Unρave$\lambda\lambda$ing the noveλ Hi$\gamma\gamma$s meχanism in (2+1)d

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Outline

Introduction

Membrane field theories and the novel Higgs mechanism

Topological mass and NHM

Diagonalisability conditions

Two-field case

Multi-field case

Difference Chern-Simons and Hitchin equations

Conclusions

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- A specific aspect of membrane field theories is a phenomenon wherein, by giving a vev to a certain scalar, the Chern-Simons nature of the theory is traded for Yang-Mills.
- This phenomenon is known as the novel Higgs mechanism (NHM).
- It is useful in understanding many aspects of multiple membrane theories.
- It also bears an intriguing similarity to some well-known features of Chern-Simons theories in $(2+1)$d, in particular topological mass generation.

- In this talk I will:

  (i) summarise the relevance of NHM to membranes in M-theory,

  (ii) explain the relation between NHM and the topological mass,

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  (iii) explore the most general conditions for NHM.

- (Work in progress, to be discussed if there is time:)
  The equations of motion of a difference Chern-Simons theory can be mapped, in a suitable gauge, to the famous Hitchin equations. After the NHM one gets a kind of deformation of the Hitchin equations.

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Membrane field theories and the novel Higgs mechanism

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- They proposed a gauge field ${A_\mu}^{ab}$ with a Chern-Simons type action:

$$\mathcal{L}_{CS} = \frac{1}{2}\Big({A^a}_b \wedge d{\widetilde{A}^b}_a + \frac{2}{3} {A^a}_b \wedge {\widetilde{A}^b}_c \wedge {\widetilde{A}^c}_a \Big)$$

where ${{\widetilde{A}_\mu}^b}_c \equiv {f^{ab}}_{cd}\, {{A_\mu}^d}_a$, f^{abcd} are the structure constants of a 3-algebra.

- With enough supersymmetry, the entire field theory is determined. For maximal $\mathcal{N} = 8$ supersymmetry the field content is:

$$\tfrac{n(n-1)}{2} \times (A_\mu; \chi), \quad n \times (8X; 4\psi)$$

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- Introducing the covariant derivative:

$$D_\mu X_a^I = \partial_\mu X_a^I - \widetilde{A}_\mu{}^b{}_a X_b^I$$

the bosonic part of the BLG action is:

$$\frac{k}{2\pi}\left(\mathcal{L}_{CS} - \tfrac{1}{2} D_\mu X^I \cdot D^\mu X^I - \tfrac{1}{12}\left(f^{abcd} X_a^I X_b^J X_c^K\right)^2\right)$$

where k is the level of the Chern-Simons term.

- The 3-algebra conditions on f^{abcd} turn out to be so restrictive that the only solution is:

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- In this case there are six fields ${A_\mu}^{ab}$ which can be broken up into two triplet gauge fields:

$$\begin{aligned} A_\mu^{a4} &= \tfrac{1}{2} C_\mu^a, \\ {\epsilon^a}_{bc} A_\mu^{bc} &= \tfrac{1}{2} B_\mu^a \end{aligned}$$

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where $a, b, c \cdots \in 1, 2, 3$.

- The 3-algebra Chern-Simons term now becomes:

$$\sim \quad \tfrac{1}{2} \left(B^a \wedge F^a(C) + \tfrac{2}{3}\, \epsilon_{abc} B^a \wedge B^b \wedge B^c \right)$$

- Making the same $3+1$ split on the scalars, we have:

$$\begin{aligned} DX^{I\,a} &= dX^{I\,a} + \varepsilon^{a}{}_{bc}\, C^{b}\, X^{I\,c} + B^{a}_{\mu}\, X^{I\,4} \\ DX^{I\,4} &= dX^{I\,4} - B^{a}\, X^{I\,a} \end{aligned}$$

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- Now giving a vev $X^{I=8,\,4} = v$ gives a mass term for the B_μ gauge field, so altogether:

$$\sim\; k\left(-\tfrac{1}{2}v^2 B^{a}_{\mu}\,B^{\mu\,a} + \tfrac{1}{2}B^{a}\wedge F^{a}(C) + \tfrac{1}{3}\epsilon_{abc}B^{a}\wedge B^{b}\wedge B^{c}\right)$$

