Mean-field Gauge Interactions in Five Dimensions I
(The Torus)

Nikos Irges, Wuppertal U.

Based on N.I . & F. Knechtli, arXiv:0905.2757 [hep-lat]

5th RMST, Kolymbari, June 2009

Background

1. J. M. Drouffe and J. B. Zuber, Phys. Rept. 102 (1983) 1

2. J. Zinn-Justin, Quantum Field Theories and Critical Phenomena

3. I. Montvay and G. Muenster, Quantum Fields on a Lattice

4. M. Creutz, Phys. Rev. Lett. 43 (1979), 553

5. B. Svetitsky and L.G. Yaffe, Nucl. Phys. B210 (1982) 423

6. Y.K. Fu and H.B. Nielsen, Nucl. Phys. B236 (1984), 167

7. S. Ejiri, J. Kubo and M. Murata, Phys. Rev. D62 (2000), 105025

8. M. Luscher and P. Weisz, JHEP 0407:014, 2204

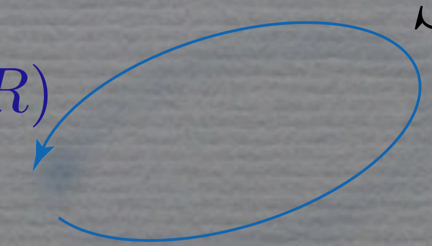
In 5 Dimensions...

$$A_M \xrightarrow{\mathcal{M}_5 = E_4 \times S^1} \{A_\mu, A_5\}$$

5D gauge field SU(2)

Z: 4D gauge field

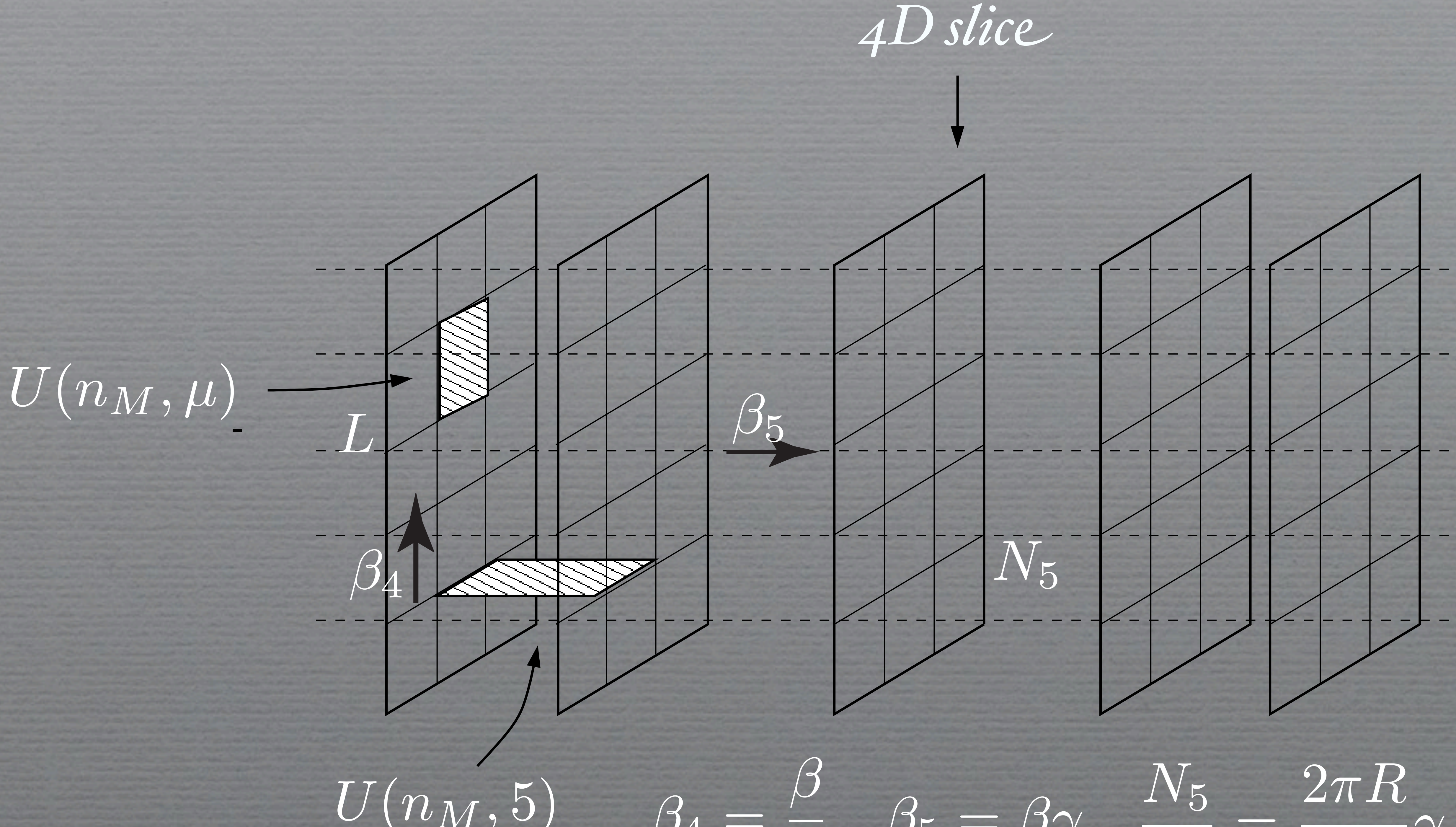
h: 4D "Higgs"



$$\phi(y) = \sum_n \phi_n \, e^{iny/R} \qquad \psi(y) = \sum_n \psi_x e^{iny/R}$$

4D Georgi-Glashow (G-S)
model + KK

The Anisotropic Lattice



$$\beta_4 = \frac{\beta}{\gamma} \quad \beta_5 = \beta\gamma \quad \frac{N_5}{L} = \frac{2\pi R}{l}\gamma$$

$\gamma =$ anisotropy parameter

The mean-field expansion

parameters: $L,\ \beta,\ \gamma$ (we keep $N_5 = L$)

$$Z = \int DU \int DV \int DH e^{(1/N)\mathrm{Re}[\mathrm{tr}H(U-V)]} e^{-S_G[V]}$$

$$Z = \int DV \int DH e^{-Seff[V,H]}, \quad Seff = S_G[V] + u(H) + (1/N)\mathrm{Retr}HV$$

$$e^{-u(H)} = \int DU e^{(1/N)\mathrm{Retr}UH}$$

To 0'th order

The background is determined by

$$\overline{V} = -\frac{\partial u}{\partial H}\bigg|_{\overline{H}} \qquad \overline{H} = -\frac{\partial u}{\partial S_G[V]}\bigg|_{\overline{V}}$$

The free energy

$$F^{(0)} = -\frac{1}{\mathcal{N}}\ln(Z[\overline{V},\overline{H}]) = \frac{S_{\text{eff}}[\overline{V},\overline{H}]}{\mathcal{N}}$$

