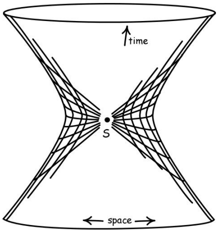
$\mathcal{I}^+$ and the Static Patch

Dionysios Anninos

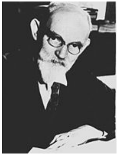
Milos, Greece, June 2011



Contents

de Sitter Space

Maximally symmetric cosmology supported by positive cosmological constant. Possibly two-de Sitter eras in our Universe: (i) Inflation, (ii) Future?



Poses an interesting theoretical problem of how holography (or quantum gravity) is defined (if it is at all!) in cosmological spacetimes.

Basic confusion is what are the correct OBSERVABLES? e.g. In flat space we have the S-matrix, in AdS we have the boundary correlation functions. In de Sitter we have causally inaccessible regions of space and no spatial infinity! Also, we must re-evaluate the tools to examine the problem, e.g. no SUSY.

de Sitter Geometry - A tale of two Observers

de Sitter space appears as a solution to Einstein gravity with a positive cosmological constant $\Lambda = +3/\ell^2$. Its metric in the GLOBAL PATCH is

$$\frac{ds^2}{\ell^2} = -d\tau^2 + \cosh^2\tau d\Omega_d^2$$

No single observer has access to the global patch (only the 'metaobserver'). The conformal boundary is SPACELIKE, and lives at $\mathcal{I}^\pm$.

The spacetime accessible to a single observer is given by the STATIC PATCH

$$\frac{ds^2}{\ell^2} = -dt^2(1-r^2) + \frac{dr^2}{(1-r^2)} + r^2 d\Omega_{d-1}^2$$

We notice the appearance of a COSMOLOGICAL HORIZON at $r = 1$.

Hawking Temperature, Entropy

Thus, the observers are immersed in a thermal bath at temperature

$$T_{dS} = \frac{1}{2\pi\ell}$$

and there is a Bekenstein-Gibbons-Hawking entropy associated to the static patch given by

$$S_{dS} = \frac{Area}{4G}$$

One can ask, what precisely this universal formula is counting? (i.e. What is static patch holography?[Banks;Susskind;Bousso,Maloney,Strominger])

When the number of spacetime dimensions exceeds THREE, one can have asymptotically de Sitter black holes, whose entropy is always LESS than the entropy of pure de Sitter.

$\mathcal{I}^+$ IN FOUR-DIMENSIONS

Asymptotic Expansion

A very general class of excited asymptotically de Sitter solutions can be put in the Fefferman-Graham expansion [Starobinsky]:

$$\frac{ds^2}{\ell^2} = -\frac{d\eta^2}{\eta^2} + \frac{dx^i dx^j}{\eta^2}\left(g^{(0)}_{ij} + \eta^2 g^{(2)}_{ij} + \dots\right) + \frac{dx^i dx^j}{\eta^2}\left(\eta^3 g^{(3)}_{ij} + \dots\right)$$

where $\mathcal{I}^+$ lives at $\eta \to 0$. Note that $\mathrm{Tr} g^{(3)}_{ij} = \nabla^i g^{(3)}_{ij} = 0$. The $g^{(k)}_{ij}$ are functions of $g^{(0)}$ and $g^{(3)}$ for $k > 3$. These solutions obey the COSMIC NO HAIR THEOREM.

Variations of initial data for the graviton lead to variations of $g^{(0)}$ and $g^{(3)}$ w.r.t. some background metric $\bar{g}$.

FG Preserving Diffeomorphisms

The Fefferman-Graham expansion is fully specified by the boundary data $(g^{(0)}, g^{(3)})$ and has a conformal structure. It is defined up to conformal transformations of $g^{(0)}$. The conformal weights of $g^{(0)}_{ij}$ and $g^{(3)}_{ij}$ are $s = 2$ and $s = -1$ respectively.

