

Exact Renormalization
Group Equation and Gauge
Invariance for QED

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Wilson's Renormalization

Group Flow:

$$Z(\lambda) = \int_D \phi e^{i S(\phi)}$$

$$\frac{\partial}{\partial t} Z(\lambda) = 0$$

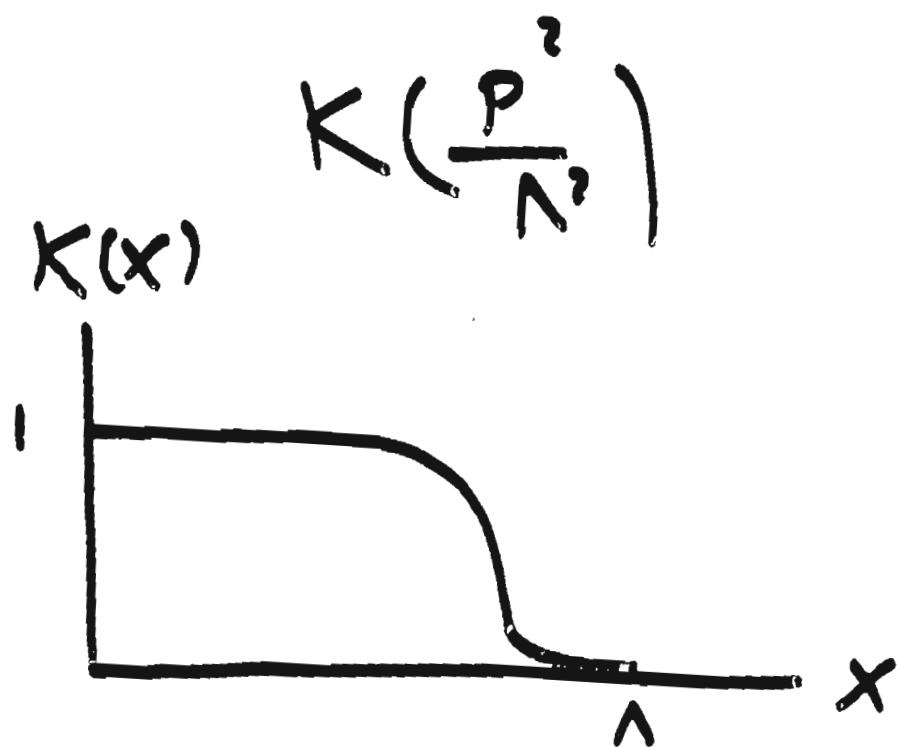
$$\frac{\partial}{\partial \lambda} \equiv \frac{\partial}{\partial t}$$

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Polchinski (1984):

Simple Renormalizability
 $\lambda\phi^4$ Th.

Cut off Function



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Estimates + Bounds

On Effective Action

$$S = \int dx [\phi (-\partial^2 + m^2) \phi + \lambda \phi^q]$$

$$\rightarrow \int dp \left[\frac{p^2 + m^2}{K(p^2/\lambda^2)} \phi_{-p} \phi_p + L \right]$$

$$\dot{Z} = 0 \rightarrow$$

$$\dot{L} = \int \frac{K}{p^2 + m^2} \left[\frac{\partial L}{\partial \phi_{-p}} \frac{\partial L}{\partial \phi_p} + \frac{\partial^2 L}{\partial \phi_{-p} \partial \phi_p} \right]$$

Polchinski ERGE

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At λ_0 :

$$L = \int dx \left[-\frac{1}{2} \dot{\varphi}_1^2 + \frac{1}{2} \dot{\varphi}_2^2 (\partial_\mu \varphi)^2 - \frac{1}{2} \dot{\varphi}_3^2 \right]$$

$$L(\varphi, \lambda, \lambda_0, \dot{\varphi}_i)$$

"Perpendicular" to $\dot{\varphi}_i$:

$$\nabla \equiv \frac{\partial L}{\partial t_0} - \sum_{i=1}^3 \frac{\partial L}{\partial \dot{\varphi}_i} \frac{\partial \dot{\varphi}_i}{\partial \varphi_j} \frac{\partial \varphi_j}{\partial t_0}$$

$$\varrho_1 = -L_2(0, 0, \lambda)$$

$$\varrho_2 = -\frac{1}{8} \frac{\partial^2}{\partial p_1^2} L_2(p_1, -p_1, \lambda) \Big|_{p_1=0}$$

$$\varrho_3 = -L(0, 0, 0, 0, \lambda)$$

$$L(\varphi, \lambda) = \sum_{n=1}^{\infty} \left\{ dp_1 \cdots dp_n L_2(p_1 \cdots p_n, \lambda) \right\}_{\varphi_n = \xi(\varepsilon p_n)} \dot{\varphi}_i$$

