Quantum black holes, wall crossing and mock modular forms

$\Sigma\alpha\mu\acute{\eta}\rho$ Μ$o\acute{\upsilon}\rho\theta\eta$

LPTHE, Paris

Ο$\rho\theta\acute{o}\delta o\xi o\varsigma$ Α$\kappa\alpha\delta\eta\mu\acute{\iota}\alpha$
Κ$\rho\acute{\eta}\tau\eta\varsigma$, July 1, 2009

Boris Pioline, Σ.M.; arXiv:0904.4253.

Atish Dabholkar, Σ.M., Don Zagier; in preparation.

What is black hole entropy?

Thermodynamic entropy

- Einstein gravity: Bekenstein-Hawking Area law $S = A/4G$.
- Higher derivative corrections (local effective action): Wald formula.
- Quantum (low energy) effects: Sen Proposal. Quantum entropy function for extremal black holes. Is this testable?

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Black hole entropy as statistical entropy

Macroscopic

- String theory being a theory of quantum gravity admits black hole solutions.
- Find a black hole solution to the effective action which carries some charges $\{Q_i\}$. Measure its Bekenstein-Hawking (Wald) entropy S_{BH}.

Microscopic

- In a different regime of parameter space (weak coupling), find a microscopic description of a generic state in the theory with the same charges $\{Q_i\}$.
- Count the number of such states $\Omega(Q_i)$ and compute the statistical entropy $S_{stat} = \ln(\Omega)$.

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Black hole entropy in string theory

- Four dimensional black holes (Q_i, P_i) (following Strominger-Vafa)

$$S_{BH} = \pi\sqrt{Q^2P^2 - (Q.P)^2}. \tag{1}$$

- Corrections to formula (Wald) in inverse powers of charges;

Progress in supergravity, starting from Cardoso, de Wit, Mohaupt 1999.

$$\Omega(q) = \exp\left(q^2 s_0 + s_1 + \frac{1}{q^2} s_2 \ldots\right) . \tag{2}$$

Here, s_1, s_2.. non trivial functions of the charges (*e.g.* Q^2/P^2).

- Detailed understanding on microscopic side as well

Progress in BPS state counting, starting from Dijkgraaf, Verlinde, Verlinde 1994.

Black hole entropy as statistical entropy

Questions 1 : Beyond perturbation theory?

- Can we compute exponentially suppressed corrections to this formula?
- Can we understand these effects from the theory of gravity?
- Test of the Quantum entropy function proposal.

Single centered black holes?

- Formulation of microscopic partition functions in flat space at weak coupling involves a representation of a generic charged state as a collection of strings, branes, momentum...
- Assumption – at strong coupling, this configuration gravitates and forms a black hole.
- However, there may exist other solutions in gravity with same charges (Multi-centered black hole bound states).
- The indexed partition function should count all these configurations.

Questions 2

- Can one single out the single centered black holes in the microscopic formulation?
- What are the symmetry properties of such a function?

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Wall Crossing

- Relatedly, the partition function has a dependence on the moduli of the theory.
- The moduli space is divided into regions bounded by walls.
- On crossing these walls, the multi-centered black holes decay, while the single centered black holes are *immortal*.

Questions 2, again

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Aim of the talk

In this talk, I will answer both these sets of questions in the context of $\mathcal{N} = 4$ string theory in four dimensions.

Outline

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The string theory setup

$\mathcal{N} = 4$ string theory in four dimensions

- Heterotic string theory on T^6,
- Or equivalently, type IIB on $K3 \times T^2$.
- U-duality group

$$G(\mathbb{Z}) = O(22, 6; \mathbb{Z}) \times SL(2, \mathbb{Z}). \tag{3}$$

1/4 BPS Dyons

- A dyonic state in the theory is specified by a charge vector

$$\Gamma^i_\alpha \equiv \begin{bmatrix} Q^i \\ P^i \end{bmatrix} \tag{4}$$

where the index $i = 1, \ldots, 28$ is in the vector representation of the T-duality group $O(22, 6; \mathbb{Z})$ and $\alpha = 1, 2$ transforms in the fundamental representation of the S-duality group $SL(2, \mathbb{Z})$.

A complete classification of dyons

Continuous invariants

- Three T-duality invariants Q^2, P^2, and $Q \cdot P$.
- Unique quartic invariant of the full U-duality group

$$\Delta = Q^2 P^2 - (Q \cdot P)^2. \tag{5}$$

Discrete invariants

- Unique U-duality invariant:

$$I = \gcd(Q \wedge P). \tag{6}$$

- This positive integer is an invariant of $G(\mathbb{Z})$ but not of $G(\mathbb{R})$.

