A-B-C APPROACHES TO SURFACE OPERATORS

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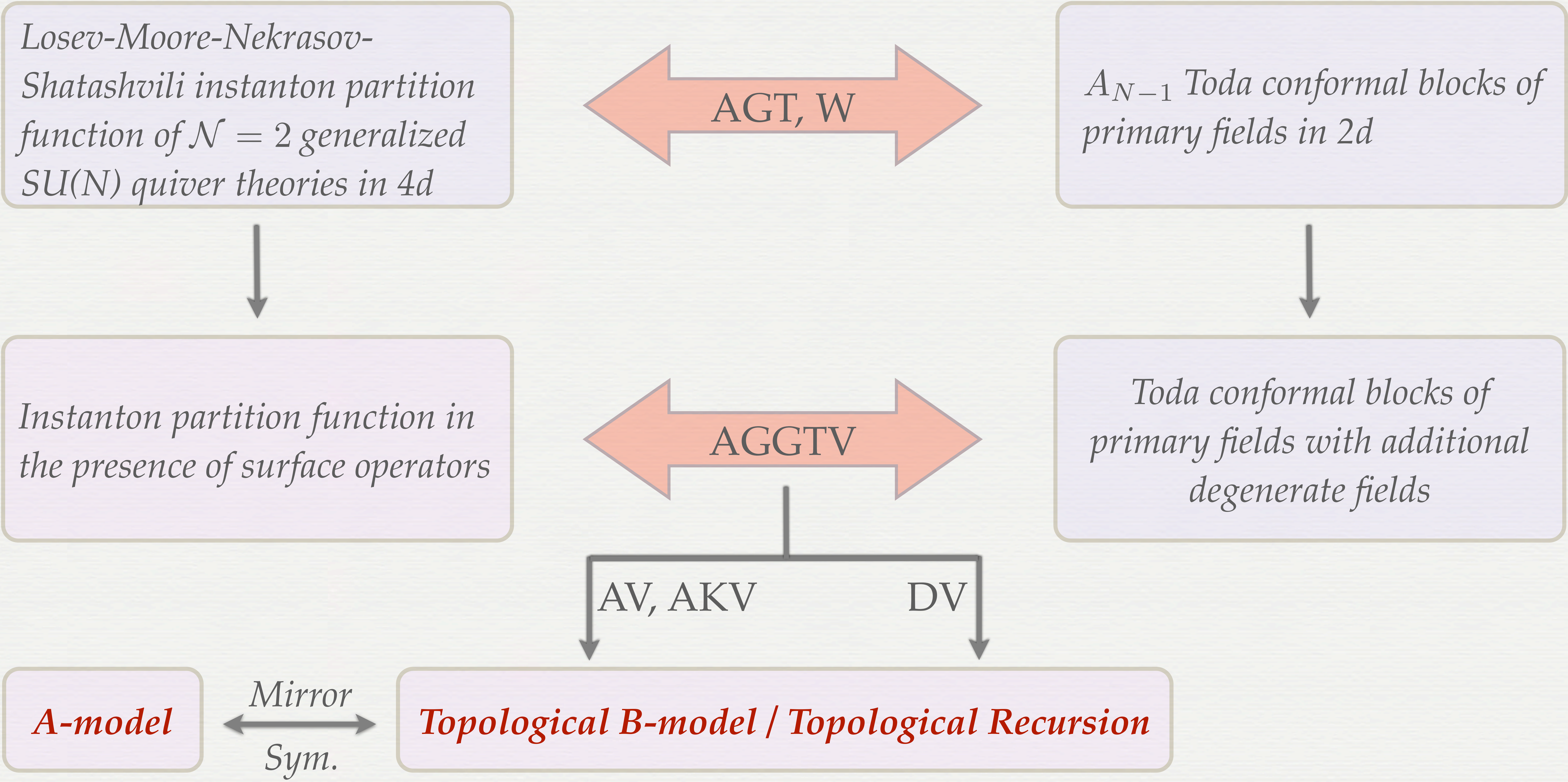
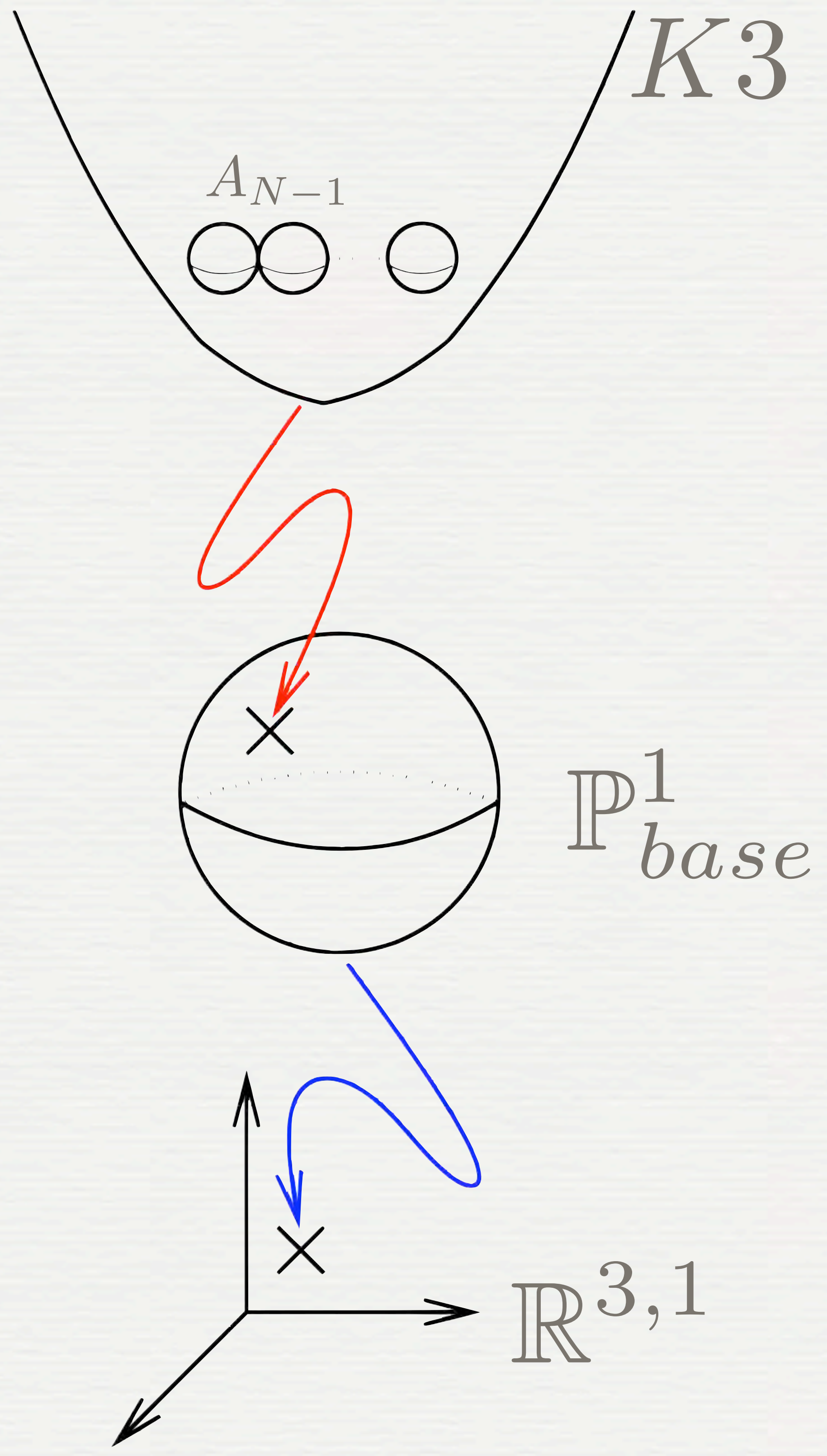
OVERVIEW & MOTIVATION