To 1st order

$$H = \bar{H} + h \qquad V = \bar{V} + v$$

$$Seff = Seff[\bar{V},\bar{H}] + \frac{1}{2}\left(\frac{\delta^2 Seff}{\delta H^2}\bigg|_{\overline{V},\overline{H}} h^2 + 2\,\frac{\delta^2 Seff}{\delta H \delta V}\bigg|_{\overline{V},\overline{H}} hv + \frac{\delta^2 Seff}{\delta V^2}\bigg|_{\overline{V},\overline{H}} v^2 \right)$$

$$\frac{\delta^2 Seff}{\delta H^2}\bigg|_{\overline{V},\overline{H}} h^2 = h_i K^{(hh)}_{ij} h_j = h^T K^{(hh)} h$$

$$\frac{\delta^2 Seff}{\delta V^2}\bigg|_{\overline{V},\overline{H}} v^2 = v_i K^{(vv)}_{ij} v_j = v^T K^{(vv)} v$$

$$\frac{\delta^2 Seff}{\delta V \delta H}\bigg|_{\overline{V},\overline{H}} v^2 = v_i K^{(vh)}_{ij} h_j = v^T K^{(vh)} h$$

The Gaussian fluctuations

$$z = \int Dv \int Dh e^{-S^{(2)}[v,h]} \qquad S^{(2)}[v,h] = \frac{1}{2}\left(h^T K^{(hh)} h + 2v^T K^{(vh)} h + v^T K^{(vv)} v\right)$$

$$z = \frac{(2\pi)^{|h|/2}(2\pi)^{|v|/2}}{\sqrt{\det[(-\mathbf{1} + K^{(hh)} K^{(vv)})]}}$$

$$Z^{(1)} = Z[\overline{V}, \overline{H}] \cdot z = e^{-S_{\text{eff}}[\overline{V}, \overline{H}]} \cdot z$$

The free energy to first order

$$F^{(1)} = F^{(0)} - \frac{1}{\mathcal{N}} \ln(z) = F^{(0)} + \frac{1}{2\mathcal{N}} \ln\Big[\det\left(-\mathbf{1} + K^{(hh)} K^{(vv)}\right) \Delta_{\text{FP}}^{-2}\Big]$$

Observables

$$\mathcal{O}[V] = \mathcal{O}[\overline{V}] + \left.\frac{\delta \mathcal{O}}{\delta V}\right|_{\overline{V}} v + \frac{1}{2} \left.\frac{\delta^2 \mathcal{O}}{\delta V^2}\right|_{\overline{V}} v^2 + \ldots$$

$$= \mathcal{O}[\overline{V}] + \frac{1}{2} \left.\frac{\delta^2 \mathcal{O}}{\delta V^2}\right|_{\overline{V}} \frac{1}{z} \int Dv \int Dh v^2 e^{-S^{(2)}[v,h]}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int Dv \int Dh \left(\mathcal{O}[\overline{V}] + \frac{1}{2} \left.\frac{\delta^2 \mathcal{O}}{\delta V^2}\right|_{\overline{V}} v^2 \right) e^{-\left(Seff[\overline{V},\overline{H}] + S^{(2)}[v,h]\right)}$$

$$< v_i v_j >= \frac{1}{z} \int Dv \int Dh\, v_i v_j e^{-S^{(2)}[v,h]} = (K^{-1})_{ij}$$

$$K = -K^{(vh)} {K^{(hh)}}^{-1} K^{(vh)} + K^{(vv)}$$

$$\boxed{< \mathcal{O} >= \mathcal{O}[\overline{V}] + \frac{1}{2} \mathrm{tr} \left\{ \left.\frac{\delta^2 \mathcal{O}}{\delta V^2}\right|_{\overline{V}} K^{-1} \right\}}$$

1st order master formula

Correlators

$$C(t) =< \mathcal{O}(t_0+t)\mathcal{O}(t_0) > - < \mathcal{O}(t_0+t) >< \mathcal{O}(t_0) >= C^{(0)}(t) + C^{(1)}(t) + \cdots$$

$$C^{(0)}(t) = 0$$

$$< \mathcal{O}(t_0+t)\mathcal{O}(t_0) >= \mathcal{O}^{(0)}(t_0+t)\mathcal{O}^{(0)}(t_0) + \frac{1}{2}\mathrm{tr}\left\{\frac{\delta^2(\mathcal{O}(t_0+t)\mathcal{O}(t_0))}{\delta^2 v}K^{-1}\right\} + \cdots$$

$$C^{(1)}(t) = \frac{1}{2}\mathrm{tr}\left\{\frac{\delta^{(1,1)}(\mathcal{O}(t_0+t)\mathcal{O}(t_0))}{\delta^2 v}K^{-1}\right\} = \frac{1}{2}\mathrm{tr}\left\{\frac{\tilde{\delta}^{(1,1)}(\mathcal{O}(t_0+t)\mathcal{O}(t_0))}{\delta^2 v}\tilde{K}^{-1}\right\}$$

$$C^{(1)}(t) = \sum_\lambda c_\lambda e^{-E_\lambda t} \qquad E_0 = m_H, \quad E_1 = m_H^*, \cdots$$

To 2nd order

$$S_{\text{eff}} = S_{\text{eff}}[\overline{V}, \overline{H}] + \frac{1}{2}\left(\frac{\delta^2 S_{\text{eff}}}{\delta H^2} h^2 + 2\frac{\delta^2 S_{\text{eff}}}{\delta H \delta V} hv + \frac{\delta^2 S_{\text{eff}}}{\delta V^2} v^2\right)$$

$$+\frac{1}{6}\left(\frac{\delta^3 S_{\text{eff}}}{\delta H^3} h^3 + \frac{\delta^3 S_{\text{eff}}}{\delta V^3} v^3\right) + \frac{1}{24}\left(\frac{\delta^4 S_{\text{eff}}}{\delta H^4} h^4 + \frac{\delta^4 S_{\text{eff}}}{\delta V^4} v^4\right) + \cdots$$

$$\mathcal{O}[V] = \mathcal{O}[\overline{V}] + \frac{\delta \mathcal{O}}{\delta V} v + \frac{1}{2}\frac{\delta^2 \mathcal{O}}{\delta V^2} v^2 + \frac{1}{6}\frac{\delta^3 \mathcal{O}}{\delta V^3} v^3 + + \frac{1}{24}\frac{\delta^4 \mathcal{O}}{\delta V^4} v^4 + \cdots$$