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$$L = \int d\tau L_n \dot{\phi}^n$$

Ren. Point :

$$\varrho_1 \approx 0, \varrho_2 \approx 0, \varrho_3 \in \lambda^R$$

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Ren. Thm:

$$\lim L(\varphi, \lambda_R, \lambda_0; \xi^*(\lambda_R, \lambda, \lambda_0))$$

Exists.

$$\left| L_{2n}^{(r)}(\lambda_0) - L_{2n}^{(r)}(\infty) \right| \leq (\dots) \left(\frac{\lambda_R}{\lambda_0} \right)^2$$

$$\left| G^{(r)}(\lambda_0) - G^{(r)}(\infty) \right| \leq (\dots) \frac{1}{\lambda_0^2}$$

...

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: Jioteshwore

$$= (\wedge_{i \in \{1, \dots, n\}} \varphi)_{ns} \sqcup (\eta s - \alpha + \frac{\beta}{\gamma})$$

$$\cdot (\wedge_{i \in \{1, \dots, n\}} \varphi)_{ns} \sqcup (\eta s, \vartheta) \}_{ns}^n \sum_{i=1}^n$$

$$(\wedge_{i \in \{1, \dots, n\}} \varphi)_{ns - \alpha s + \eta s}.$$

$\left\{ \text{निपात नामांकित} \right.$

$$(\eta s - \alpha s, \eta s - \alpha s)_{s+ns} \sqcup (\eta s, \vartheta)_{ns} \}_{ns}^n -$$

$$(\vartheta) \overset{s}{\underset{n+s}{\wedge}}, \frac{1}{s} = (\eta s, \vartheta), \quad ; \vartheta \overset{1-\eta s}{\underset{1-\eta s}{\wedge}} = \vartheta$$

$$ns \sqcup \overset{\eta s - \alpha}{\wedge} \leftarrow ns \sqcup$$

$$\left(\frac{\partial}{\partial s} \vartheta \right)_{ns} \geq \left/ (\wedge_{i \in \{1, \dots, n\}} \varphi)_{ns} \right|_{ns}^{(T)} \quad \longleftarrow$$

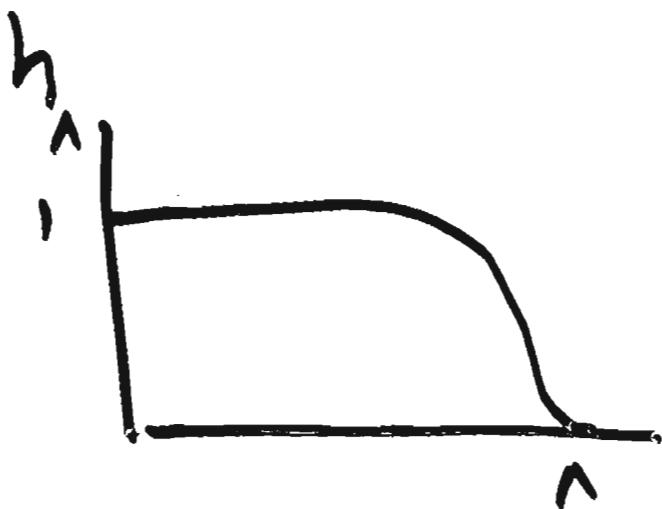
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Another Cut off Method.

$$\Phi(p) \rightarrow h_\lambda(p) \Phi(p)$$



Then

$$K_\lambda(p) = \frac{1}{h_\lambda^2(p)}$$

Range
 $p < \lambda^2$

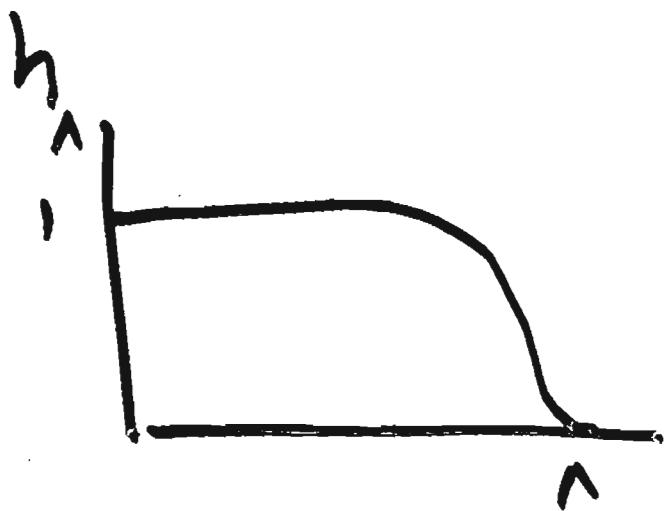
$$L(\phi, \lambda) = \sum_n \int dp L_{zn} \hat{h}_\lambda^{zn} \hat{\phi}^{zn}$$

Bounds as before

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$$L(\phi, \lambda) = \sum_n \int dp L_{2n} h_\lambda^{2n} \phi^{2n}$$

Bounds as before

QED

Same as $\lambda\psi^4$

Need gauge invariance

But cut off breaks it.

and violates Ward Identities.