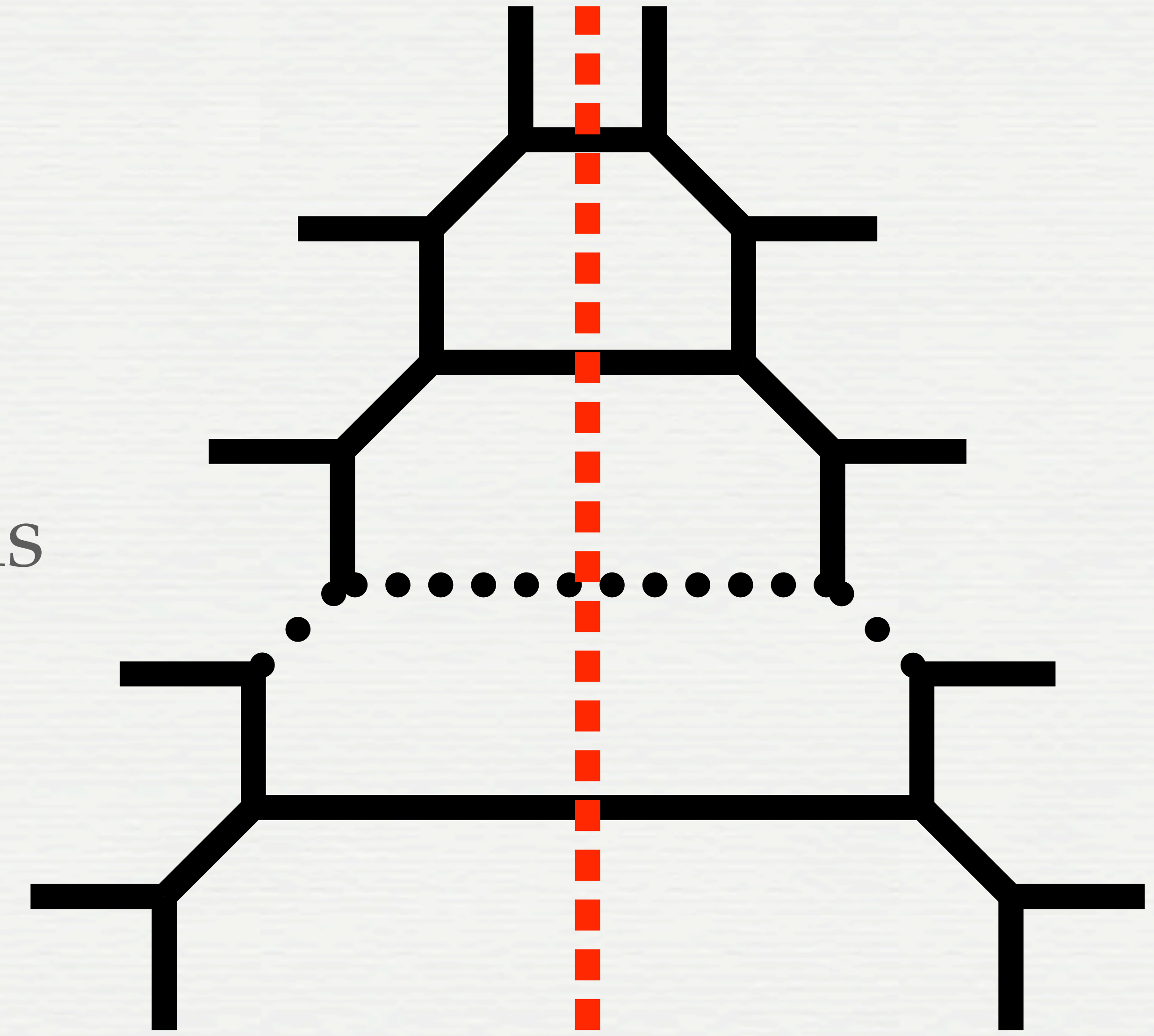
OVERVIEW & MOTIVATION

$\mathcal{N}=2$ supersymmetric gauge theories can be *geometrically engineered* in type IIA setup. For *SU(N)* theoris with different matter multiplets the internal space is a *toric Calabi-Yau 3fold, i.e.,* (refined) topological vertex can be employed.

