\
 & & & & \downarrow \\
\{\pi_{(2n+2)}\} & \xleftarrow{\delta} & \mathcal{L}_{(2n+2)} & \xleftarrow{\int} & \mathcal{R}_{(2n+2)} \\
\downarrow & & & & \\
\mathcal{R}_{(2n+4)} & \xrightarrow{\int} & \mathcal{L}_{(2n+4)} & \ldots &
\end{array}
$$

Summary & Conclusions

- We have an algorithmic procedure for constructing an EFT of any gravitational theory in terms of a symplectic space of boundary data at infinity.
- This symplectic space is identified with the symplectic space of (a sector of) gauge invariant operators in any potential holographically dual theory.
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