Furthermore, the Fefferman-Graham preserving diffeororphisms are given by:

$$\begin{aligned} \xi^\eta &= \eta \delta\sigma(\vec{x}) \\ \xi^i &= \phi^i(\vec{x}) + \frac{\eta^2}{2} g^{(0)ij} \partial_j \delta\sigma(\vec{x}) + \dots \end{aligned}$$

where the purely $\delta\sigma$ diffeomorphisms give rise to scale transformations and the ϕ^i diffeomorphisms are tangent to $\mathcal{I}^+$.

Boundary Conditions

We propose that the appropriate boundary conditions for dS_4^+ are all spacetimes obeying the Fefferman-Graham expansion [D.A.,Ng,Strominger].

These boundary conditions differ from those of AdS_4 which freeze $g^{(0)}$, i.e. Dirichlet boundary conditions. In AdS_4 we are allowed to do this because $g^{(0)}$ lives at spacelike infinity where we can freely impose boundary conditions.

In a sense, our picture is more reminiscent of the asymptotically flat case where gravitational radiation can also leak through the boundary...

ASG

The asymptotic symmetry group is defined as:

$$\text{ASG} \equiv \frac{\text{allowed diffeomorphisms}}{\text{trivial diffeomorphisms}} \ .$$

For dS_4^+, the geometric structure of $\mathcal{I}^+$ given our boundary conditions leads to:

$$\xi_{ASG} = \phi^i(x^i)\partial_i \ .$$

i.e. diffeomorphisms tangent to $\mathcal{I}^+$. We would like to render diffeomorphisms corresponding to conformal transformations of the boundary metric TRIVIAL, given that they are maps between members of the same conformal class.

The infinite dimensional group is closer in spirit to the BMS group of asymptotically flat space in four-dimensions than the finite dimensional group of AdS_4.

Charges

Employing the techniques developed by [Compere,Marolf,Iyer,Wald,Zoupas] we have managed to define a set of finite and integrable charges associated with each asymptotic symmetry.

The on-shell charges are found to be:

$$Q_\xi = \int_{\partial\Sigma} d^2x \sqrt{\sigma} n^i \xi^j T^{BY}_{ij} = Q_{BY}, \quad \xi^i \in \xi^i_{ASG}.$$

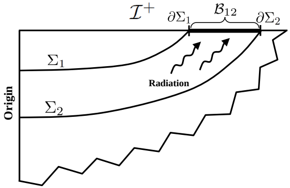
These are the BROWN-YORK charges defined on a 2d submanifold $\partial\Sigma$ of $\mathcal{I}^+$ for the regularized Brown-York stress tensor $T^{BY}_{ij} = -3g^{(3)}_{ij}/16\pi\ell G$ satisfying $\mathrm{Tr}\, T^{BY}_{ij} = \nabla^i T^{BY}_{ij} = 0$.

Conservation Equation

The charges obey a CONSERVATION EQUATION:

$$Q_\xi[\partial\Sigma_2] - Q_\xi[\partial\Sigma_1] = \frac{1}{2}\int_{\mathcal{B}_{12}} d^3x\sqrt{g^{(0)}}\,T^{ij}\mathcal{L}_\xi g_{ij}^{(0)}.$$

This is reminiscent of the flux going through $\mathcal{I}^+$ in flat space where T^{ij} would be the Bondi 'news' tensor, indicating the presence of gravitational radiation.



Dirichlet Boundary Conditions?

We could impose more stringent boundary conditions [D.A.,Ng,Strominger], which freeze $g^{(0)}$, in de Sitter space by introducing a ‘de Sitter demon’. Such a demon sends modes from an causally disconnected region of space to precisely cancel the variations on the metric on $\mathcal{I}^+$ caused by arbitrary signals sent by a physical observer.

The above procedure has been explicitly performed in the linearized gravity approximation. Interestingly, with such Dirichlet boundary conditions the de Sitter invariant two-point function becomes an analytic continuation of the anti-de Sitter one with $\ell_{AdS} \to i\ell_{dS}$.

In this case the symmetries at $\mathcal{I}^+$ become simply $SO(4,1)$, i.e. the Euclidean conformal group in three-dimensions. The geometry up to $\mathcal{I}^+$ (with fixed $g^{(0)}$) is fully fixed by data in a single static patch.