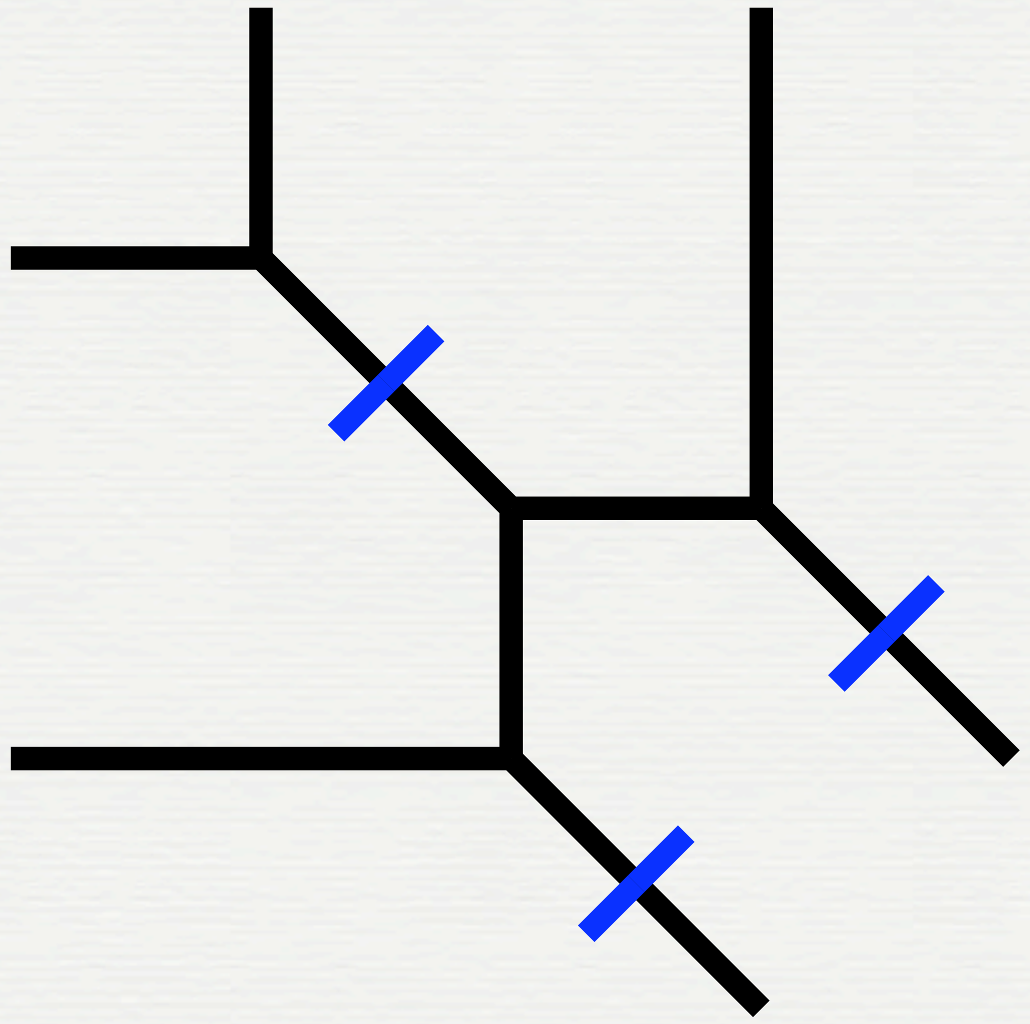


toric geometry
can be encoded
in toric diagrams

KKV



$SU(N_c)$ with $N_f = 2N_c$



closed topological vertex,
T_2 geometry

RESULTS & OVERVIEW

- The B-model topological recursion can be used to compute CFT correlation function with degenerate insertions
  - the partition function in the presence of a surface operators
  - beyond the semi-classical limit taken by AGGTV, exact in α', perturbative in g_s
  - explicit computations can be made
- The B-model computation can be mirrored to the A-model, *i.e.*, the (refined) topological vertex can be used
  - valid only at the large volume limit
  - exact in g_s, perturbative in α'
  - explicit computations can be made

RESULTS & OVERVIEW

- The Conformal theory approach is based on AGT and AGGTV conjectures and is a new approach to *'perform'* gauge theory computations
  - relates the gauge theory with a surface operator insertion to a quiver gauge theory without a surface operator in a particular limit
  - has the contributions from the conventional instantons as well as *'two-dimensional'* instantons due the presence of surface operators
- All of the A-B-C approaches give results consistent with each other

OUTLINE

- AGT & AGGTV
- Remodeling & B-model computations
- Refined topological vertex & A-model computations
- Conformal field theory computations

AGT & AGGTV

AGT

A large class of *4d* superconformal gauge theories with $\mathcal{N} = 2$ supersymmetry is constructed by Gaiotto starting with *6d* $\mathcal{N} = (2,0)$ superconformal A_{N-1} type theories upon compactifying on a Riemann surface *C*.

The space of coupling constants is identified with the (universal cover) of the moduli space of complex structures of the punctured Riemann surface *C*.

This construction can be realized in M-theory by wrapping a M5-brane on a Riemann surface with punctures, such as on a sphere with 4 punctures. In the IR, the world volume theory will give rise to the gauge theory.

A single M5-brane wrapping a sphere with 4 punctures will give rise to *SU(2)* theory with 4 hypermultiplets supported on the punctures.

M2-branes can on M5-branes, and depending on the embedding, they can introduce *2d* defects in the *4d* transverse space. In the gauge theory, it introduces breaking of the gauge group in a smaller group over the defect.

AGT

AGT conjecture states that the instanton part of the LMNS partition function of *SU(2)* quiver theory with matter can be identified with the Virasoro conformal blocks of Liouville theory on a sphere or a torus

External momenta	$\longleftrightarrow$	Mass of the matter multiplets
Internal momenta	$\longleftrightarrow$	Coulomb branch parameters
Central charge	$\longleftrightarrow$	Deformation parameters

$2d$ A_r Toda theory	$4d$ A_r quiver gauge theory
Conformal block	Instanton partition function
Three-point function	1-loop partition function
Level k	Instanton number k
b	ϵ_1
$\frac{1}{b}$	ϵ_2
External α's	Masses, μ_j
Internal σ's	Coulomb moduli, a_i

AGT

AGT defined the quadratic differential by

$$\psi_2(z)dz^2 = \frac{\langle T(z)\prod_i \mathcal{O}_i(z_i)\rangle}{\langle \prod_i \mathcal{O}_i(z_i)\rangle}$$