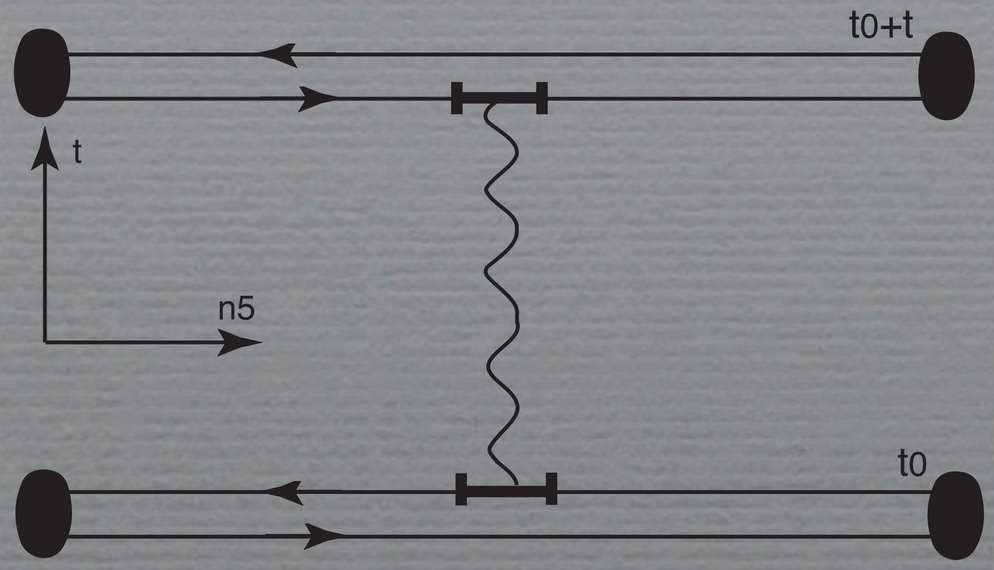
2nd order master formula

$$\boxed{\begin{aligned} < \mathcal{O} > &= \mathcal{O}[\overline{V}] + \frac{1}{2}\left(\frac{\delta^2 \mathcal{O}}{\delta V^2}\right)_{ij} \left(K^{-1}\right)_{ij} \\ &+\frac{1}{24}\sum_{i,j,l,m}\left(\frac{\delta^4 \mathcal{O}}{\delta V^4}\right)_{ijlm} \left((K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl}\right)\end{aligned}}$$

$$C^{(2)}(t) = \frac{1}{24}\sum_{i,j,l,m}\left(\frac{\delta^4 \mathcal{O}^c(t)}{\delta v^4}\right)_{ijlm} \left((K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl}\right)$$

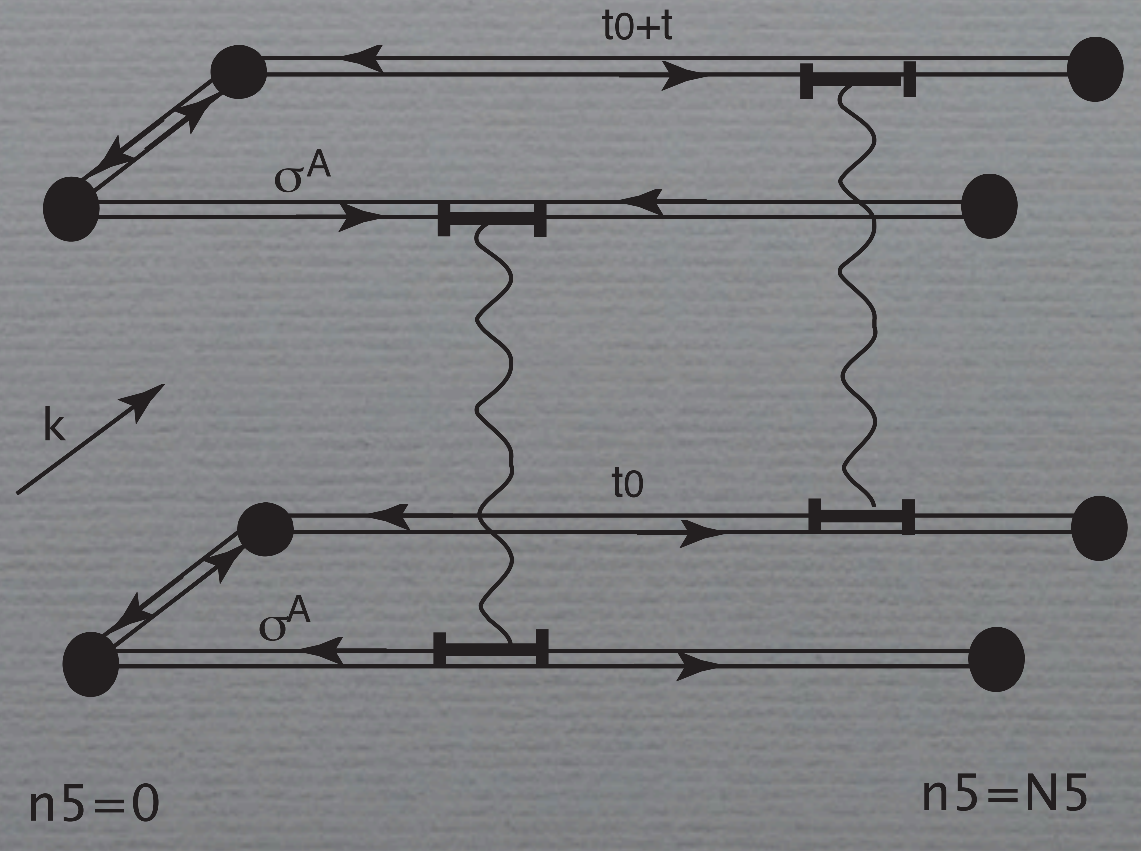
Lattice Observables (Polyakov loops)