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- The field B is algebraic and we can self-consistently integrate it out by initially neglecting the cubic term:

$$B = \frac{k}{v^2}{}^{*}F(C) + \mathcal{O}\left(\frac{1}{v^4}\right)$$

- Inserting this back we get an $SU(2)$ Yang-Mills theory with the 7 remaining scalars, and the fermions, in the adjoint (also X^8 disappears):

$$\frac{k}{v^2}\mathrm{tr}\left\{-\frac{1}{4}\boldsymbol{F}\wedge{}^*\boldsymbol{F}-\tfrac{1}{2}D_\mu\boldsymbol{X}^iD^\mu\boldsymbol{X}^i+\tfrac{1}{4}[\boldsymbol{X}^i,\boldsymbol{X}^j]^2+\cdots\right\}$$

where $\boldsymbol{F},\boldsymbol{X}$ are now 2×2 matrices.

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  where $\boldsymbol{F},\boldsymbol{X}$ are now 2×2 matrices.
- The dots represent fermion terms and also corrections suppressed by inverse powers of v^2.
- We see that $v/\sqrt{k}$ plays the role of g_{YM}.

- The Chern-Simons action can be written in a form that has now become very familiar. Under:

$$\boldsymbol{B} = \tfrac{1}{2}(\boldsymbol{A}^1 - \boldsymbol{A}^2), \qquad \boldsymbol{C} = \tfrac{1}{2}(\boldsymbol{A}^1 + \boldsymbol{A}^2)$$

  the Lagrangian:

$$\mathrm{tr}\left(\boldsymbol{B} \wedge \boldsymbol{F}(\boldsymbol{C}) + \tfrac{1}{3}\boldsymbol{B} \wedge \boldsymbol{B} \wedge \boldsymbol{B}\right)$$

  becomes:

$$\sim \tfrac{1}{2}\,\mathrm{tr}\Big(\boldsymbol{A}^1 \wedge d\boldsymbol{A}^1 + \tfrac{2}{3}\boldsymbol{A}^1 \wedge \boldsymbol{A}^1 \wedge \boldsymbol{A}^1 - \boldsymbol{A}^2 \wedge d\boldsymbol{A}^2 - \tfrac{2}{3}\boldsymbol{A}^2 \wedge \boldsymbol{A}^2 \wedge \boldsymbol{A}^2\Big)$$

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- The covariant derivative is then:

$$D\boldsymbol{X} = d\boldsymbol{X} - \boldsymbol{A}^1\boldsymbol{X} + \boldsymbol{X}^T\boldsymbol{A}^2$$

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- Here we focus on $\mathcal{N} = 8$ [BLG] and $\mathcal{N} = 6$ [ABJM,ABJ] theories.
- In this context the NHM has provided a few different illuminations about membranes and M-theory, which I will now briefly review.

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- With maximal or near-maximal supersymmetry:

$$\mathcal{L}_{diff-CS}\Big|_v = \frac{k}{v^2}\mathcal{L}_{SYM} + \mathcal{O}\left(\frac{k}{v^4}\right)$$

where the first term on the RHS is $\mathcal{N}=8$ supersymmetric Yang-Mills theory and, as noted, $v/\sqrt{k}$ plays the role of g_{YM}.

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- If we take $v \to \infty$ then the higher-order terms drop out and we find that:

$$\mathcal{L}_{diff-CS}\Big|_{v\to\infty} = \lim_{g_{YM}\to\infty} \frac{1}{g_{YM}^2}\mathcal{L}_{SYM}$$

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- This amounts to a proof that somewhere on its moduli space, and therefore presumably everywhere, the [BLG,ABJM] and probably many other theories describe multiple membranes.

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- The NHM was originally worked out at fixed level k. If we carry out the same procedure and take $v \to \infty, k \to \infty$ keeping $v/\sqrt{k}$ fixed, then:

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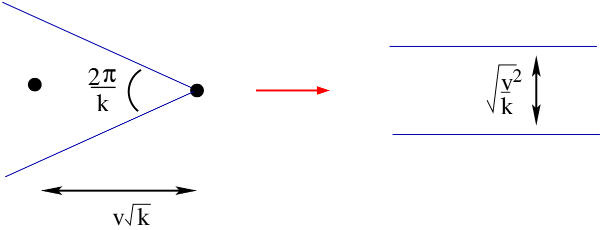
- This is explained by analogy with deconstruction. Again, it works equally well for BLG and ABJM.