In the semi-classical limit, *i.e.*, $\epsilon_{1,2} \ll a_i, m_i$, the Seiberg-Witten curve of the theory is obtained:

$$\boxed{x^2 = \psi_2^{SW}(z)}$$

This limit is checked using

$$\oint_{\beta_a} \sqrt{\psi_2(z)} \to \oint_{\beta_a} x dz = m_a$$

β_a is a small loop around the *a-th* puncture

$$\oint_{\gamma_i} \sqrt{\psi_2(z)} \to \oint_{\gamma_i} x dz = a_i$$

γ_i is a cycle around the long thin *i-th* neck

AGGTV

AGGTV argue for *SU(2)* theory with four hypermultiplets, the inclusion of a surface operator is given by

$$Z = \frac{\langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(\zeta) V_{-b/2}(z) | \alpha_4 + b/2 \rangle}{\langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(\zeta) | \alpha_4 \rangle} = e^{-\frac{b}{\hbar} G_0(z) + b^2 G_1(z) + b^3 \hbar G_2(z) + \mathcal{O}(\hbar^2)}$$

The degenerate fields satisfy the following null state condition $\left(L_{-1}^2 + b^2 L_{-2}\right) V_{-b/2} = 0$

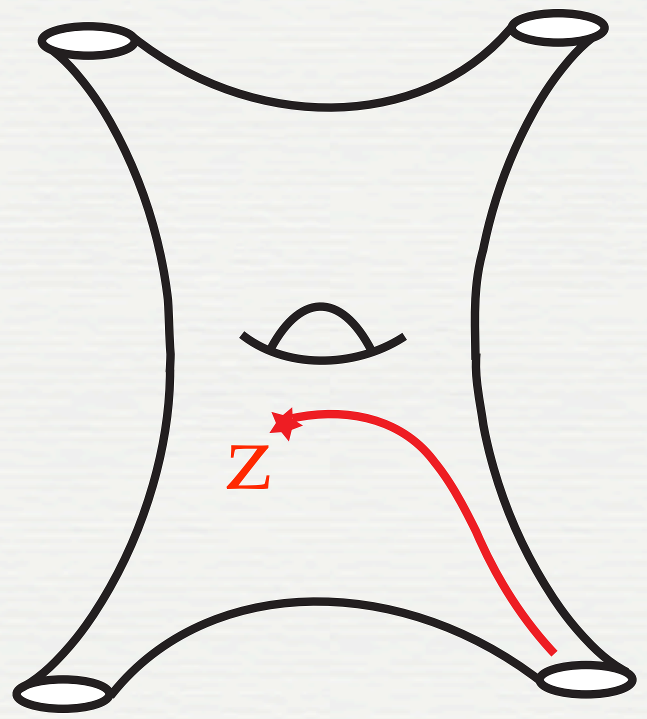
$$\partial_z^2 \langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(\zeta) V_{-b/2}(z) | \alpha_4 \rangle + b^2 \langle \alpha_1 | V_{\alpha_2}(1) V_{\alpha_3}(\zeta) T(z) V_{-b/2}(z) | \alpha_4 \rangle = 0$$

The null state condition combined with the AGT relation leads to

$(\partial_z G_0)^2 + \psi_2(z) = 0$ or equivalently $\boxed{G_0(z) = \int^z x(z') dz'}$

$\big\downarrow$ AV

Disc amplitude



REMODELLING & B-MODEL COMPUTATIONS

REMODELING THE B-MODEL

Topological recursion (Eynard & Orantin) is developed to solve the loop equations of matrix models in a systematic way. This method uses the spectral curve of the matrix model together with the basic ingredients associated to the curve and creates differentials in a recursive way

$$\mathcal{W}_k^{(g)}(z_1, \ldots, z_k) dp_1 \ldots dp_k$$

These differentials can be integrated to obtain genus g open k-point amplitudes

$$A_k^{(g)}(z_1, \ldots, z_k) = \int^{z_1} \cdots \int^{z_k} dp_1 \ldots dp_k \, \mathcal{W}_k^{(g)}(z_1, \ldots, z_k)$$

These amplitudes can be organized into the partition function (for one insertion)

$$\begin{aligned} Z_{\rm null}(z)\big|_{\mathcal{Q}=0} &= \exp\Big[\sum_{g,k} \hbar^{2g-2+k} \frac{1}{k!} A_k^{(g)}(z, \cdots, z)\Big] \\ &= \exp\Big[\underbrace{\frac{1}{\hbar} A_1^{(0)}(z)}_{G_0(z)\big|_{\epsilon_1+\epsilon_2=0}} + \underbrace{\frac{1}{2!} A_2^{(0)}(z,z)}_{G_1(z)\big|_{\epsilon_1+\epsilon_2=0}} + \hbar \Big(\underbrace{A_1^{(1)}(z) + \frac{1}{3!} A_3^{(0)}(z,z,z)}_{G_2(z)\big|_{\epsilon_1+\epsilon_2=0}}\Big) + \cdots\Big] \end{aligned}$$

REMODELING THE B-MODEL

The amplitudes can be easily generalized to multiple insertions

$$A_j^{(g)}(z,\ldots,z) \to \sum_{i_1,\ldots,i_j=1}^{k} A_j^{(g)}(z_{i_1},\ldots,z_{i_j})$$

We checked explicitly that indeed this generalization works by comparing with the AGGTV computation

$$G_1(z) \to G_1(z_1,z_2)\big|_{\epsilon_1+\epsilon_2=0} = A_2^{(0)}(z_1,z_2) + \frac{1}{2}(A_2^{(0)}(z_1,z_1) + A_2^{(0)}(z_2,z_2))$$

We focused on two theories, T_2 theory and *SU(2)* theory with four flavors. T_2 theory is a free theory with four hypermultiplets and is also used in Gaiotto's construction as a building block for generalized quiver theories.

