Resolution of the Ghost Problem in non-Linear Massive Gravity

Fawad Hassan

Stockholm University, Sweden

SFH, R. A. Rosen, arXiv:1103.6055
SFH, R. A. Rosen, arXiv:1106.3344

June 24, 2011

Outline of the talk

Motivation for massive gravity

Linear theory: Fierz-Pauli mass

Construction of higher order terms

Potentially ghost free non-linear actions

Absence of ghosts in massive gravity

Discussions

Outline of the talk

Motivation for massive gravity

Linear theory: Fierz-Pauli mass

Construction of higher order terms

Potentially ghost free non-linear actions

Absence of ghosts in massive gravity

Discussions

Motivation for massive gravity

- **Observational:**
- **Theoretical:**
- **Relation to String Theory:** ?,

Motivation for massive gravity

- **Observational:**

  Small graviton mass $\Rightarrow$ IR modifications of GR

  Affects dark energy

  Ameliorates the Cosmological Constant Problem

- **Theoretical:**

- **Relation to String Theory:** ?,

Motivation for massive gravity

- **Observational:**

  Small graviton mass $\Rightarrow$ IR modifications of GR

  Affects dark energy

  Ameliorates the Cosmological Constant Problem

- **Theoretical:**

  Massive spin-2 in Poincare / (A)dS representations

  Linear theory: Fierz-Pauli (1939)

  Challenge: ghost free non-linear extension

  [Boulware-Deser ghost (1972)]

- **Relation to String Theory:** ?,

Motivation for massive gravity

- **Observational:**

  Small graviton mass $\Rightarrow$ IR modifications of GR

  Affects dark energy

  Ameliorates the Cosmological Constant Problem

- **Theoretical:**

  Massive spin-2 in Poincare / (A)dS representations

  Linear theory: Fierz-Pauli (1939)

  Challenge: ghost free non-linear extension

  [Boulware-Deser ghost (1972)]

- **Relation to String Theory:** ?,
  (However see talk by C. Bachas)

Outline of the talk

Motivation for massive gravity

Linear theory: Fierz-Pauli mass

Construction of higher order terms

Potentially ghost free non-linear actions

Absence of ghosts in massive gravity

Discussions

Linear Theory: Fierz-Pauli Mass (1939)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\mathcal{L}_{FP} = m^2 M_p^2 \left(h_{\mu\nu} h^{\mu\nu} - a\, h^\mu_\mu h^\nu_\nu\right)$$

Linear Theory: Fierz-Pauli Mass (1939)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\mathcal{L}_{FP} = m^2 M_p^2 \left(h_{\mu\nu} h^{\mu\nu} - a \, h^\mu_\mu h^\nu_\nu \right)$$

- $a = 1$: **5 propagating modes (massive spin-2)**
- $a \neq 1$: (above) + 1 ghost $[m_{ghost} \sim m^2/(a-1)]$

Linear Theory: Fierz-Pauli Mass (1939)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\mathcal{L}_{FP} = m^2 M_p^2 \left(h_{\mu\nu} h^{\mu\nu} - a\, h^\mu_\mu h^\nu_\nu \right)$$

- $a = 1$: **5 propagating modes (massive spin-2)**
- $a \neq 1$: (above) + 1 ghost $[m_{ghost} \sim m^2/(a-1)]$
- **Even for $a = 1$, the ghost generically returns at non-linear level/higher orders in $h_{\mu\nu}$**
  (Boulware, Deser 1972)

Questions

- **What is a reasonable non-linear action ?**

- **Is it ghost free at the non-linear level ?**

Questions

- **What is a reasonable non-linear action ?**

  (serious proposals: 2010)

- **Is it ghost free at the non-linear level ?**

  (answer: more recent)

Ghost as a Stuckelberg mode

Restoring covariance of $\mathcal{L}_{FP}$ by Stuckelberg trick:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu$$

$$\pi_\mu = \partial_\mu \phi + \pi^t_\mu, \qquad\qquad (\partial^\mu \pi^t_\mu = 0)$$

Helps see the ghost for $a \neq 1$:

$$\mathcal{L}_{FP} \to \mathcal{L}_{FP} + F_{\mu\nu} F^{\mu\nu} + (a-1)\ \Box\phi\Box\phi$$

Ghost as a Stuckelberg mode

Restoring covariance of $\mathcal{L}_{FP}$ by Stuckelberg trick:

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu$$

$$\pi_\mu = \partial_\mu \phi + \pi^t_\mu, \qquad\qquad (\partial^\mu \pi^t_\mu = 0)$$

Helps see the ghost for $a \neq 1$:

$$\mathcal{L}_{FP} \to \mathcal{L}_{FP} + F_{\mu\nu}F^{\mu\nu} + (a-1)\ \Box\phi\Box\phi$$

Even at higher orders in $h_{\mu\nu}$ the ghost can be inferred from ϕ-terms with higher time derivatives

(*Arkani-Hamed etal, 2003*)

Outline of the talk

Motivation for massive gravity

Linear theory: Fierz-Pauli mass

Construction of higher order terms

Potentially ghost free non-linear actions

Absence of ghosts in massive gravity

Discussions

Construction of higher order terms

Procedure:

- Start with most general expansion in powers of $h_{\mu\nu}$
- Restore covariance by Stuckelberg trick,

$$h^{\mu}_{\nu} \quad \to \quad h^{\mu}_{\nu} + \partial^{\mu}\pi_{\nu} + \partial_{\nu}\pi^{\mu} \\ + \partial^{\mu}\pi_{a}\partial_{\nu}\pi^{a} + h^{\mu}_{\rho}\left(\partial_{\nu}\pi^{\rho} + \partial^{\rho}\pi_{\nu} + \partial^{\rho}\pi_{a}\partial_{\nu}\pi^{a}\right)$$

- Fix coefficients to make ϕ healthy

Two parameter family of potentially ghost free mass terms to quintic order in h

(C. de Rham and G. Gabadadze, 2010)

Ghost free to first order in h and all orders in ϕ

Outline of the talk

Motivation for massive gravity

Linear theory: Fierz-Pauli mass

Construction of higher order terms

Potentially ghost free non-linear actions

Absence of ghosts in massive gravity

Discussions

Potentially ghost free non-linear action

The potentially ghost free expansions of the mass term in powers of h can be resummed into complete non-linear actions

(de Rham, Gabadadze, Tolley, 2010)

Features:

- Ghost free to first order in h and all orders in ϕ
- 2-parameter family
- Contains the square-root matrix $\sqrt{g^{-1}\eta}$

Systematics of non-Linear Massive actions

Generalites: (SFH, R.A. Rosen, 2011)

- $\det g$, $\mathrm{Tr}\, g = 4$ not enough for a covariant mass
- Need an extra "metric" $f_{\mu\nu}$ $[m^2 F(g^{-1} f)]$

Systematics of non-Linear Massive actions

Generalites: (SFH, R.A. Rosen, 2011)

- $\det g$, $\operatorname{Tr} g = 4$ not enough for a covariant mass
- Need an extra “metric” $f_{\mu\nu}$ $[m^2 F(g^{-1} f)]$

  How to choose f ?

  - *flat (FP; de Rham, Gabadadze)*
  - *(A)dS metric (see talk by C. Bachas)*
  - *dynamical (bigravity, strong gravity)*
- Perturbative ghost analysis suggests $\sqrt{g^{-1} f}$

The minimal massive gravity action

The minimal massive gravity action:

$$S_{min} = -M_p^2 \int d^4x \sqrt{-g} \left[R + 2m^2 \operatorname{Tr} \sqrt{g^{-1}f} + \Lambda' \right]$$

- General action: 2-parameter deformation of S_{min}
- Very useful for ghost analysis

The general massive gravity action

"Deformed determinant":

$$\widehat{\det}(\mathbb{1} + \mathbb{K}) = \sum_{n=0}^{4} \frac{-\alpha_n}{\#} \epsilon_{\mu_1 \cdots \mu_n \lambda_{n+1} \cdots \lambda_4} \epsilon^{\nu_1 \cdots \nu_n \lambda_{n+1} \cdots \lambda_4} \mathbb{K}^{\mu_1}{}_{\nu_1} \cdots \mathbb{K}^{\mu_n}{}_{\nu_n}$$