The scalar



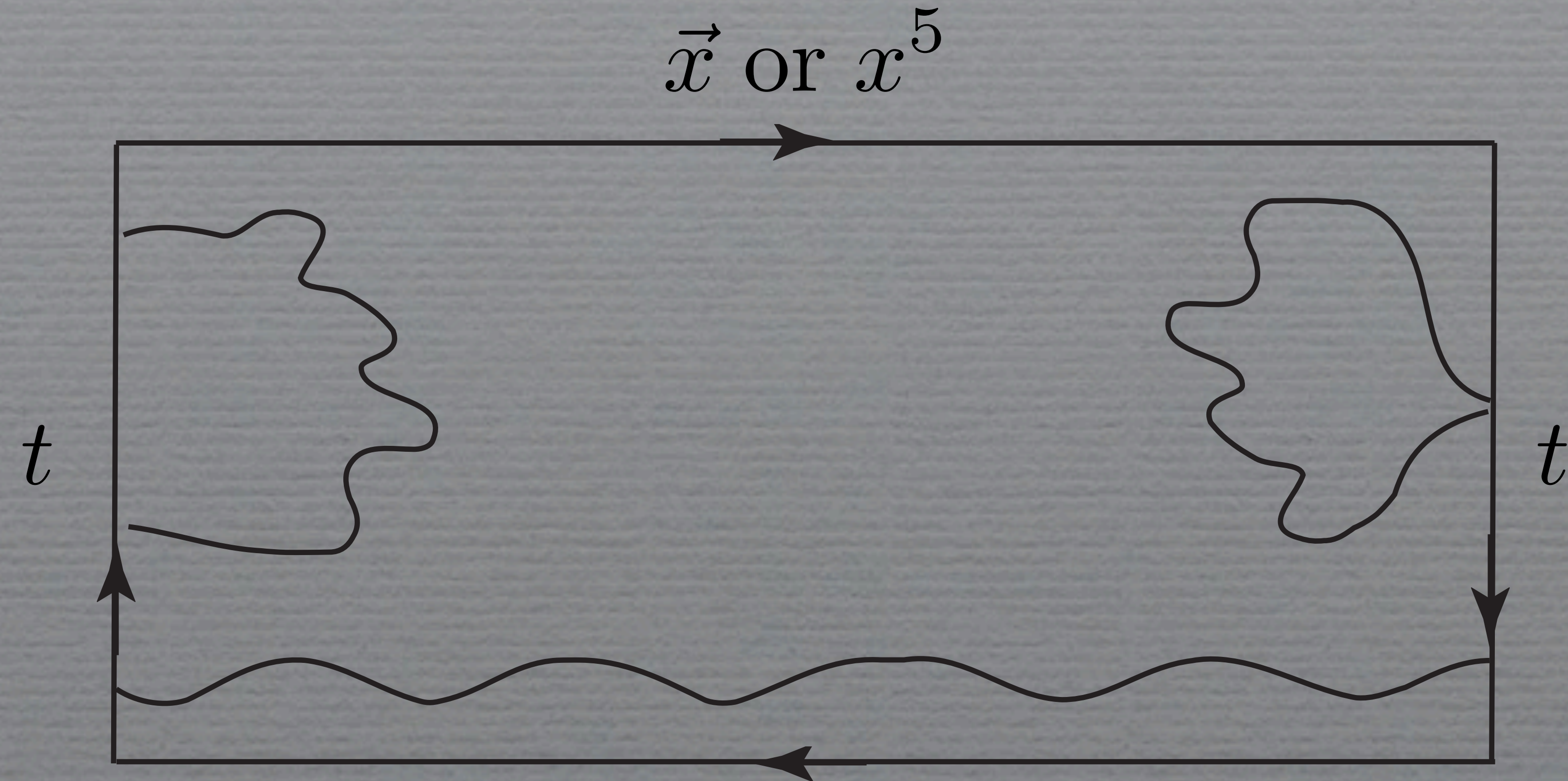
$$m = \lim_{t\to\infty} \ln \frac{C^{(1)}(t)}{C^{(1)}(t-1)}$$

The vector



$$m = \lim_{t\to\infty} \ln \frac{C^{(2)}(t)}{C^{(2)}(t-1)}$$

The Wilson loop



$$\mathrm{t} \to \infty : \qquad \mathrm{e}^{-Vt} \simeq < \mathcal{O}_W >$$

The SU(2) model

Mean-field parametrization

$$U(n,M) = u_0(n,M)\mathbf{1} + i\sum_k u_k(n,M)\sigma^k \qquad u_\alpha \in \mathbb{R}, u_\alpha u_\alpha = 1$$

$$V(n,M) = v_0(n,M)\mathbf{1} + i\sum_k v_k(n,M)\sigma^k\,, \quad v_\alpha \in \mathbb{C}$$

$$H(n,M) = h_0(n,M)\mathbf{1} - i\sum_k h_k(n,M)\sigma^k\,, \quad h_\alpha \in \mathbb{C}$$

The propagator (in momentum space)

$$\tilde{K}(p',M',\alpha';p'',M'',\alpha'') = \delta_{p'p''}\delta_{\alpha'\alpha''}C_{M'M''}(p',\alpha')$$

$$C_{M'M''}(p',\alpha') = [A\delta_{M'M''} + B_{M'M''}(1-\delta_{M'M''})]$$

$$A = -\left[\frac{1}{b_2}(1-\delta_{\alpha'0}) + \frac{1}{b_1}\delta_{\alpha'0}\right] - 2\beta\overline{v}_0^2\left[\sum_{N\neq M'}\cos(p'_N) + \frac{1}{\xi}\sin^2(p'_{M'}/2)\right]$$

$$B_{M'M''} = -4\beta\overline{v}_0{}^2\left[\delta_{\alpha'0}\cos\left(\frac{p'_{M'}}{2}\right)\cos\left(\frac{p'_{M''}}{2}\right) + (1-\delta_{\alpha'0})\sin\left(\frac{p'_{M'}}{2}\right)\sin\left(\frac{p'_{M''}}{2}\right)\right]$$

$$b_1 = -\frac{1}{\overline{h}_0 I_1(\overline{h}_0)}\left(I_2(\overline{h}_0) - \overline{h}_0\left(\frac{I_2(\overline{h}_0)^2}{I_1(\overline{h}_0)} - I_3(\overline{h}_0)\right)\right)$$

$$b_2 = -\frac{\overline{v}_0}{\overline{h}_0}$$

The static potential

$$V(r) = -2\log(\overline{v}_0) - \frac{1}{2\overline{v}_0^2}\frac{1}{L^3 N_5} \times \left\{ \sum_{{p'}_{M\neq 0}, p'_0=0} \left[\frac{1}{4}\sum_{N\neq 0}(2\cos(p'_N r)+2)\right] C_{00}^{-1}(p',0)\right]$$

$$+3 \sum_{{p'}_{M\neq 0}, p'_0=0} \left[\frac{1}{4}\sum_{N\neq 0}(2\cos(p'_N r)-2)\right] \frac{1}{C_{00}(p',1)}\right\}$$

The scalar mass

$$C_H^{(1)}(t) = \frac{1}{\mathcal{N}}(P_0^{(0)})^2 \sum_{p'_0} \cos(p'_0 t) \sum_{p'_5} |\tilde{\Delta}^{(\mathcal{N}_5)}(p'_5)|^2 \tilde{K}^{-1}\Big((p'_0, \vec{0}, p'_5), 5, 0; (p'_0, \vec{0}, p'_5), 5, 0\Big)$$

$$\Delta^{(m_5)}(n_5) = \sum_{r=0}^{m_5-1} \frac{\delta_{n_5 r}}{\overline{v}_0(\hat{r})}, \qquad \hat{r} = r + 1/2$$

The vector mass

$$C_V^{(2)}(t) = \frac{2304}{\mathcal{N}^2}(P_0^{(0)})^4(\overline{v}_0(0))^4 \sum_{\vec{p}'} \sum_k \sin^2(p'_k)\left(\overline{\overline{K}}^{-1}(t,\vec{p}',1)\right)^2$$

$$\overline{K}^{-1}((p'_0,\vec{p}'),5,\alpha) = \sum_{p'_5,p''_5} \tilde{\Delta}^{(N_5)}(p'_5)\tilde{\Delta}^{(N_5)}(-p''_5)K^{-1}(p'',5,\alpha;p',5,\alpha)$$

$$\overline{\overline{K}}^{-1}(t,\vec{p}',\alpha) = \sum_{p'_0} e^{ip'_0 t}\overline{K}^{-1}((p'_0,\vec{p}'),5,\alpha)$$

