Supersymmetric Heterotic Backgrounds

George Papadopoulos

King's College London

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Work in collaboration with

Jan Gutowski and Ulf Gran

Motivation

- AdS/CFT
- CFT, (conformal deformations)
- String, M-theory compactifications with or without fluxes
- Black hole physics, (Mathur's proposal)
- M-theory , (supersymmetry algebra)
- Geometry; special geometric structures

There is a need for constructing increasingly general supersymmetric backgrounds.

Outline

- Solve the Killing spinor equations of the heterotic supergravity in all cases
- Give all $1/2$ and $1/4$ supersymmetric solutions of the heterotic supergravity
- Worldvolume theory and conformal deformations of $AdS_3 \times S^3$
- A uniqueness theorem for fuzzballs
- Report on the status of the classification in type II
- Puzzle with the M-theory supersymmetry algebra

Method

Originally the supersymmetric solutions 4-D Einstein-Maxwell supergravity have been found using twistor methods, [Tod].

The Killing spinor equations (KSEs) of simple 5-D supergravity have been solved using the Killing spinor form bi-linears, closely related to G-structures, [Gauntlett, Gutowski, Hull, Pakis, Reall].

Throughout, the Spinorial Geometry method to solving KSEs is used, [Gillard, Gran, GP].
This is based on the gauge symmetry of KSEs and a description of spinors in terms of forms.

Killing spinor equations

The Killing spinor equations of Heterotic supergravities are

$$\mathcal{D}\epsilon = \hat{\nabla}\epsilon = \nabla\epsilon + \frac{1}{2}H\epsilon + \mathcal{O}(\alpha') = 0\,, \quad \mathcal{F}\epsilon = F\epsilon + \mathcal{O}(\alpha') = 0\,,$$
$$\mathcal{A}\epsilon = d\Phi\epsilon - \frac{1}{2}H\epsilon + \mathcal{O}(\alpha') = 0$$

These are valid up to 2-loops in the sigma model calculation.
It is convenient to solve them in the order

gravitino $\rightarrow$ gaugino $\rightarrow$ dilatino

The gravitino and gaugino have a straightforward Lie algebra interpretation while the solution of the gaugino is more involved. All have been solved [Gran, Lohrmann, GP; hep-th/0510176], [Gran, Roest, Sloane, GP; hep-th/0703143].

Gravitino and dilatino

The gravitino Killing spinor equation is

$$\mathcal{D}\epsilon = \hat{\nabla}\epsilon = \nabla\epsilon + \frac{1}{2}H\epsilon = 0$$

where $\hat{\nabla}$ is a metric connection with skew-symmetric torsion H, and so for generic backgrounds

$$\mathrm{hol}(\hat{\nabla}) = G = Spin(9,1)$$

In addition

$$\hat{\nabla}\epsilon = 0 \Rightarrow \hat{R}\epsilon = 0$$

So either

$$\mathrm{Stab}(\epsilon) = \{1\} \Longrightarrow \hat{R} = 0$$

all spinors are parallel and M is parallelizable (group manifold if $dH = 0$) or

$$\mathrm{Stab}(\epsilon) \neq \{1\} \Longrightarrow \epsilon \text{ singlets}$$

$\mathrm{Stab}(\epsilon) \subset Spin(9,1)$ and $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{Stab}(\epsilon)$.

Solutions of both gravitino and dilatino KSEs can be summarized as follows:

L	$\mathrm{Stab}(\epsilon_1,\dots,\epsilon_L)$	N
1	$Spin(7)\ltimes\mathbb{R}^8$	$1(1)$
2	$SU(4)\ltimes\mathbb{R}^8$	$1(1),\ 2(1)$
3	$Sp(2)\ltimes\mathbb{R}^8$	$1(1),\ 2(1),\ 3(1)$
4	$(\times^2 SU(2))\ltimes\mathbb{R}^8$	$1(1),\ 2(1),\ 3(1),\ 4(1)$
5	$SU(2)\ltimes\mathbb{R}^8$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1)$
6	$U(1)\ltimes\mathbb{R}^8$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1)$
8	$\mathbb{R}^8$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1),\ 7(1),\ 8(1)$
2	G_2	$1(1),\ 2(1)$
4	$SU(3)$	$1(1),\ 2(2),\ 3(1),\ 4(1)$
8	$SU(2)$	$1(1),\ 2(2),\ 3(3),\ 4(6),\ 5(3),\ 6(2),\ 7(1),\ 8(1)$
16	$\{1\}$	$8(2),\ 10(1),\ 12(1),\ 14(1),\ 16(1)$