General action:

$$\begin{aligned} S &= -M_p^2 \int d^4x \sqrt{-g} \left[R - 2m^2 \, \widehat{\det}(\sqrt{g^{-1} f}) \right] \\ &= -M_p^2 \int d^4x \sqrt{-g} \left[R - 2m^2 \sum_{n=0}^{3} \beta_n \, e_n(\sqrt{g^{-1} f}) \right] \end{aligned}$$

e_n: Elementary symmetric polynomials of eigenvalues

Outline of the talk

Motivation for massive gravity

Linear theory: Fierz-Pauli mass

Construction of higher order terms

Potentially ghost free non-linear actions

Absence of ghosts in massive gravity

Discussions

Absence of ghosts

Review: GR propagating modes in ADM formulation

$$g^{\mu\nu} = \begin{pmatrix} N^{-2} & N^j/N^2 \\ N^i/N^2 & \gamma^{ij} - N^i N^j/N^2 \end{pmatrix}$$

- 6 propagating modes: γ_{ij} $\qquad (\gamma_{ij}, \pi^{ij})$
- 4 non-propagating modes: N, N_i

First order action:

$$\pi^{ij}\partial_t \gamma_{ij} + N R^0 + N^i R_i$$

N^μ equations of motion: $\qquad R_\mu(\gamma, \pi) = 0$
$+$ symmetries $\Rightarrow$

$$6 \to 2 \text{ (massless graviton)}$$

Ghost in generic massive gravity

First order action:

$$\pi^{ij}\partial_t\gamma_{ij} + N R^0 + N^i R_i - m^2 V(\gamma, N, N^i)$$

N^μ equations of motion: $\quad R_\mu(\gamma,\pi) = m^2 V_\mu(\gamma, N, N^i)$

$+$ no symmetries $\Rightarrow$

$$6 \to 5 + 1 \text{ (massive spin-2 + ghost)}$$

(Boulware, Deser, 1972)

Can the $\widehat{det}$ actions avoid this problem?

Absence of ghost in the **minimal** massive gravity

For $f_{\mu\nu} = \eta_{\mu\nu}$, $\Lambda = 0$,

$$\mathcal{L}_{min} = \pi^{ij}\partial_t\gamma_{ij} + N R^0 + N^i R_i - 2m^2\sqrt{\gamma}\, N \left(\mathrm{Tr}\sqrt{g^{-1}\eta} - 3\right)$$

We show: of the 4 N^μ eqns. one combination is independent of the N^μ,

$$R^0(\gamma,\pi) = m^2\widetilde{V}_\mu(\gamma,\pi)$$

Hamiltonian constraint + a secondary constraint $\Rightarrow$

$6 \to 5$ **(massive spin-2 + no ghost!)**

The remaining 3 eqns determine $N^i(N,\gamma,\pi)$ and N.

Details

New Variables: $N^i = (\delta^i_j + N\, D^i{}_j) n^j$
Requirement: $N\sqrt{g^{-1}\eta} = \mathbb{A} + N\mathbb{B}$
D Matrix: $(\sqrt{1 - n^T \mathbf{I}\, \mathbf{n}})\, D = \sqrt{(\gamma^{-1} - Dnn^T D^T)\mathbf{I}}$

$$\begin{aligned} \mathcal{L}_{min} &= \pi^{ij}\partial_t\gamma_{ij} + NR^0 + R_i(\delta^i_j + ND^i{}_j)n^j \\ &\quad -2m^2\sqrt{\gamma}\left[\sqrt{1 - n^T\mathbf{In}} + N\,\mathrm{Tr}(\sqrt{\gamma^{-1}\mathbf{I} - \mathbf{Dnn^TD^TI}}) - 3N\right] \end{aligned}$$

Momentum constraints:

$$(\sqrt{1 - n^r\delta_{rs}n^s})\, R_i + 2m^2\sqrt{\gamma}\, n^l\delta_{li} = 0$$

Solution: $n^i = -R_j\delta^{ji}\left[m^4\sqrt{\gamma} + R_k\delta^{kl}R_l\right]^{-1/2}$

Outline of the talk

Discussions

- The minimal action $e_1 \sim (\sqrt{g^{-1}\eta})$ is ghost free
- Works for the next term $e_2 \sim (\sqrt{g^{-1}\eta})^2$,

$$e_2(\sqrt{g^{-1}\eta}) = \frac{1}{2}\left[(\mathrm{Tr}\sqrt{g^{-1}\eta})^2 - \mathrm{Tr}\, g^{-1}\eta\right] .$$

- For the last term, $e_3 \sim (\sqrt{g^{-1}\eta})^3$ the outcome is not clear
- Positivity of Hamiltonian
- More general $f_{\mu\nu}$: FRW, dynamical
- Classical solutions