The free energy

$$F^{(1)} = F^{(0)} + \frac{1}{2\mathcal{N}}\sum_p \ln\left[\det\left(-\mathbf{1} + \tilde{K}^{(hh)}_{\alpha'=0}\tilde{K}^{(vv)}_{\alpha'=0}\right)\det\left(-\mathbf{1} + \tilde{K}^{(hh)}_{\alpha'\neq 0}\tilde{K}^{(vv)}_{\alpha'\neq 0}\right)^3 \Delta_{\mathrm{FP}}^{-2}\right]$$

A few remarks on the calculation

1. We fix the Lorentz gauge;
We pick up a Fadeev-Popov determinant but no ghost loops at this order; Gauge dependent free energy but no worries

2. Torons appear always as 0/0 --> "0/0=0 regularization"

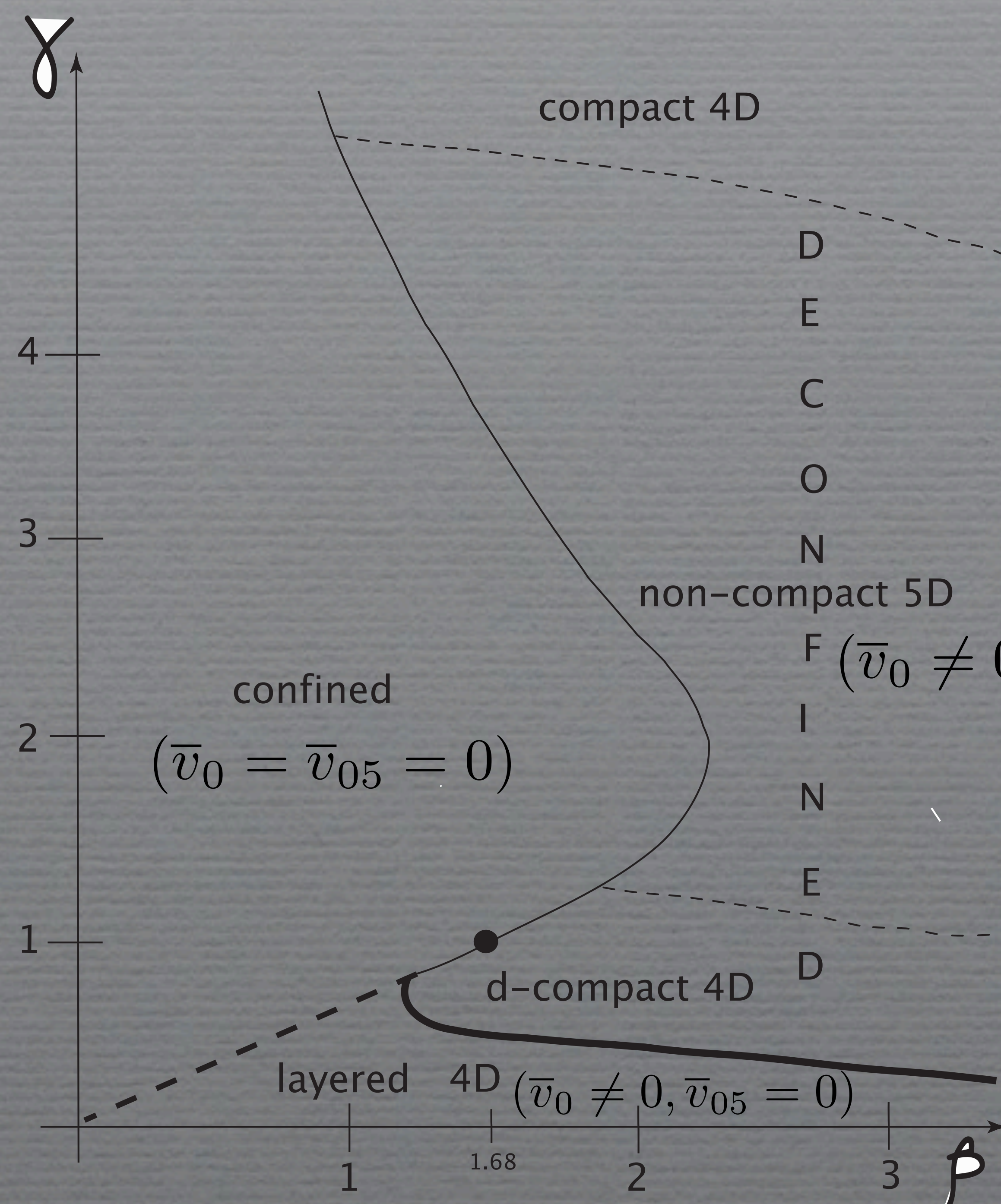
3. Formulas are generalized to the anisotropic lattice -->
2 independent Wilson loops since the background is different along 4D and the 5th dimension