- L is the number of parallel spinors, ie solutions of the gravitino and N is the number of solutions of both gravitino and dilatino, so $N\le L$.
- The number in parenthesis denotes the different geometries for a given N.

- There are differences with the holonomy groups that appear in the Berger classification
- There are compact and non-compact isotropy groups which lead to geometries with different properties
- There is a restriction on the number of parallel spinors.
- The isotropy group of more than 8 spinors is $\{1\}$
- The conditions on the geometry of the spacetime which arise from the KSEs are known in all cases

L	$\mathrm{Stab}(\epsilon_1, \ldots, \epsilon_L)$	N
1	$Spin(7) \ltimes \mathbb{R}^8$	$1(1)$
2	$SU(4) \ltimes \mathbb{R}^8$	$1(1),\ 2(1)$
3	$Sp(2) \ltimes \mathbb{R}^8$	$1(1),\ 2(1),\ 3(1)$
4	$(\times^2 SU(2)) \ltimes \mathbb{R}^8$	$1(1),\ 2(1),\ 3(1),\ 4(1)$
5	$SU(2) \ltimes \mathbb{R}^8$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1)$
6	$U(1) \ltimes \mathbb{R}^8$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1)$
8	$\mathbb{R}^8$	$1(1),\ 2(1),\ 3(1),\ 4(1),\ 5(1),\ 6(1),\ 7(1),\ 8(1)$
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- If the isotropy group is the identity, the $1/2$ susy solutions are WZNW models.

$N = 8, SU(2)$

The solution of the conditions that arise from the Killing spinor equations [GP] give

$$\begin{aligned} ds^2 &= \eta_{ab}\lambda^a\lambda^b + h\, ds^2_{\rm hk}\ , \quad e^{2\Phi} = h \\ H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b - \star_{\rm hk}\tilde{d}h \end{aligned}$$

- The spacetime is a Principal bundle $M = P(G, B; \pi)$ equipped with local frame (λ^a, e^i), where λ^a is an anti-self dual instanton connection and

$$\mathcal{F}^a \equiv d\lambda^a - \frac{1}{2}H^a{}_{bc}\lambda^b\lambda^c = \frac{1}{2}H^a{}_{ij}e^i \wedge e^j$$

- The base space B is a hyper-Kähler 4-manifold
- $\mathfrak{Lie}G = \mathbb{R}^{6,1}\ ,\quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)\ ,\quad \mathfrak{cw}_6,$ self-dual, [Chamseddine,Figueroa, Sabra]
- Φ depends only on the coordinates of B

- Field equations require

$$\star_{\mathrm{hk}} dH \equiv -\tilde{\nabla}^2_{\mathrm{hk}} h - \frac{1}{2}\eta_{ab}\,\mathcal{F}^a_{ij}\;\mathcal{F}^{bij} = \frac{\alpha'}{8}\big(\mathrm{tr}\check{R}_{ij}\check{R}^{ij} - \mathrm{tr}F_{ij}F^{ij}\big) + \mathcal{O}(\alpha'^2)$$

where $\check{\nabla} = \nabla - \frac{1}{2}H$.

Solutions

Suppose that λ is flat, $\mathcal{F} = 0$, and $F = \check{R}$, so $dH = 0$. There are two classes of solutions.

$$G \times B_{\mathrm{hk}} \ ,$$

where

$$G = \mathbb{R}^{5,1} \ , \quad AdS_3 \times S^3 \ , \quad CW_6$$

These include vacua for compactifications to 6 and 3 dimensions.