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- All coefficients were uniquely determined by this procedure.
- Presumably the same procedure can (and should) be carried out for ABJM theory.

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$$\frac{k_1}{2\pi} S_{CS}(A_+) + \frac{k_2}{2\pi} S_{CS}(A_-)$$

They argued that $k_1 + k_2$ corresponds in the dual type IIA theory on $AdS_4 \times CP^3$ to a Romans mass.

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- The NHM confirms this proposal: for unequal levels, it creates a Yang-Mills theory plus a residual Chern-Simons theory of level $k_1 + k_2$.

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- The NHM confirms this proposal: for unequal levels, it creates a Yang-Mills theory plus a residual Chern-Simons theory of level $k_1 + k_2$.

- The latter reproduces the coupling $\int F_0 \, S_{CS}(A)$ on a D2-brane in the presence of the Romans mass.

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- The propagating modes have a single degree of freedom with spin $+1$ but no corresponding spin -1 state.
- This is possible because Chern-Simons theory is parity violating.

- Subsequently a different model called the "self-dual" theory was considered [Townsend et al]:

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- In fact it was shown that the two theories are equivalent:

$$S_{CS} + S_{mass} \;\sim\; S_{YM} + S_{CS}$$

- In the above equivalence, the mass on the LHS becomes the Yang-Mills coupling on the RHS.

- By contrast, the NHM involves a pair of gauge fields having Chern-Simons terms with opposite signs, as well as an explicit mass term of a specific form (possibly arising via a Higgs mechanism). The theory is equivalent to a (classically massless) Yang-Mills theory:

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- We will now examine these equivalences in a little more detail.
- We will not require supersymmetry. Also, since we want to understand the spectrum of the theory, we work at the linearised level.

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- This theory has a single on-shell degree of freedom that is massive and has spin $+1$. If we change the sign of the mass term we instead get spin -1.

- On the other hand, the Lagrangian:

$$\mathcal{L}_2 = \frac{1}{2} A \wedge dA + \frac{1}{2} m \, A \wedge {}^*A$$

is said to be self-dual. The equations of motion are:

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- A related point is that $\mathcal{L}_1$ has a smooth massless limit while $\mathcal{L}_2$ becomes purely topological and thereby loses a degree of freedom as $m \to 0$.
- In $\mathcal{L}_2$, if the mass term comes from a Higgs field then of course the full theory has gauge invariance realised in the Higgs mode.

- The two Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ give equivalent theories. Classically, this is shown as follows. First,

$$^*dA = mA \implies d\,^*dA = m\,dA$$

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- For the converse,

$$\begin{aligned} d\,^*dA = m\,dA \qquad &\implies d\,(^*dA - m\,A) = 0 \\ &\implies {}^*dA - m\,A = d\lambda \end{aligned}$$

and a field re-definition

$$A \to A - \frac{1}{m} d\lambda$$

gives $\mathcal{L}_2$.

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- While $\mathcal{L}_1$ has the form of a generalised topologically massive theory, $\mathcal{L}_2$ is instead a massless Maxwell Lagrangian.

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- Now let us consider $\mathcal{L}_1$ in the difference-Chern-Simons basis, by writing:

$$\begin{aligned} A^1 &= C + B \\ A^2 &= C - B \end{aligned}$$

after which it becomes:

$$\mathcal{L}_1 = k\left(\frac{1}{2}A^1 \wedge dA^1 - \frac{1}{2}A^2 \wedge dA^2 + \frac{1}{4}m(A^1 - A^2) \wedge {}^*(A^1 - A^2)\right)$$

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- In this basis there is a non-diagonal mass term:

$$m_{IJ} \sim \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

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- More precisely it leads to topological mass non-generation...!
- The crucial new ingredients are to have more than one gauge field, a difference of two Chern-Simons actions, and a suitable non-diagonal mass term.

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Topological mass and NHM

Diagonalisability conditions

Two-field case

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- A necessary (but not sufficient) condition for this comes from a conflict between the simultaneous diagonalisability of the kinetic and mass terms.
- This phenomenon is peculiar to Chern-Simons gauge theories with a mass term and does not have an analogue in scalar or Maxwell-type vector theories.