4. Analytical formulas are computed numerically

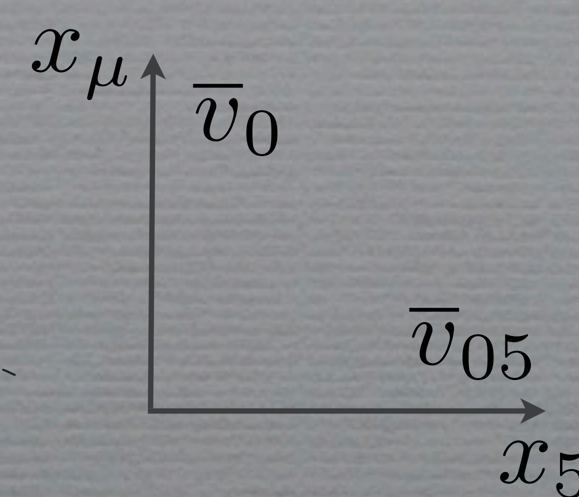
Keep this in mind

5. Our working assumption:
whenever non-trivial, physical

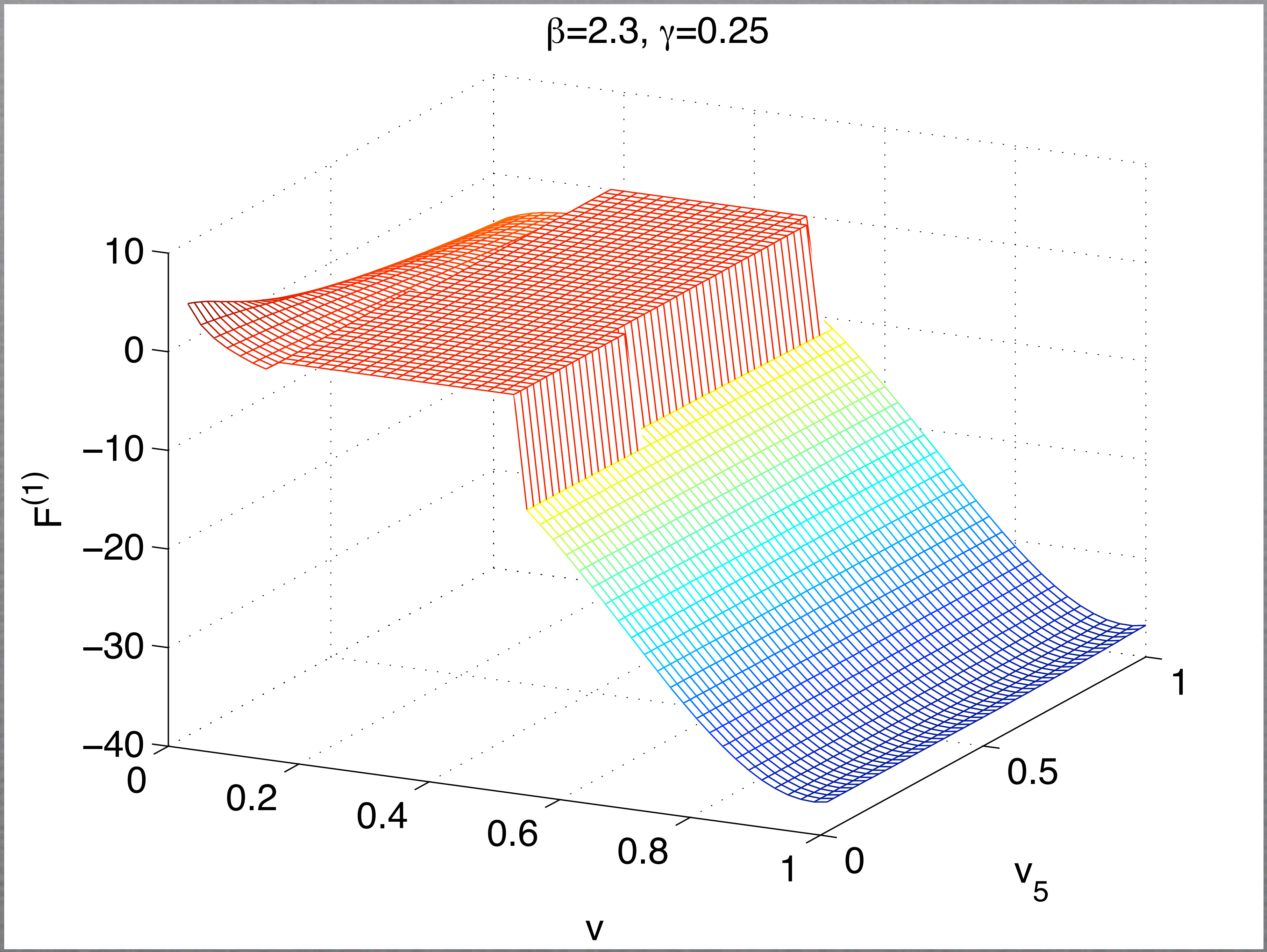
The phase diagram

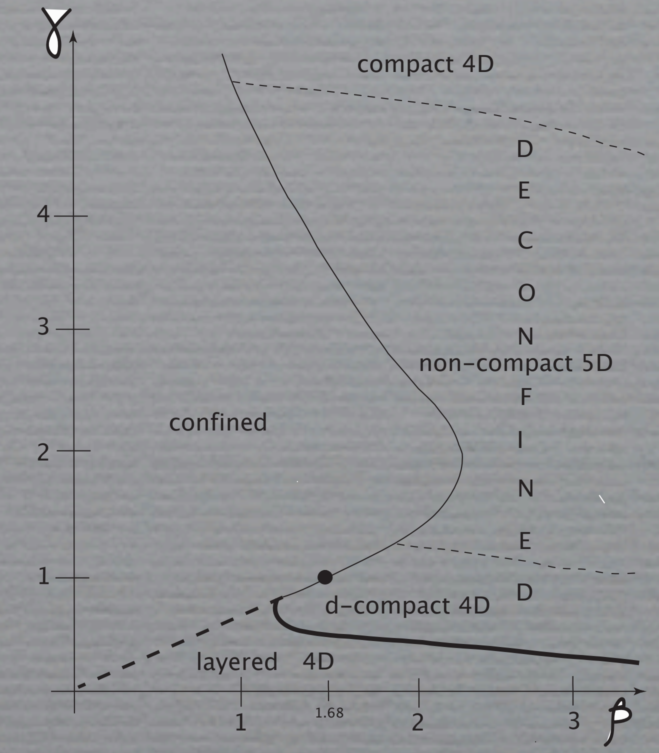


the mean-field background



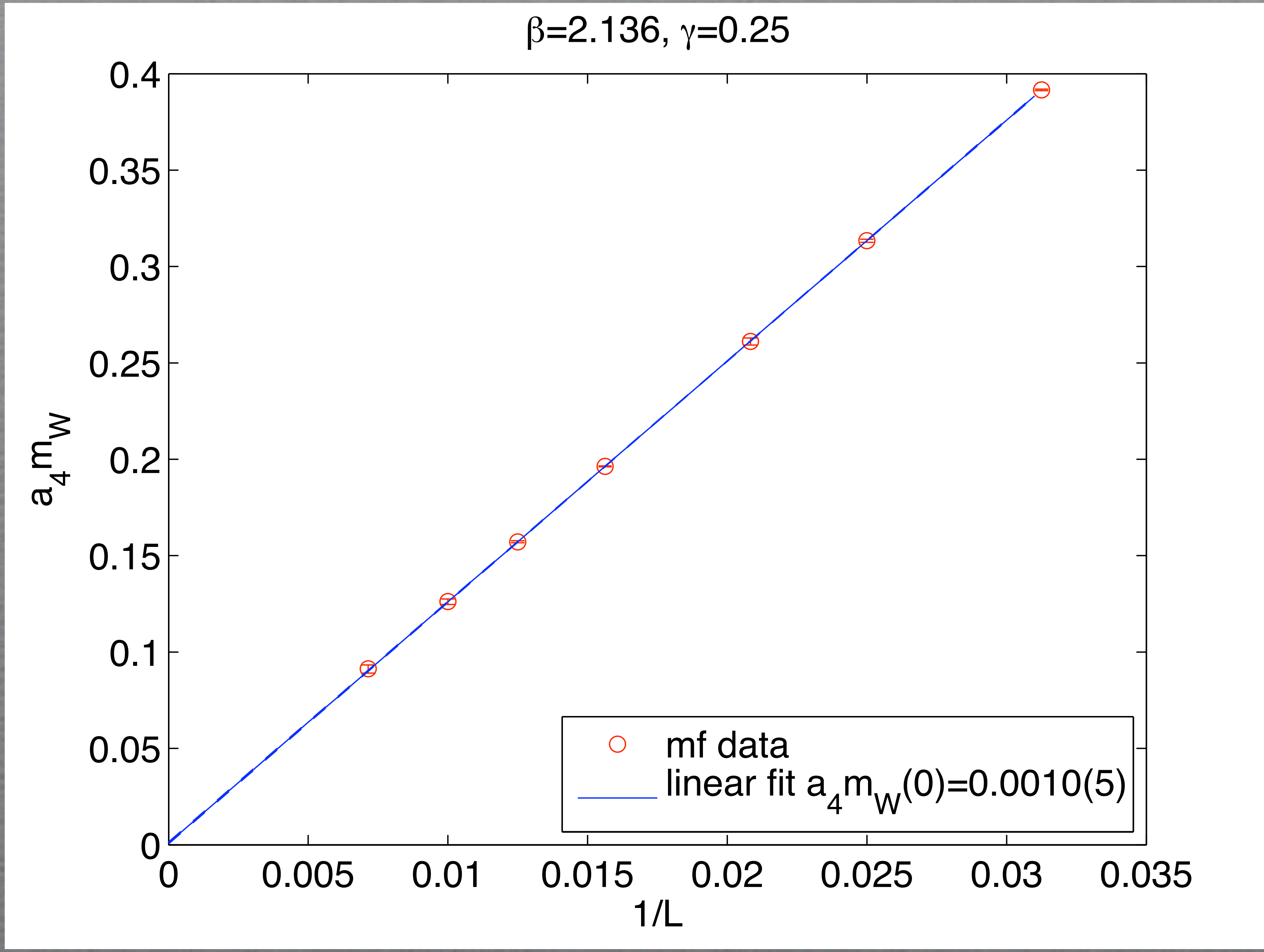
The d-compact phase is stable:

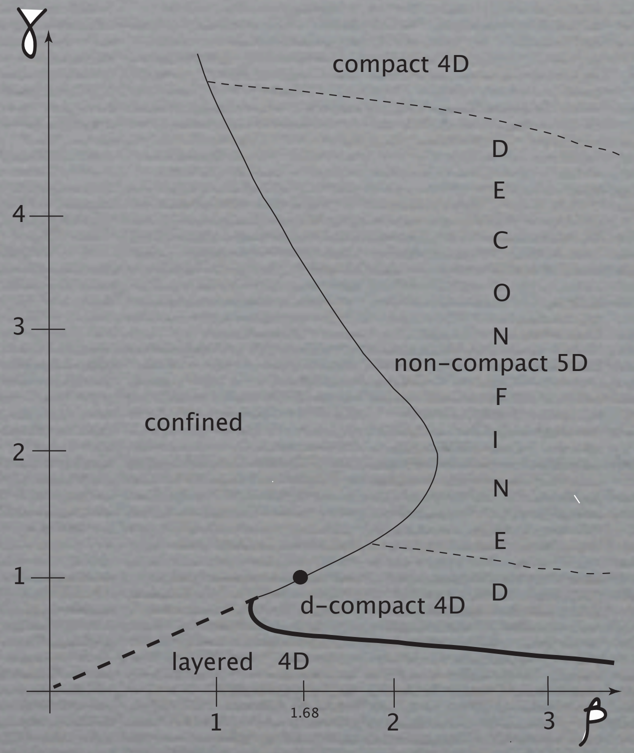




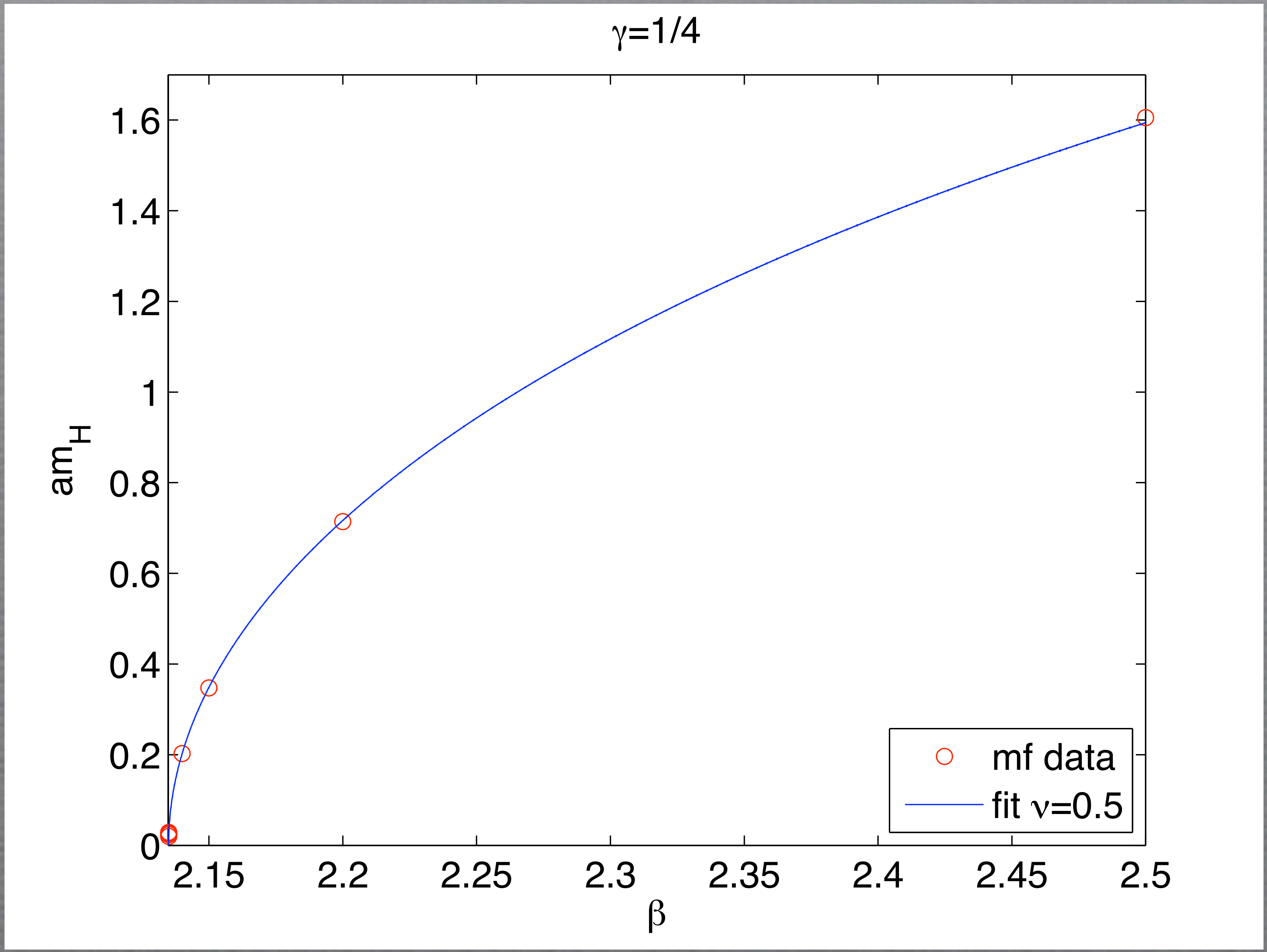
The compact phase (for small β) is unstable to this order