The 5-brane solution with flat [Callan, Harvey, Strominger] or curved worldvolume and transverse space $B_{\rm hk}$. For $B_{\rm hk}=\mathbb{R}^4$

$$
\begin{aligned}
ds^2 &= ds^2(G)+h ds^2(\mathbb{R}^4)\,, \quad H=\frac{1}{3}\eta_{ab}\lambda^a\wedge d\lambda^b-\star_{\rm hk}\tilde{d}h\,,\\
e^{2\Phi} &= h\,, \quad h=1+\sum_\ell \frac{N_\ell}{|x-x_\ell|^2}
\end{aligned}
$$

At infinity is $G\times\mathbb{R}^4$, and $G\times S^3\times\mathbb{R}$ with linear dilaton near the position of the branes. The 5-brane charge at infinity is $p=\sum_\ell N_\ell$

New solutions

Again suppose that $F = \check{R}$, ie $dH = 0$. Take $G = AdS_3 \times S^3$ and λ a $SU(2) = S^3$ instanton on $B_{\rm hk} = \mathbb{R}^4$. Using t'Hooft's ansatz to set

$$C^r_i = (I_r)^j{}_i \partial_j \log(1 + \frac{\rho^2}{|x|^2}) \ , \quad r = 1, 2, 3$$

the smooth 1-instanton solution, $\lambda = dgg^{-1} - gCg^{-1}$, $g \in SU(2)$,

$$\begin{aligned} ds^2 &= ds^2(AdS_3) + \delta_{rs}\lambda^r\lambda^s + h ds^2(\mathbb{R}^4) \ , \\ H &= d\mathrm{vol}(AdS_3) + \frac{1}{3}\delta_{rs}\lambda^r \wedge d\lambda^s + \frac{2}{3}\delta_{rs}\lambda^r \wedge \mathcal{F}^s - \star_{\rm hk}\tilde{d}h \ , \\ e^{2\Phi} &= h \ , \quad h = 1 + 4\frac{|x|^2 + 2\rho^2}{(|x|^2 + \rho^2)^2} \end{aligned}$$

where ρ is the size of the instanton.

This solution $M = AdS_3 \times X_7$ is smooth with one $AdS_3 \times S^3 \times \mathbb{R}^4$ asymptotic region. There is no throat at $|x| = 0$, the location of the instanton. The dilaton is bounded everywhere on spacetime. Using the ADHM construction, solutions can be constructed that depend on $8\nu - 3$ continuous parameters.

Worldvolume theory

The worldvolume theory of the $N=8$, $SU(2)$ backgrounds can be described in terms of gauged 2-d sigma models with an anomalous gauging [Hull, Spence].

Minimal coupling is not sufficient to gauge the Wess-Zumino term W, $dW=0$. To begin, minimal coupling gives

$$W=\frac{1}{6}W_{pqr}(dX^p+C^{a_1}\xi^p_{a_1})\wedge(dX^q+C^{a_2}\xi^q_{a_2})\wedge(dX^r+C^{a_3}\xi^r_{a_3})\ ,$$

where ξ Killing vector fields, and C is a connection which gauges the isometries. Next

$$dW=\frac{1}{2}\xi^p_a H_{pqr}F(C)^a\wedge(dX^q+C^{a_2}\xi^q_{a_2})\wedge(dX^r+C^{a_3}\xi^r_{a_3})\neq 0\ .$$

Add non-minimal coupling as

$$\begin{aligned} W &= \frac{1}{6}W_{pqr}(dX^p+C^{a_1}\xi^p_{a_1})\wedge(dX^q+C^{a_2}\xi^q_{a_2})\wedge(dX^r+C^{a_3}\xi^r_{a_3}) \\ &- u_{ap}(dX^p+C^b\xi^p_b)\wedge F(C)^a\ ,\quad i_{\xi_a}W=du_a \end{aligned}$$

Then

$$dW = -c_{ab} F(C)^a \wedge F(C)^b \ , \quad c_{ab} = \xi^M_{(a} u_{b)M}$$

If $c \neq 0$, the coupling is anomalous.