- Consider a collection of vector fields $A^I, I = 1, 2, \cdots n$ described by the most general abelian Chern-Simons Lagrangian with a mass term:

$$\mathcal{L} = \frac{1}{2} k_{IJ} A^{(I)} \wedge dA^{(J)} + \frac{1}{2} m_{IJ} A^{(I)} \wedge {}^{*}A^{(J)}$$

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- Both k_{IJ} and m_{IJ} are constant real symmetric matrices.
- k_{IJ} is taken to be non-degenerate, while m_{IJ} is allowed to have zero eigenvalues.
- Let us now try to bring this action into standard form.

- For comparison, we first consider a generic free scalar field theory with Lagrangian:

$$-\frac{1}{2}g_{IJ}\partial_\mu\phi^I\partial^\mu\phi^J - \frac{1}{2}(m^2)_{IJ}\phi^I\phi^J$$

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- Here g_{IJ} and $(m^2)_{IJ}$ are constant real symmetric matrices, and g_{IJ} is positive-definite (otherwise the theory has ghosts).
- To bring the Lagrangian into its standard form, first perform an orthogonal transformation on ϕ^I to diagonalise g_{IJ}, which then takes the form $\mathrm{diag}(g_1, g_2, \cdots, g_n)$ with $g_I > 0$ for all I.

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- Next one re-scales the fields:

$$\phi^I \to \frac{\phi^I}{\sqrt{g_I}}$$

  so that the kinetic form has the identity metric δ_{IJ}.

- Finally one performs another orthogonal transformation on ϕ^I that diagonalises m^2 while preserving the kinetic term, ending up with:

$$-\frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^I - \frac{1}{2}m_I^2\phi^I\phi^I$$

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  with some m_I possibly equal to 0.
- Thus the theory has been reduced to a collection of independent fields, some massive and others massless (some of the masses can be tachyonic as long as the full potential is bounded below).

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- In fact, as we just saw, M2-brane field theories have levels of both signs, which ensures that parity is conserved (similar actions arise for the Chern-Simons formulation of 3d gravity).
- To be completely general we therefore assume k_{IJ} has p negative and q positive eigenvalues with $p + q = n$.

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  where η_{IJ} is the Lorentzian metric preserved by $O(p,q)$.
- Hence the linear transformations $A^I \to \Lambda_{IJ}A^J$ which preserve the kinetic term are given by matrices Λ_{IJ} satisfying:

$$\Lambda^T \eta \Lambda = \eta$$

  namely the $O(p,q)$ Lorentz transformations.

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- Therefore in the basis where k_{IJ} is diagonal, we start by seeking the conditions on m_{IJ} such that it can be diagonalised by a Lorentz transformation.
- Whenever this is possible, the theory will reduce to a collection of decoupled Chern-Simons actions with definite masses, and there will be no novel Higgs mechanism.

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- This has been analysed in the GR literature. The possibilities are categorised as algebraically general and algebraically special, with the latter having sub-cases.
- As a necessary condition, we will see that only the algebraically special cases can have an NHM.

Outline

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Diagonalisability conditions

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Multi-field case

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- In this simple example one can explicitly find the diagonalisability conditions.
- We simply ask what is the most general 2×2 matrix that can be obtained from a diagonal matrix by a Lorentz boost.

- In terms of components:

$$m_{IJ} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

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- Theories that exhibit the novel Higgs mechanism must therefore fail to satisfy this inequality. As a check we notice that the mass matrix we originally displayed in an example,

$$m_{IJ} \sim \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

just barely fails to satisfy the above inequality.

- The above condition can be reformulated in terms of eigenvalues of the (non-symmetric) matrix

$$(\eta\, m)^I{}_J = \eta^{IK} m_{KJ} = \begin{pmatrix} -a & b \\ -b & c \end{pmatrix}$$

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- The analogue of this question for $T_{\mu\nu}$ in general relativity is well-studied in (3+1)d and the possible cases classified (see for example the book of [Stephani et al]).
- We can adapt this classification to (1+1)d.

- In $1+1$ dimensions there are precisely three possibilities:

	Eigenvalues	Eigenvectors
(i)	Two distinct, real	Two distinct, real (one space-like, one time-like)
(ii)	Two coincident	One
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- The last two cases are termed algebraically special.
- Case (i) allows us to make an $SO(1,1)$ matrix:

$$\Lambda = \begin{pmatrix} v_t & v_s \end{pmatrix}$$

where v_t, v_s are the orthonormalised eigenvectors, the first one time-like and the second space-like. Clearly Λ diagonalises $\eta\, m$ by a similarity transformation:

$$\Lambda^{-1}\, \eta\, m\, \Lambda = \eta\, m_{\text{diag}}$$

where we have labelled the diagonal matrix as $\eta\, m_{\text{diag}}$.