REMODELING THE B-MODEL

Let us focus on T_2 theory more closely. According to the remodeling

$$\mathcal{W}_1^{(0)}(z_1)dz_1 = \lambda_{SW}(z_1)$$

where we use the Seiberg-Witten differential coming from the *M*-theory description. We compute 1-, 2- and 3-functions using this differential

$$A_1^{(0)}(z) = \alpha_1 \log(z) + \frac{-\alpha_0^2 + \alpha_1^2 + \alpha_2^2}{2\alpha_1} z - \frac{(\alpha_0^4 - 3\alpha_1^4 - 6\alpha_1^2\alpha_2^2 + \alpha_2^4 + 2\alpha_0^2(\alpha_1^2 - \alpha_2^2))}{16\alpha_1^3} z^2 + \cdots$$

The computation of higher point function, even at genus *g=0*, is more involved and requires the Bergman kernel

$$\mathcal{W}_2^{(0)}(z_1, z_2)\, dz_1 dz_2 = B(z_1, z_2) - \frac{dz_1 dz_2}{2(z_1 - z_2)^2}$$

and gives

$$A_2^{(0)}(z_1, z_2) = \frac{\alpha_0^4 + (\alpha_1^2 - \alpha_2^2)^2 - 2\alpha_0^2(\alpha_1^2 + \alpha_2^2)}{16\alpha_1^4} z_1 z_2 + \\ + \frac{(\alpha_0^2 + \alpha_1^2 - \alpha_2^2)(\alpha_0^2\alpha_1^2 + (\alpha_1^2 - \alpha_2^2)^2 - 2\alpha_0^2(\alpha_1^2 + \alpha_2^2))}{32\alpha_1^6}(z_1^2 z_2 + z_1 z_2^2) + \cdots$$

REMODELING THE B-MODEL

We computed the 3-point function at genus *g=0,* and one point function at genus *g=1*. The corresponding differentials are quite involved but after integrating

$$A_3^{(0)}(z_1, z_2, z_3) = \frac{(\alpha_0^2 - \alpha_2^2)(\alpha_0^4 + (\alpha_1^2 - \alpha_2^2)^2 - 2\alpha_0^2(\alpha_1^2 + \alpha_2^2)}{\alpha_1^7} z_1 z_2 z_3 + \cdots$$

and

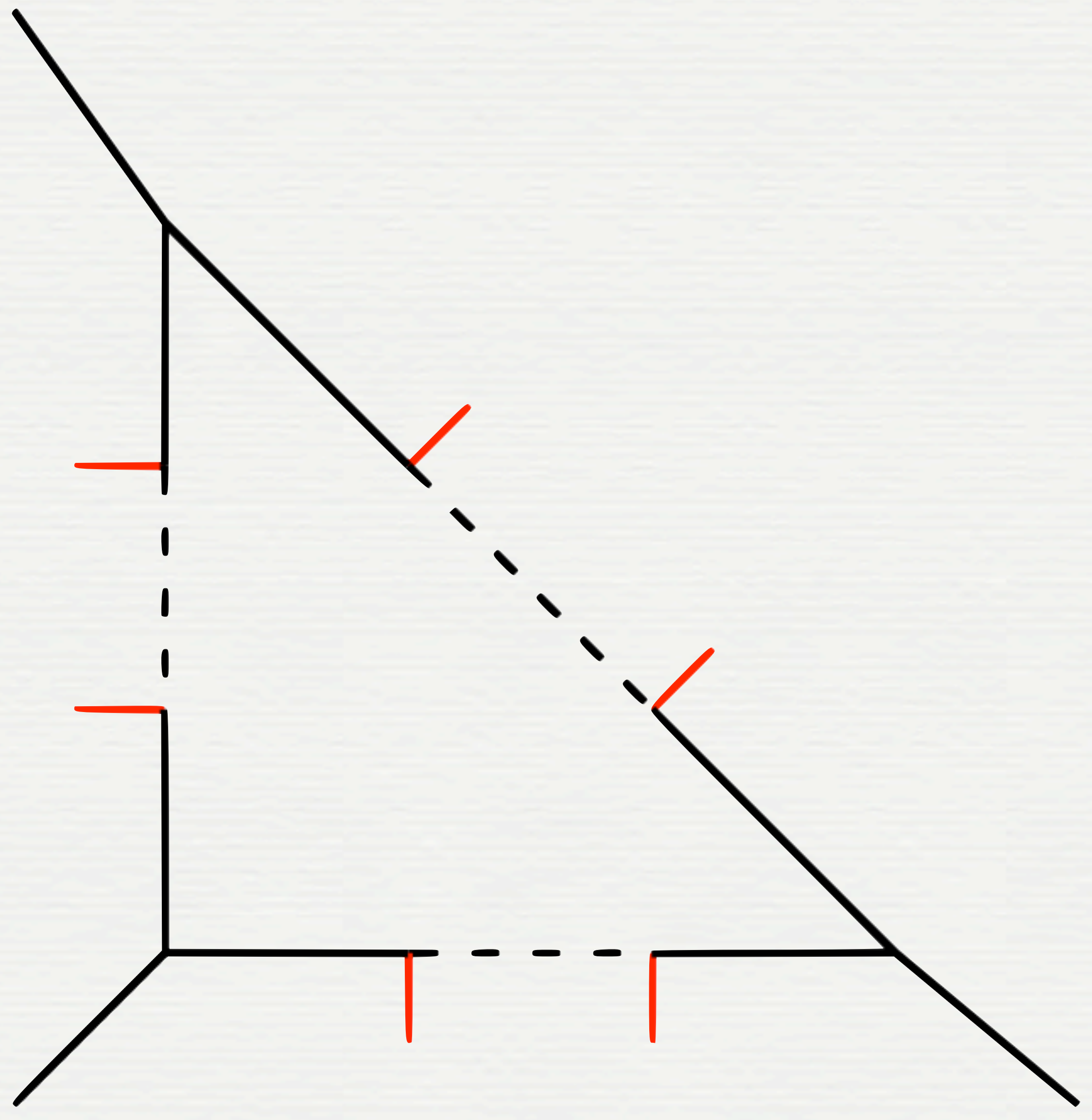
$$A_1^{(1)}(z) = \frac{\alpha_0^4 + (\alpha_1^2 - \alpha_2^2)^2 - 2\alpha_0^2(\alpha_1^2 + \alpha_2^2)}{32\alpha_1^5} z^2 + \\ + \frac{(3\alpha_1^2 + 5(\alpha_0^2 - \alpha_2^2)^2)(\alpha_0^4 + (\alpha_1^2 - \alpha_2^2)^2 - 2\alpha_0^2(\alpha_1^2 + \alpha_2^2)}{96\alpha_1^7} z^3 + \cdots$$

After plugging in these results in the proposed way we found complete agreement with our computations based on AGGTV