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The dyon degeneracy formula

We focus on $I = 1$ *c.f.* arXiv:0803.2692; Atish Dabholkar, João Gomes, Σ.M. for general answer.

$$\Omega(Q_i, P_i)|_\phi = (-1)^{Q.P+1} \oint_{\mathcal{C}(\phi)} d\sigma d\rho dv \, e^{-i\pi\left(Q^2\rho + P^2\sigma + Q.Pv\right)} \, \mathrm{Z}(\rho, \sigma, v) \,, \tag{7}$$

with

$$\mathrm{Z}(\rho, \sigma, v) = \frac{1}{\Phi_{10}(\rho, \sigma, v)} \,. \tag{8}$$

and $\mathcal{C}$ is a specific moduli dependent contour arXiv:0706.2363; Cheng,Verlinde.

Φ_{10} is the *Igusa cusp form*, which is the unique weight 10 Siegel modular form of $Sp(2, \mathbb{Z})$.

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Black hole solutions

- The (two-derivative) effective action is given by $\mathcal{N} = 4$ supergravity.
- This admits a *BPS* black hole solution:

$$ds^2 = e^{-2f(r)}dt^2 + e^{2f(r)}\left(dr^2 + r^2 d\Omega^2\right), \tag{9}$$

  with flux and scalar fields turned on. Near the horizon, the geometry is $AdS_2 \times S^2$ and the scalars get fixed.
- These black holes carry charges (Q_i, P_i) and have entropy

$$S_{BH} = \sqrt{\Delta}\left(1 + O(1/Q^2)\right) . \tag{10}$$

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Multi-center solutions

- In the $\mathcal{N} = 4$ theory, the only multi-centered configurations that contribute to the index are the two-centered small black hole bound states A. Dabholkar, M.Guica, Σ.M.,S.Nampuri arxiv:0903.2481.
- These are multi-center configurations with each center being a half-BPS state and vanishing horizon area in the classical theory.
- In the higher-derivative corrected theory, they develop a string scale horizon.
- They carry entropy $S = \sqrt{Q^2}$.

Evaluation of microscopic degeneracy (7)

- The evaluation proceeds by performing one contour integral + two saddle point expansions.
- Divisors of Φ_{10} labelled by four independent integers (n_2, n_1, m, j), with $n_2 \geq 0$.
- The contour always encircles the divisors $n_2 \geq 1$ for any value of the moduli.
- The contribution to the degeneracy is then simply the residue at these divisors which is $\Omega \sim \exp(\sqrt{\Delta}/n_2)$.
- Leading degeneracy given by $n_2 = 1$ indeed matches the supergravity computation including higher derivative corrections.

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Walls

- Moving around in moduli space corresponds to deforming the Fourier contour.
- This does not change the degeneracy except when one encounters a pole of the partition function. Crossing a pole corresponds to crossing a wall in the moduli space.
- Here, the only poles which one crosses correspond to the divisors with $n_2 = 0$.
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Immortal degeneracies

- To extract the immortal degeneracies, we should choose a contour that avoids all the poles which contribute to jumps.
- Such a contour is given by setting the asymptotic moduli to the attractor values $\phi_\infty = \phi_*$.
- Generating function for the immortal dyon degneracies with fixed $(P^2 = 2m, Q.P = l)$

$$h^*_{m,l}(\tau) = e^{-\pi i l^2/2m} \sum \Omega^*(m,n,l) q^n \,. \tag{11}$$

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Outline

Quantum entropy function

- Wald's formalism applies for any local theory of gravity, and computes power law corrections to the classical entropy formula.
- To understand the contributions $\exp(S_0/n_2)$, $n_2 = 2, 3..$, we need a formalism which goes beyond a local theory of gravity.
- For extremal black holes, there has been such a proposal called the *quantum entropy function* (QEF) Sen, arXiv:0809.3304.
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Quantum entropy function

- The quantum entropy function $d(Q_i)$ is a Euclidean path integral over asymptotically AdS_2 field configurations with fixed electric charge Q_i, fixed value of the scalar fields at infinity (this includes magnetic fluxes P_i), and a Wilson line insertion.
- The functional integral runs over all fields in the dimensionally reduced two-dimensional field theory.
- The Euclidean path integral is dominated by the field configuration corresponding to pure AdS_2. In general, there are other saddle points approaching AdS_2 asymptotically, and lead to exponentially suppressed contributions.
- These saddle points do not necessarily correspond to smooth geometries, but may include (e.g. orbifold) singularities allowed by the UV completion Banerjee,Sen, arxiv:0810.3472.