THE ROTATING NARIAI OBSERVER

Rotating Nariai Universe

Gravity with $\Lambda > 0$ admits a class of non-asymptotically de Sitter solutions whose vacuum is known as the rotating Nariai universe with metric [Nariai;Booth,Mann]:

$$ds^2 = \Gamma(\theta)\left(-r(\tau - r)dt^2 + \frac{dr^2}{r(\tau - r)}\right) + \gamma(\theta)\left(d\phi + krdt\right)^2 + \alpha(\theta)d\theta^2$$

This one-parameter family of solutions violates the cosmic no-hair theorem!

This is a one parameter family of cosmological solutions to Einstein gravity with $\Lambda > 0$ arising as the near (cosmological) horizon region of a rotating black hole in de Sitter, whose horizon tends to the cosmological horizon.

Geometry of Rotating Nariai

It is an S^2 fibration over dS_2.

Constant time slices of the global geometry have an $S^1 \times S^2$ topology.

There is an $SL(2,\mathbb{R}) \times U(1)$ four-dimensional isometry group.

The parameter τ is a near-extremality parameter which when non-zero allows for the black hole and cosmological horizons to be preserved.

At constant polar angle θ the geometry becomes that of warped de Sitter space, which is a solution to topologically massive gravity with positive cosmological constant.

Asymptotic Symmetries I

We would now like to explore the asymptotic symmetry group of the rotating Nariai geometry [D. A., Hartman].

We propose the following boundary conditions at $\mathcal{I}^+_{RN}$ (recall large r is a time coordinate):

$$\begin{aligned} h_{tt} &\sim r^2, \quad h_{\phi\phi} \sim h_{t\phi} \sim 1, \quad h_{t\theta} \sim h_{\phi\theta} \sim h_{\theta\theta} \sim h_{\phi r} \sim 1/r, \\ h_{tr} &\sim h_{\theta r} \sim 1/r^2, \quad h_{rr} \sim 1/r^3. \end{aligned}$$

The diffeomorphisms preserving the above boundary conditions are given by:

$$\zeta_n = e^{-in\phi} \left(-\partial_\phi + inr\partial_r \right), \quad \zeta_t = \partial_t .$$

such that $[\zeta_m, \zeta_n] = -i(m-n)\zeta_{m+n}$.

Asymptotic Symmetries II

One can find the asymptotic symmetry group of the rotating Nariai geometry, to be a single copy of the Virasoro algebra with REAL, POSITIVE central charge:

$$c_L = \frac{12 r_c^2 \sqrt{(1 - 3r_c^2/\ell^2)(1 + r_c^2/\ell^2)}}{-1 + 6r_c^2/\ell^2 + 3r_c^4/\ell^4}.$$

The ASG comes from an enhancement of the $U(1)$ isometries.

Notice further that the central charge vanishes when $r_c^2 = \ell^2/3$, this is the non-rotating Nariai geometry.

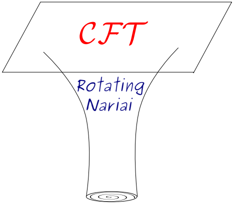
Rotating Nariai/CFT Proposal

One is thus led to propose that the rotating Nariai geometry is HOLOGRAPHICALLY DUAL to a two-dimensional conformal field theory.

Indeed, somewhat mysteriously, the cosmological entropy is given by the Cardy formula (with $T_R = 0$):

$$S_c = \frac{\pi^2}{3} T_L c_L .$$

T_L is the chemical potential conjugate to the angular momentum $\mathcal{Q}_{\partial_\phi}$.



Scalar Waves I - ANSATZ

One finds further evidence for the correspondence by studying free scalar perturbations in the rotating Nariai background [D. A.,T.Anous]:

$$\nabla^2\Phi = 0.$$

We choose the following ansatz: $\Phi(t, r, \phi, \theta) = e^{-i\omega t + im\phi} R(r) Y_{lm}(\theta)$.

Then the equations of motion SEPARATE and can be solved exactly in terms of HYPERGEOMETRIC FUNCTIONS.