REFINED TOPOLOGICAL VERTEX
&
A-MODEL COMPUTATION

TOPOLOGICAL VERTEX

- Divide the toric diagram into trivalent vertices
- Compute the amplitude of each vertex
- Glue the amplitudes with appropriate propagators to obtain the full amplitude



$$Z(V_1, V_2, V_3) = \sum_{\lambda,\mu,\nu} C_{\lambda\mu\nu}\, \mathrm{tr}_\lambda V_1\, \mathrm{tr}_\mu V_2\, \mathrm{tr}_\nu V_3$$

$$V_i = \mathrm{P}\ \exp\left[\oint A\right]$$

I,AKMV

REFINED TOPOLOGICAL VERTEX

According to the Gopakumar&Vafa formulation of the topological string theory, the free energy can be written in following form

$$F = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L} (-1)^{2j_L} \boxed{N_\beta^{j_L}} e^{-kT_\beta} \left(\frac{q^{-2j_L k} + \ldots + q^{+2j_L k}}{k(q^{k/2} - q^{-k/2})^2} \right), \; q = e^{ig_s} \qquad \boxed{\epsilon_1 + \epsilon_2 = 0}$$

The *refinement* (Hollowood, Iqbal & Vafa) of this form is motivated by the LMSN partition function

$$F = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L, j_R} e^{-kT_\beta} \frac{(-1)^{2j_L + 2j_R} \boxed{N_\beta^{(j_L, j_R)}} \left((t\,q)^{-kj_L} + \ldots + (t\,q)^{+kj_L} \right) \left(\left(\frac{t}{q}\right)^{-kj_R} + \ldots + \left(\frac{t}{q}\right)^{+kj_R} \right)}{k(q^{k/2} - q^{-k/2})(t^{k/2} - t^{-k/2})}$$

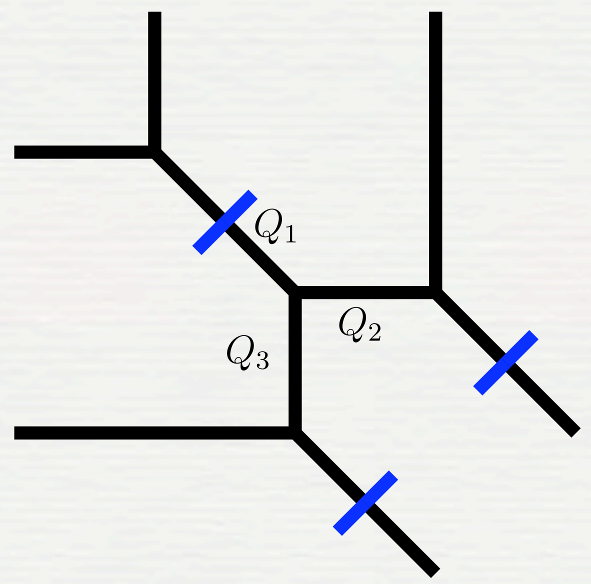
$$\boxed{q = e^{i\epsilon_1}, \text{ and } t = e^{-i\epsilon_2}}$$

with $N_\beta^{(j_L, j_R)}$ denoting the degeneracy of particles of spin $(j_L, j_R) \in SU(2)_L \times SU(2)_R \simeq SO(4)$ coming from a specific curve β in *M*-theory compactification down to 5d.

CHIRAL TODA 3-POINT FUNCTIONS

Benini, Benvenuti & Tachikawa proposed that a certain N-junction (web-diagram of N NS5,D5 and (1-1) 5-branes) in type IIB describe the 5d version of T_N. The dual diagrams turn out to be toric diagrams of certain Calabi-Yau 3folds.

N=2



The refined topological string partition function is (up to the refined MacMahon factor)

$$Z=\prod_{i,j=1}^{\infty}\frac{(1-Q_1Q_2\,q^{\rho_i+\frac{1}{2}}t^{-\rho_j-\frac{1}{2}})(1-Q_1Q_3\,q^{\rho_i+\frac{1}{2}}t^{-\rho_j-\frac{1}{2}})(1-Q_2Q_3\,q^{\rho_i-\frac{1}{2}}t^{-\rho_j+\frac{1}{2}})}{(1-Q_1\,q^{\rho_i}t^{-\rho_j})(1-Q_2\,q^{\rho_i}t^{-\rho_j})(1-Q_3\,q^{\rho_i}t^{-\rho_j})(1-Q_1Q_2Q_3\,q^{\rho_i}t^{-\rho_j})}$$

This expression is the q-deformed version of the Liouville theory chiral 3-point function

Choosing $Q=e^{-2Rm}$, $q^{\rho_i}=e^{2R(i-\frac{1}{2})\epsilon_1}$ and $t^{\rho_j}=e^{-2R(j-\frac{1}{2})\epsilon_2}$, and taking the limit $R\to 0$, the partition function can be written in terms of Barnes double gamma function

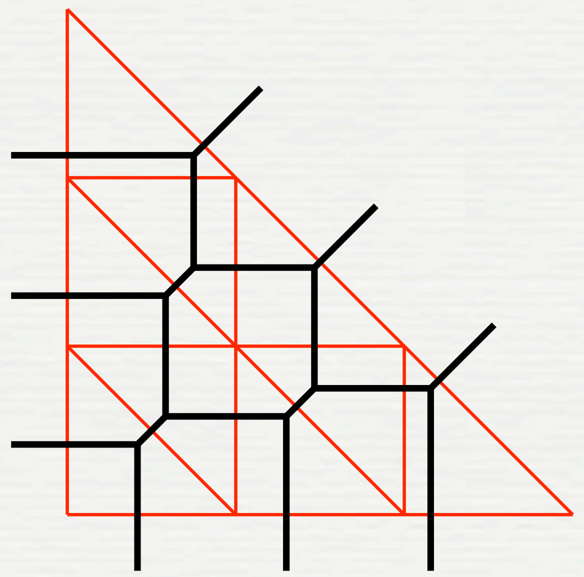
$$\frac{\Gamma_2(-\alpha_1+\alpha_2+\alpha_3)\Gamma_2(\alpha_1-\alpha_2+\alpha_3)\Gamma_2(\alpha_1+\alpha_2-\alpha_3)\Gamma_2(\alpha_1+\alpha_2+\alpha_3-\mathcal{Q})}{\Gamma_2(\mathcal{Q}/2)\Gamma_2(2\alpha_1)\Gamma_2(2\alpha_2)\Gamma_2(2\alpha_3)}$$

Chiral 3-point function of the Liouville theory

$$\Gamma_2(x)\equiv\Gamma_2(x|\epsilon_1,\epsilon_2)\propto\prod_{i,j=0}^{\infty}(x+i\,\epsilon_1+j\,\epsilon_2)^{-1}$$