The gauging that occurs in the $N = 8$, $SU(2)$ models is that of the left action on G, and so it is anomalous. However, the gauge fields are composite and depend on the scalars of the base space B. The anomaly cancels due to an opposite contribution from the base space. Since $F = \check{R}$, the worldvolume supersymmetry is (1,1).

- All the solutions which are of the type $AdS_3 \times X_7$ are exact string solutions, ie the models are ultraviolet finite.

Deformations of $AdS_3 \times S^3 \times \mathbb{R}^4$?

The backgrounds $AdS_3 \times X_7$ and $AdS_3 \times S^3 \times \mathbb{R}^4$ preserve 8 spacetime supersymmetries and both are conformal. Moreover $AdS_3 \times S^3 \times \mathbb{R}^4$ is a WZNW model.

- Is there a CFT description of $AdS_3 \times X_7$ as deformation of $AdS_3 \times S^3 \times \mathbb{R}^4$?
- The supergravity analysis suggests that such deformations are the only ones that admit 8 spacetime supersymmetries

$N = 8$, $\mathbb{R}^8$

There is a choice of coordinates [GP] such that

$$\begin{aligned} ds^2 &= 2e^-e^+ + ds^2(\mathbb{R}^8)\ , \quad H = d(e^- \wedge e^+)\ , \\ e^- &= h^{-1}dv\ , \quad e^+ = du + Vdv + n_i dx^i\ , \quad e^{2\Phi} = h^{-1} \end{aligned}$$

All components depend on v and x, and $e_+ = \partial_u$ is Killing.
The field equations imply that

$$\partial_i^2 h = \partial_i^2 V = 0\ , \quad \partial^2 n_i\ , \quad \partial^i n_i = \partial_v h$$

The solutions is a superposition of fundamental strings [Dabholkar, Gibbons, Harvey, Ruis-Ruis], pp-waves and null rotations (also known as chiral null models).

Are the 2 charge fuzzballs unique?

These are mostly investigated for the $D1 - D5$ brane system.

Fuzzball solutions are constructed using a chain of dualities from the heterotic solution of a fundamental string with a pp-wave.

The heterotic solution is found using an ansatz and it is believed that nevertheless is the most general one.

Assuming that the fuzzballs

- preserve 8 spacetime supersymmetries
- and have the asymptotic charges of a fundamental string with a pp-wave

the classification theorem for holonomy $\mathbb{R}^8$ backgrounds implies that there are no other solutions.

If one of the assumptions is removed, the theorem does not hold.

$N=4, SU(2)$

The solution of the conditions that arise from the Killing spinor equations give

$$
\begin{aligned}
ds^2 &= \eta_{ab}\lambda^a\lambda^b + h\, ds_4^2 \,, \\
H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b - \star_4 \tilde{d}h
\end{aligned}
$$

- The spacetime is a Principal bundle $M = P(G, B; \pi)$ equipped with an anti-self dual connection $\lambda^a \equiv e^a$ with curvature, $\mathcal{F}$ is either an anti-self-dual instanton or $\mathcal{F} \in \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.
- The base space B is either hyper-Kähler or Kähler 4-manifold
- $\mathfrak{Lie}G =$ $\mathbb{R}^{6,1}\,, \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)\,, \quad \mathfrak{cw}_6\,, \quad \mathbb{R}^{2,1} \oplus \mathfrak{su}(2)\,, \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}^3\,, \quad \mathfrak{cw}_4 \oplus \mathbb{R}^2$
- Φ depends not only on the coordinates of B but also on the coordinates of the fibre G
- These solutions can be thought of as WZNW models with linear dilaton twisted over B.