- Noting that $\Lambda^{-1} = \eta\Lambda^T\eta$, we see that:

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- As a confirmation of this picture, one can check explicitly that the above three cases correspond to:

$$\begin{array}{rccc} \text{(i)} & 2|b| & < & |a+c| \\ \text{(ii)} & 2|b| & = & |a+c| \\ \text{(iii)} & 2|b| & > & |a+c| \end{array}$$

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- As we have already seen by direct computation, only the first case admits diagonalisation of m_{IJ} by a Lorentz transformation.

Two-field case: Sufficient conditions for NHM

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- It turns out the basis in which k_{IJ} is diagonal is not the most convenient. Instead, the useful basis is the one in which k_{IJ} is purely off-diagonal.
- This is just the light-cone basis, in which the Lorentzian space is taken to be spanned by two independent null vectors and the metric on field space therefore takes the form:

$$k_{IJ} = k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Suppose that in this basis the mass matrix is given by:

$$m = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

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- The Lagrangian is then:

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- If $\alpha \neq 0$ then the first equation can be solved for A^1. Inserting the solution back into the action, we find:

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- The resulting theory has a massless propagating gauge field if and only if $\beta = \gamma = 0$.

- To arrive at this action we assumed that $\alpha \neq 0$, but of course we could instead assume $\gamma \neq 0$ and eliminate A^2 in the same manner.

- To arrive at this action we assumed that $\alpha \neq 0$, but of course we could instead assume $\gamma \neq 0$ and eliminate A^2 in the same manner.
- Thus the final condition for the novel Higgs mechanism in the two-field case is that one of α, γ be nonzero and the other one, along with β, vanish.

Outline

Introduction

Membrane field theories and the novel Higgs mechanism

Topological mass and NHM

Diagonalisability conditions

Two-field case

Multi-field case

Difference Chern-Simons and Hitchin equations

Conclusions

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  Theorem: Two real quadratic forms A_{ij} and B_{ij} can be simultaneously diagonalised by a change of basis if and only if they have no common zeros along the diagonal in any basis.
- For us the two matrices are η_{IJ} and m_{IJ}. The above theorem suggests choosing a maximally off-diagonal basis for the former. If we have p timelike and q spacelike directions with $p < q$ (the analysis is similar for $p \geq q$) we can bring η to the form:

$$\eta_{IJ} = \begin{pmatrix} 0 & \mathbb{I}_p & 0 \\ \mathbb{I}_p & 0 & 0 \\ 0 & 0 & \mathbb{I}_{q-p} \end{pmatrix}$$

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- Therefore a necessary (but not sufficient) condition for the NHM is that m_{IJ} in this basis has at least one zero along the diagonal.

Multi-field case: Sufficient conditions for NHM

- Sufficient conditions can be found by following the same procedure as in the two-field case. For simplicity let us take $p = q$.

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- In the basis for η_{IJ} above, we divide the $A^{(I)}, I = 1, 2, \cdots, 2p$ into two sets:

$$\begin{aligned} A^i &= B^i,\ i = 1, 2, \cdots, p \\ A^{p+i} &= C^i,\ i = 1, 2, \cdots, p \end{aligned}$$

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- Then the free Lagrangian takes the form:

$$\mathcal{L} = B^i \wedge dC_i + \frac{1}{2}\alpha_{ij} B^i \wedge {}^*B^j + \beta_{ij} B^i \wedge {}^*C^j + \frac{1}{2}\gamma_{ij} C^i \wedge {}^*C^j$$

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- The corresponding equations of motion are:

$$\begin{aligned} dC_i + \alpha_{ij}\, {}^*B^j + \beta_{ij}\, {}^*C^j &= 0 \\ dB_i + \beta_{ij}\, {}^*B^j + \gamma_{ij}\, {}^*C^j &= 0 \end{aligned}$$

- Now suppose the matrix α_{ij} is invertible. In that case we can solve the first equation for B^i and insert this back into the original Lagrangian to get:

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\alpha^{-1}_{ij} dC^i \wedge {}^*dC^j - (\alpha^{-1}\beta)_{ij} C^i \wedge dC^j \\ &+ \tfrac{1}{2}\left(\gamma - \beta\alpha^{-1}\beta\right)_{ij} C^i \wedge {}^*C^j\end{aligned}$$

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- We conclude that there is one massless propagating vector field for every simultaneous zero eigenvector of the matrices β_{ij} and γ_{ij}, under the condition that α_{ij} is invertible.
- As in the two-field case, the roles of α_{ij} and γ_{ij} can be interchanged.