The large r, i.e. late time, behavior goes as:

$$\Phi \sim r^{-h_\pm}, \quad h_\pm = \frac{1}{2} \pm \frac{i}{2}\sqrt{4m^2k^2 + 4j_{lm} - 1}.$$

Thus, conformal weights are COMPLEX.

Scalar Waves II - QUASINORMAL MODES

Demanding that the waves have ingoing flux at the black hole horizon and outgoing flux at the cosmological horizon, our spectrum becomes quantized and takes the form:

$$\begin{aligned} m &= -2\pi i T_L(n + h_L) , \quad n = 0, 1, 2, 3, \ldots \\ \omega &= -2\pi i T_R\,(n + h_R) , \quad n = 0, 1, 2, 3, \ldots \end{aligned}$$

where $T_R = \tau/4\pi$ is the temperature of the cosmological horizon in the rotating Nariai geometry and $h_L = h_R = h_\pm$.

These are retrieved by the poles of boundary correlators at $\mathcal{I}^+_{RN}$ and are equivalent to what one expects in a CFT$_2$.

Scalar Waves III - α-Vacua

As in pure de Sitter space, we can define a complex parameter worth of quantum vacua for the free scalar field known as the α-vacua.

Three interesting vacua are:

- the $|in\rangle$ vacuum (defined by positive frequency modes at $\mathcal{I}^-_{RN}$),
- the $|out\rangle$ vacuum (defined by positive frequency modes at $\mathcal{I}^+_{RN}$) and
- the $|E\rangle$ vacuum (defined by modes which are analytic in the lower hemisphere of the Eucledeanized dS_2).

Interestingly, the rotating Nariai geometry has cosmological particle creation for scalars in ALL dimensions. We find

$$\langle in|a^\dagger_{out} a_{out}|in\rangle = \cosh^2(\pi m k)\,\mathrm{csch}^2(\pi\mu/2)\,.$$

Brief word on warped dS$_3$ - I

A toy model in three dimensions is given by warped dS$_3$ [D. A., S. Detournay, S. de Buyl]. This is a vacuum of a theory known as topologically massive gravity with positive cosmological constant. The metric has an $SL(2,\mathbb{R}) \times U(1)$ isometry and is:

$$ds^2 = \frac{\ell^2}{(3-\nu^2)} \left[-\frac{dt^2}{(1+t^2)} + d\phi^2 (1+t^2) + \frac{4\nu^2}{(3-\nu^2)} (du + t d\phi)^2 \right] .$$

Interestingly, quotients of this geometry give rise to a smooth solution with a cosmological and internal horizon equivalent to the rotating Nariai geometry at fixed polar angle. These quotients are a two-parameter family where we denote the parameters by T_L and T_R. When $T_R = 0$, the two horizons coincide and we can take a near horizon limit, this is the Nariai limit!

Brief word on warped dS_3 - II

The entropy of the cosmological horizon is $S_c = \frac{\pi^2 \ell}{3}(T_L c_L + T_R c_R)$, where c_R is computed from the ASG and c_L can be obtained from the Nariai limit. We see that turning on T_R pushes out the cosmological horizon and increases its entropy.

Furthermore, the internal horizon has entropy $S_{BH} = \frac{\pi^2 \ell}{3}(T_L c_L - T_R c_R)$ hinting toward a level matching condition.

Interestingly, $c_R = 0$ for $\mu^2 \ell^2 = 27/5$ and theory may simplify at this special point, e.g. $S_{BH} = S_c$, $E_R = 0$ etc.

Conclusions and Challenges

- There is a rich structure residing at $\mathcal{I}^+$ of dS_4 which begs for an interpretation
- There may be a way in which we can physically motivate tighter boundary conditions, i.e. 'de Sitter demons'
- The rotating Nariai universe has a very rich symmetry group at $\mathcal{I}_{RN}$.
- What is the non-zero T_R extension of our result?
- Why does Cardy's formula give the right answer?
- More broadly, given recent progress in AdS/CFT the time may be ripe for a precise dS/CFT duality.