$N = 4$, $SU(3)$

The solution of the conditions that arise from the Killing spinor equations give

$$
\begin{aligned}
ds^2 &= \eta_{ab}\lambda^a\lambda^b + ds^2(B)\,, \\
H &= \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b + \pi^*\tilde{H}
\end{aligned}
$$

- The spacetime is a Principal bundle $M = P(G, B; \pi)$ equipped with an connection $\lambda^a \equiv e^a$ with curvature $\mathcal{F}$.
- $\mathfrak{Lie}G = \mathbb{R}^{3,1}\,,\quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}\,,\quad \mathfrak{su}(2) \oplus \mathbb{R}\,,\quad \mathfrak{cw}_4$
- Φ depends only on the coordinates of B
- If G is abelian, B is a complex, conformally balanced, with an $SU(3)$-structure compatible with a skew-symmetric connection with torsion, and $\mathcal{F} \in \mathfrak{su}(3)$. (CYs with torsion)
- If G non-abelian, B is a complex , conformally balanced, with a $U(3)$-structure compatible with a skew-symmetric connection with torsion, and $\mathcal{F} \in \mathfrak{su}(3) \oplus u(1)$ (Hermitian-Einstein).

$N = 4, \times^2 SU(2) \ltimes \mathbb{R}^8$

The solution of the conditions that arise from the Killing spinor equations give

$$
\begin{aligned}
ds^2 &= 2e^- e^+ + h_1\, d\sigma^2_{\rm hk} + h_2\, d\delta^2_{\rm hk}\ , \\
H &= d(e^- \wedge e^+) + \frac{1}{2} h_{ij}\, e^- \wedge e^i \wedge e^j - \star_\sigma dh_1 - \star_\delta dh_2 \\
e^- &= (dv + m_i x^i)\ , \quad e^+ = du + V dv + n_i dx^i
\end{aligned}
$$

where $e_+ = \partial_u$ is a null Killing vector field and e^i is a transverse frame to the lightcone.

- These solutions are in the same universality class as those of rotating intersecting 5-branes with transverse space a hyper-Kähler manifold and superposed with a pp-wave and a fundamental string .

$D = 11$

- $N = 1$, the KSEs have been solved in all case [Gauntlett, Pakis, Gutowski]
- $N > 29$, all solutions are maximally supersymmetric [Gutowski, Gran, Roest, GP]
- $N = 32$, the maximally supersymmetric solutions, classified by [Figueroa, GP], are $\mathbb{R}^{10,1}$, $AdS_4 \times S^7$, $AdS_7 \times S^4$, [Freund, Rubin], and the maximally supersymmetric plane wave [Kowalski-Glikman]
- IIA, $N = 31$, all solutions are maximally supersymmetric [Bandos, Azcarraga, Varela]

A puzzle

It has been proposed that the M-theory supersymmetry algebra is

$$\{Q, Q\} = P_M \Gamma^M + Z_{MN} \Gamma^{MN} + Z_{M_1 \dots M_5} \Gamma^{M_1 \dots M_5}$$

where Z's are central charges associated with the brane solitons.

- The brane solitons, M2 and M5, arise as BPS states
- It also predicts states that preserve 31 and 30 supersymmeries but there are no associated supergravity backgrounds.
- Is the above superalgebra the right algebraic structure to describe the BPS states of M-theory?

IIB

[Gutowski, Gran, Roest, GP]

- $N = 1$, the KSEs have been solved in all cases
- $N = 2$, the KSEs have been solved provided $P = G = 0$
- $N = 28$, there is a unique plane wave solution constructed by Bena and Roidan
- $N > 28$, all solutions are maximally supersymmetric
- $N = 32$, the maximally supersymmetric solutions, classified by [Figueroa, GP], are $\mathbb{R}^{9,1}$, $AdS_5 \times S^5$ [Freund, Rubin; Schwarz], and the maximally supersymmetric plane wave [Blau, Hull, Figueroa, GP]

Conclusions

- The Killing spinor equations of heterotic supergravity have been solved in ALL cases.
- The $1/2$-supersymmetric heterotic solutions are either fundamental rotating strings superposed with pp-waves, or can be constructed from hyper-Kähler 4-manifolds and their anti-self dual instantons.
- The 2 charge fuzzball solution is unique
- The $1/4$-supersymmetric heterotic solutions are associated with either hyper-Kähler or Kähler 4-manifolds, or suitable 6-manifolds with a either $U(3)$ or $SU(3)$ structure and their anti-self dual instantons or Hermitian-Einstein connections.