Outline

Introduction

Membrane field theories and the novel Higgs mechanism

Topological mass and NHM

Diagonalisability conditions

Two-field case

Multi-field case

Difference Chern-Simons and Hitchin equations

Conclusions

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- In this connection, a curious mathematical observation ([in collaboration with David Tong]) is that before turning on a Higgs vev, the equations can be recast in a suitable gauge as the famous Hitchin equations.
- Therefore the equations one gets after giving a Higgs vev might perhaps be thought of as some deformation of the Hitchin system.

- We start with the usual difference action:

$$L_{diff-CS} \sim \mathrm{tr}\Big(A^1 \wedge dA^1 + \tfrac{2}{3}A^1 \wedge A^1 \wedge A^1 \\ -A^2 \wedge dA^2 - \tfrac{2}{3}A^2 \wedge A^2 \wedge A^2\Big)$$

with the infinitesimal gauge symmetries:

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- Upon changing variables:

$$B = \tfrac{1}{2}(A^1 - A^2), \quad C = \tfrac{1}{2}(A^1 + A^2)$$

we have seen that the action becomes:

$$\sim \mathrm{tr}\Big(B \wedge F(C) + \tfrac{1}{3}B \wedge B \wedge B\Big)$$

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- In these variables, the gauge transformations are:

$$\begin{aligned} \delta B &= d\Lambda_B + [C, \Lambda_B] + [B, \Lambda_C] \\ \delta C &= d\Lambda_C + [C, \Lambda_C] + [B, \Lambda_B] \end{aligned}$$

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$$\begin{aligned} F(C)_{12} + [B_1, B_2] &= 0 \\ F(C)_{20} + [B_2, B_0] &= 0 \\ F(C)_{01} + [B_0, B_1] &= 0 \end{aligned}$$

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- Since this gauge condition depends on both A^1 and A^2, it has the effect of coupling the two formerly decoupled systems.

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- From the first of these equations, the space components $B_i, i = 1, 2$ are also time-independent.

- Thus we have finally reduced the system to four quantities $C_i, B_i, i = 1, 2$. One of the equations they satisfy is:

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- Next we define $z = x^1 + ix^2$ and $\Phi = B_1 - iB_2$. Consider:

$$D^C_{\bar{z}}\Phi \sim (D^C_1 + iD^C_2)(B_1 - iB_2)$$

  For this to vanish, we must have:

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- Hence at the end, the quantities $C_i, \Phi, \bar{\Phi}$ depend on the space variables x^1, x^2 or equivalently $z, \bar{z}$ and satisfy:

$$\begin{aligned} F(C)_{12} &= \frac{i}{2}[\Phi, \bar{\Phi}] \\ D^C_{\bar{z}}\Phi &= D^C_z\bar{\Phi} = 0 \end{aligned}$$

These are precisely the Hitchin equations.

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- The last two equations are:

$$\begin{aligned} \dot{C}_2 &= [B_2, B_0] - \tfrac{v^2}{2} B_1 \\ \dot{C}_1 &= [B_1, B_0] + \tfrac{v^2}{2} B_2 \end{aligned}$$

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  and they determine the time-dependence of C_1, C_2.
- And... we have to leave the story here!

Outline

Introduction

Membrane field theories and the novel Higgs mechanism

Topological mass and NHM

Diagonalisability conditions

Two-field case

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Conclusions

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- This phenomenon is possible only because of the possibility of indefinite signature Chern-Simons terms.
- It has implications for multiple membranes in M-theory, and should have more general implications for the dynamics of gauge theories in (2+1)d.
- It would also be interesting to see if analogous results hold in (2+1)d gravity, given that topological mass generation can occur there and the action is of difference-Chern-Simons form.

$\epsilon\upsilon\chi\alpha\rho\iota\sigma\tau\acute{\omega}!$