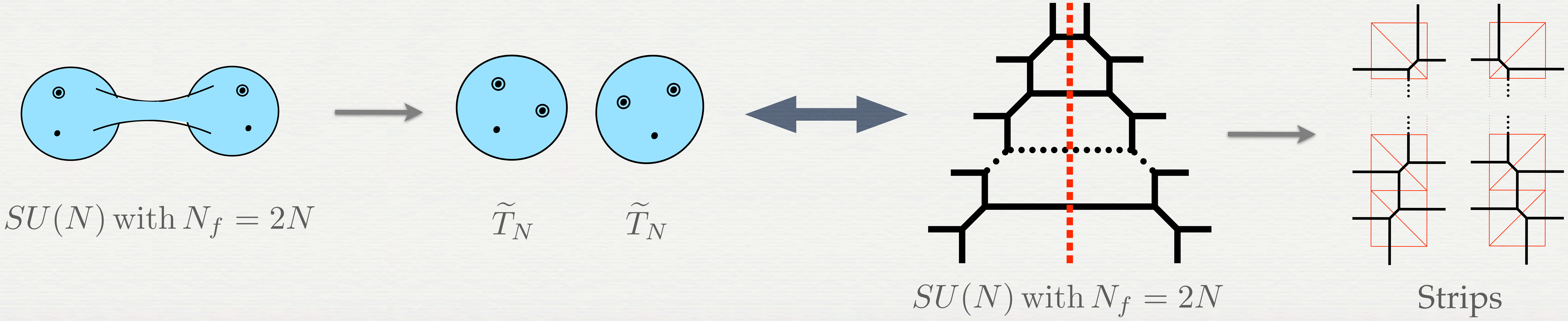
CHIRAL TODA 3-POINT FUNCTIONS

Although this approach can be continued to higher rank, the computations are not only technically more involved but also the general 3-point functions of higher rank Toda theories are not known to compare with and 4d limit is very subtle.



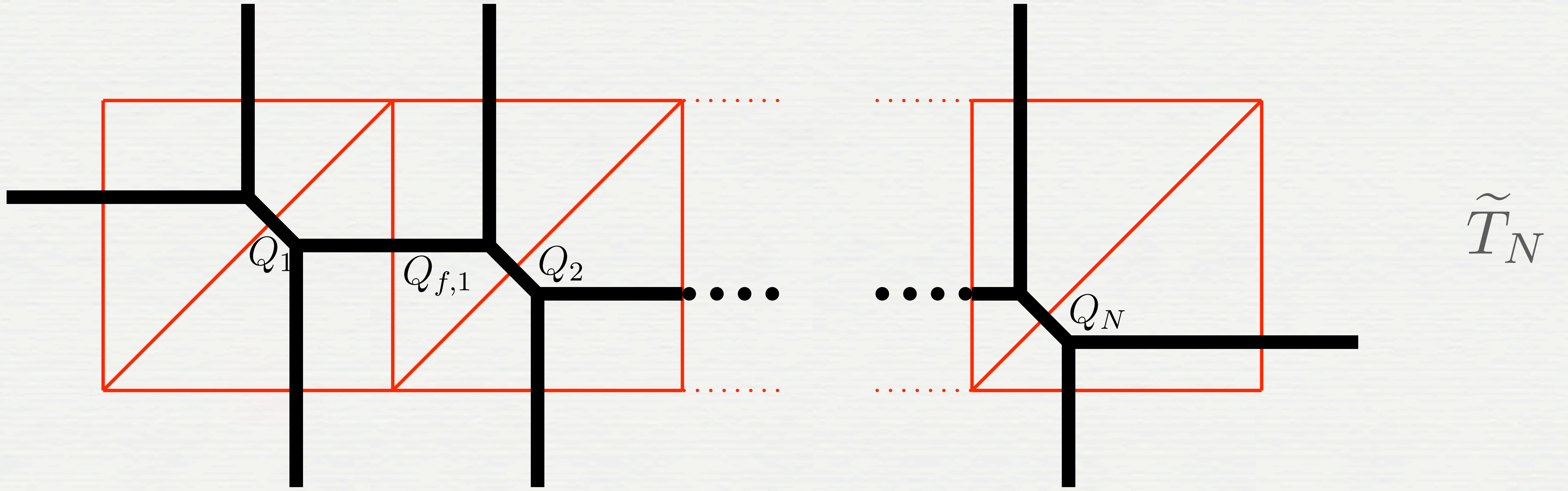
T_3

However, to get more insight we can consult Gaiotto's construction



CHIRAL TODA 3-POINT FUNCTIONS

We can compute the topological string partition function explicitly using the refined topological vertex