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Subleading configurations

- In our problem, the black hole is a compactification of an effective black string.
- The near horizon geometry is $AdS_2 \times S^1 \times M$.
- Family of saddle points labelled by integers (c, d): $\mathbb{Z}/c\mathbb{Z}$ orbifold of the Euclidean AdS_2, accompanied by a translation of angle $2\pi d/c$ along the circle S^1. Σ.M.,B.Pioline, arxiv:0904.4253.
- For $c > 1$ and $1 \leq d < c$, the resulting geometry is *smooth*, and gives a subleading contribution of order

$$\exp\left(\frac{S_0}{c} + 2\pi \mathrm{i}\, Q^2 \frac{d}{c}\right) \qquad (12)$$

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From AdS_2 to AdS_3

- These geometries are the very-near-horizon limit of the $SL(2,\mathbb{Z})$ family of black holes in Euclidean AdS_3 Maldacena and Strominger hep-th/9804085 related to the Rademacher (Farey tail) expansion of Jacobi forms Dijkgraaf, Maldacena, Moore, Verlinde hep-th/0005003.
- The AdS_3 path integral has fixed electric potential (complex structure of the boundary torus) $\Rightarrow$ Canonical partition function.
- The AdS_2 path integral has fixed electric charge $\Rightarrow$ Microcanonical partition function.
- Our construction can be thought of as the Laplace transform in the bulk theory.
- In the microscopic theory, the integers (c, d) correspond precisely to (n_2, n_1). The other two integers (m, j) correspond to similar orbifolds involving $M = S^2 \times \widetilde{S}^1 \times K3$.

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The immortal partition function

- We are interested in the counting function (11) $h^*_{m,l}(\tau)$ for immortal black holes.
- This counting function can be thought of as the partition function of an effective black string, which upon compactification gives our black hole.
- There is a near horizon AdS_3 with an associated $SL(2,\mathbb{Z})$ global diffeomorphism symmetry.
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- Recall that the counting function (11) $h^*_{m,l}(\tau)$ for immortal black holes is a Fourier coefficient of a *meromorphic* Jacobi form.

$$h^*_{m,l}(\tau) = e^{-\pi i l^2 \tau/2m} \int_{z_*}^{z_*+1} \psi_m(\tau, z) e^{-2\pi i l z} dz \; . \qquad (13)$$

- Because of the poles, the Fourier coefficient depends on the contour of integration, and is not modular.

- However, theorem of Zwegers + our improvement $\Rightarrow$ $h^*_{m,l}(\tau)$ is a mock modular form.

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- A mock modular form is a holomorphic function which transforms under modular transformations almost but not quite as a modular form.

- The holomorphic counting function $h^*_{m,l}(\tau)$ is not modular, but it can be completed to a modular function $\widehat{h^*_{m,l}}(\tau,\bar{\tau})$ by adding a certain specific non-holomorphic function.

- It obeys the holomorphic anomaly equation

$$\frac{8\pi i}{\sqrt{m}}\,\tau_2^{3/2}\,\frac{\partial\widehat{h^*_{m,l}}(\tau,\bar{\tau})}{\partial\bar{\tau}} = \frac{p_{24}(m+1)}{\eta^{24}(\tau)}\,\overline{\vartheta_{m,l}(\tau)}\,. \tag{14}$$

- Meromorphy is a reflection of wall crossing. The factors $p_{24}(m+1)$ and $\frac{1}{\eta^{24}(\tau)}$ are precisely the degeneracy of electric and magnetic small black holes.

Outline

Summary 1

1. Universal series of exponentially suppressed corrections to the degeneracy of extremal black holes $\exp(S_0/c)$, $c = 2, 3...$
2. When the black hole arises from a black string, this is related to the Rademacher (Farey tail) expansion of the associated modular form.

Summary 2

1. One can define a holomorphic counting function for counting the microstates of a single-centered black hole.
2. This counting function is a *mock* modular form in that it fails to be modular but in a very specific way. The failure of modularity is governed by a *shadow*, which is a holomorphic modular form. Physically, the shadow governs the jumps in the spectrum upon crossing a wall in the moduli space.

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2. This counting function is a *mock* modular form in that it fails to be modular but in a very specific way. The failure of modularity is governed by a *shadow*, which is a holomorphic modular form. Physically, the shadow governs the jumps in the spectrum upon crossing a wall in the moduli space.

Summary 1

1. Universal series of exponentially suppressed corrections to the degeneracy of extremal black holes $\exp(S_0/c)$, $c = 2, 3...$
2. When the black hole arises from a black string, this is related to the Rademacher (Farey tail) expansion of the associated modular form.

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2. This counting function is a *mock* modular form in that it fails to be modular but in a very specific way. The failure of modularity is governed by a *shadow*, which is a holomorphic modular form. Physically, the shadow governs the jumps in the spectrum upon crossing a wall in the moduli space.

Further questions

- *Canonical* mock modular form? – Relation to Hurwitz class numbers.
- Jacobi-Rademacher expansion $\Rightarrow$ Post-modern Farey tail?
- More general physical systems with meromorphic Jacobi forms.
- $\mathcal{N} = 2$ compactifications? (Here the wall-crossings may be large).

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ΩΡΑ ΦΑΓΗΤΟΥ !