Approaches :

- 1 - Violations go away at the end of the flow.
- 2 - Covariant cut-off
- 3 - Background Field Method
- 4 - Use SUSY, then Brk it.

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Cut off Gauge Invariance

$$S = \int dP \ h_\lambda \bar{t}_P(iP + m) h_\lambda t_P$$

$$+ e \int dx (\tilde{h}_\lambda \circ F)(\tilde{h}_\lambda \circ A)(\tilde{h}_\lambda \circ t)$$

Convolution : $(f \circ g)(x) = \int f(y-x)g(y) dy$

$$\psi(x) \rightarrow (\tilde{h}^{-1}) \circ [(\tilde{h}_\lambda \circ g)(\tilde{h}_\lambda \circ \psi)]$$

$$A_{\mu}(x) \rightarrow A_{\mu}(x) + (\tilde{h}^{-1}) \circ [(\tilde{h}_\lambda \circ g)(\tilde{h}_\lambda \circ dg^{-1})]$$

$$g(x) = 1 + ie\epsilon(x) + \frac{1}{2!}(\tilde{h}^{-1}) \circ [(\tilde{h} \circ ie\epsilon)(\tilde{h} \circ ie\epsilon)] \\ + \dots$$

$h=1 \Rightarrow$ usual gauge ins.

h sharp \Rightarrow discrete gauge transf.

$h = e^{\eta}, \eta$, small \Rightarrow Ward. Id. Violation

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Ward Identity

$$\begin{aligned} & -\frac{1}{3} \vec{P}^* P_\mu h^2 A_{-\vec{P}}^\mu - P_\mu \frac{\delta \Gamma}{\delta A_{\mu, -\vec{P}}} \\ & + e h \left\{ d\vec{q} \left[\vec{h}_q^\dagger \vec{h}_{q-\vec{P}} \bar{\psi}_{q-\vec{P}} \frac{\delta \Gamma}{\delta \bar{\psi}_q} \right. \right. \\ & \quad \left. \left. + \vec{h}_q \vec{h}_{q-\vec{P}}^\dagger \frac{\delta \Gamma}{\delta \psi_{q-\vec{P}}} \psi_q \right] = 0 \right. \end{aligned}$$

Origins of The Symmetry :

Translationally Inv. * Product

$$(f * g)(x) = \int dp dq K(p,q) f(p-q) g(q) \cdot e^{ip \cdot x}$$

Associativity :

$$K(p,q)K(q,r) = K(p,r)K(p+q, q+r)$$

Solution :

$$K(p,q) = h^{-1}(p) h(q) h(p-q) e^{i\omega}$$

$$\omega(p,q) = \theta_{\mu\nu} p^\mu q^\nu + \eta(q) - \eta(p)$$

$$\text{here } \eta \text{ even, } \eta \text{ odd.} \quad + \eta(p-q)$$

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$$\theta = 0, \eta = 0, h \neq 0$$

$$(f * g)(x) = \int dp dq e^{ipx}.$$

$$h(p) \tilde{h}(p-q) \tilde{f}(p-q) h(q) \tilde{g}(q)$$

$$\therefore h(p) \widetilde{(f * g)}(p) = (h\tilde{f}) \circ (h\tilde{g})$$

$$(f * g)(x) = (\tilde{h}) \{ (\tilde{h} \circ f)(\tilde{h} \circ g)\}$$

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* QED

$$S = \int \bar{\psi} * (\imath \partial + e A) * \psi + F * F$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e(A_{\mu\nu} - A_{\nu\mu})$$

$$g = e^{i\epsilon} = 1 + i\epsilon + \frac{1}{2!} i\epsilon * i\epsilon + \dots$$

$$\psi \rightarrow g * \psi, A \rightarrow g * A * g^{-1}$$

$$-g * \partial_\mu g^{-1}$$

$$\psi \rightarrow g * \psi \Rightarrow$$

$$\psi \rightarrow \tilde{(h^{-1})} \circ [(\tilde{h} \cdot g) (\tilde{h} \cdot \psi)]$$