Let us focus on N=2 and compare with T_2, at first they do not agree

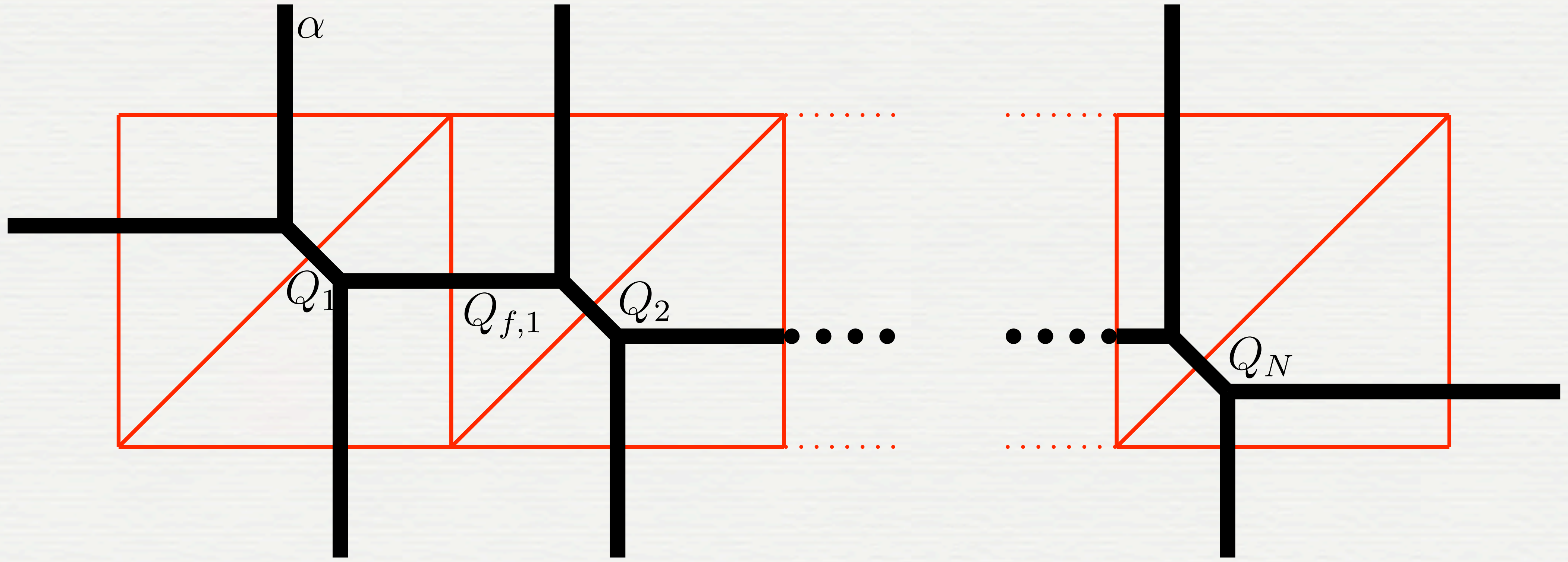
$$Z'_{\widetilde{\mathrm{T}}_2} = \prod_{i,j=1}^{\infty} \frac{(1-Q_1\, q^{-\rho_i} t^{-\rho_j})(1-Q_f\, q^{-\rho_i} t^{-\rho_j})(1-Q_2\, q^{-\rho_i} t^{-\rho_j})(1-Q_1 Q_f Q_2\, q^{-\rho_i} t^{-\rho_j})}{(1-Q_1 Q_f\, q^{-\rho_i+1/2} t^{-\rho_j-1/2})(1-Q_f Q_2\, q^{-\rho_i-1/2} t^{-\rho_j+1/2})} \neq Z_{\mathrm{T}_2}$$

However, we are allowed to rescale each vertex operators by an arbitrary function of their momenta and AGT is not sensitive to this rescaling, *i.e.*, $\widetilde{T}_2$ *strip* also agrees with the chiral 3-point function.

This geometry allows to compute open topological string amplitudes in the presence of toric branes

TORIC BRANES ON THE STRIP

We can insert a single toric brane on one of the external legs of the strip and label it with partition of a single column



$\widetilde{T}_N$ with a brane

For N=2, in the unrefined case we recover (in the 4d limit) the conformal block with three primary and one degenerate operator insertions

$$Z_{\text{open}}(z) = \sum_{n=0}^{\infty} \underbrace{z^n Z_{(n)}(Q_1, Q_2, Q_f, q)} \quad \text{with} \quad Z_{(n)}(Q_1, Q_2, Q_f, q) = \prod_{k=1}^{n} \underbrace{\frac{(1 - Q_1\, q^k)(1 - Q_1 Q_2 Q_f\, q^k)}{(1 - q^k)(1 - Q_1 Q_f\, q^k)}}$$

In the 4d limit, this gives the hypergeometric function ${}_2F_1$

In the 4d limit, this gives 3 Pochhammer symbols

This computation is generalized to the refined case with multiple surface operator insertions as previously anticipated by Gukov

CONFORMAL FIELD THEORY APPROACH

CFT APPROACH

The instanton counting needs to be extended to include surface operators for a full fledged gauge theory understanding of AGGTV. We still can gain some insight using AGT & AGGTV together from CFT's.

In the perturbative approach, for *SU(2)* with four hypermultiplets, AGGTV tells us to look at

$$\langle\alpha_1|V_{\alpha_2}|\sigma\rangle\langle\sigma|V_{\alpha_3}|\alpha_4+b/2\rangle\langle\alpha_4+b/2|V_{-b/2}|\alpha_4\rangle = \boxed{\langle\alpha_1|V_{\alpha_2}|\sigma\rangle\langle\sigma|V_{\alpha}|\tilde{\sigma}\rangle\langle\tilde{\sigma}|V_{\alpha_3}|\alpha_4\rangle}\Big|_{\boxed{\tilde{\sigma}=\alpha_4+b/2,\alpha_3=-b/2}}$$

Through AGT this 5-point function is related to $SU(2)\times SU(2)$ quiver gauge theory

These restrictions can be translated into gauge theory

This can be generalized immediately to higher rank, *i.e.*, $SU(N)\times SU(N)$

$$\sum_{\ell=0}^{\infty}\sum_{\vec{Y}} \underbrace{\boxed{y^{|\vec{Y}|}}}_{4\mathrm{d}}\,\underbrace{\boxed{z^{\ell}}}_{2\mathrm{d}} \prod_{n,m=1}^{2}\prod_{s\in Y_n}\frac{[E(\hat{a}^1_n,Y_n,W,s)-m_4-\epsilon]\prod_{f=1}^{3}P(\hat{a}^1_n,Y_n,s,m_f)}{E(\hat{a}^1_n-\hat{a}^1_m,Y_n,Y_m,s)[E(\hat{a}^1_n-\hat{a}^1_m,Y_n,Y_m,s)-\epsilon]}$$

$$\times\prod_{t\in W}\frac{[-E(-\hat{a}^1_m,W,Y_m,t)-m_4]}{[m_4-m_3+\epsilon+\epsilon_2\,(p-1)][\epsilon_2\,p]}$$

CFT APPROACH

We can focus on the terms with $\ell = 0$ then we obtain the usual instanton expansion of *SU(N)* theory.

We can also focus on the terms with $|\vec{Y}| = 0$ then we reproduce *'2d instanton part'*

$$Z_{\text{2d inst}} = \sum_{\ell=0}^{\infty} \frac{(A_1)_\ell (A_2)_\ell}{(B_1)_\ell} \frac{z^\ell}{\ell!}$$

Note the agreement with the topological vertex computation

Dimofte, Gukov & Hollands and Taki proposed a bubbling description supporting this CFT approach

CONCLUSION

- The B-model topological recursion can be used to compute CFT correlation function with degenerate insertions
- The A-model computations are mirror of the B-model, and the (refined) topological vertex can be used
- The Conformal theory approach is based on AGT and AGGTV conjecture and a new approach to *'perform'* gauge theory computations

THANK YOU