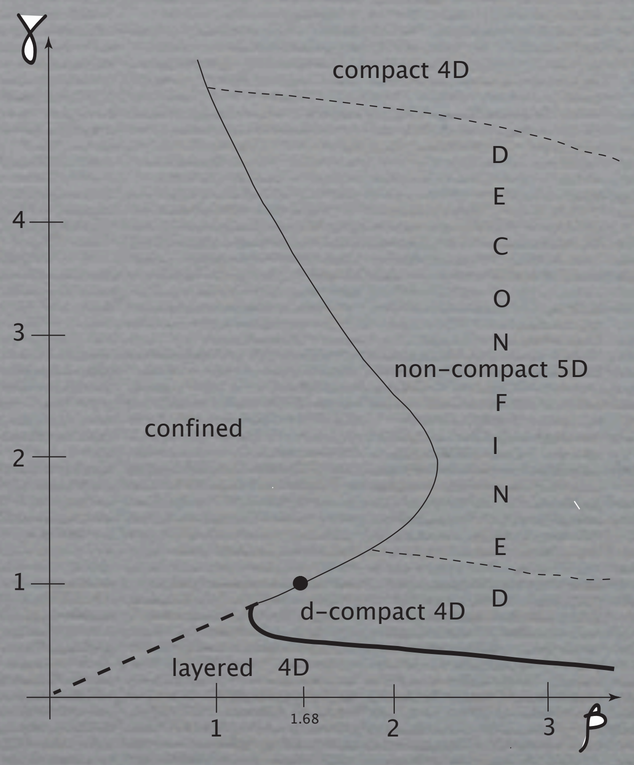
The W is massless or perhaps exponentially light:



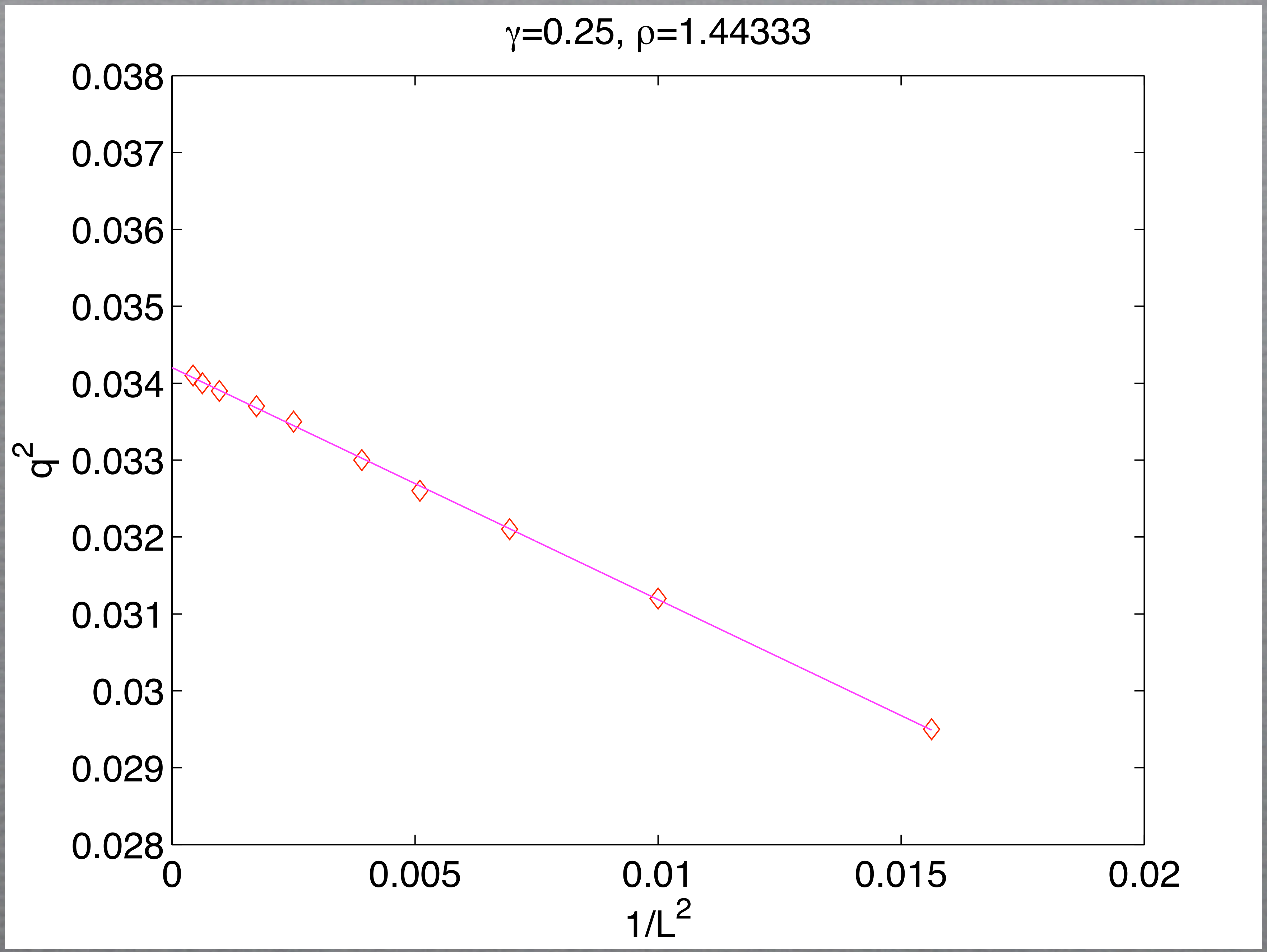


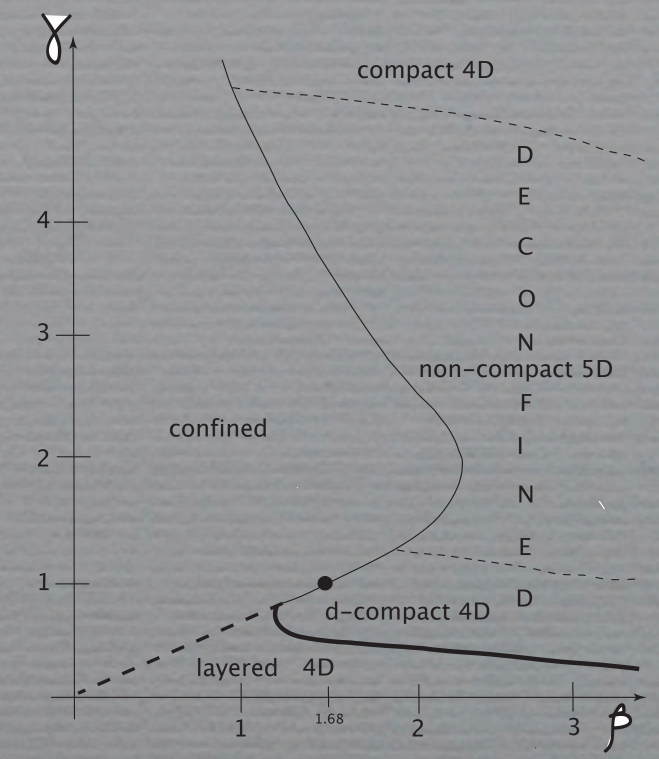
The d-compact phase is separated from the layered phase by a 2nd order phase transition:





The continuum limit can be taken:

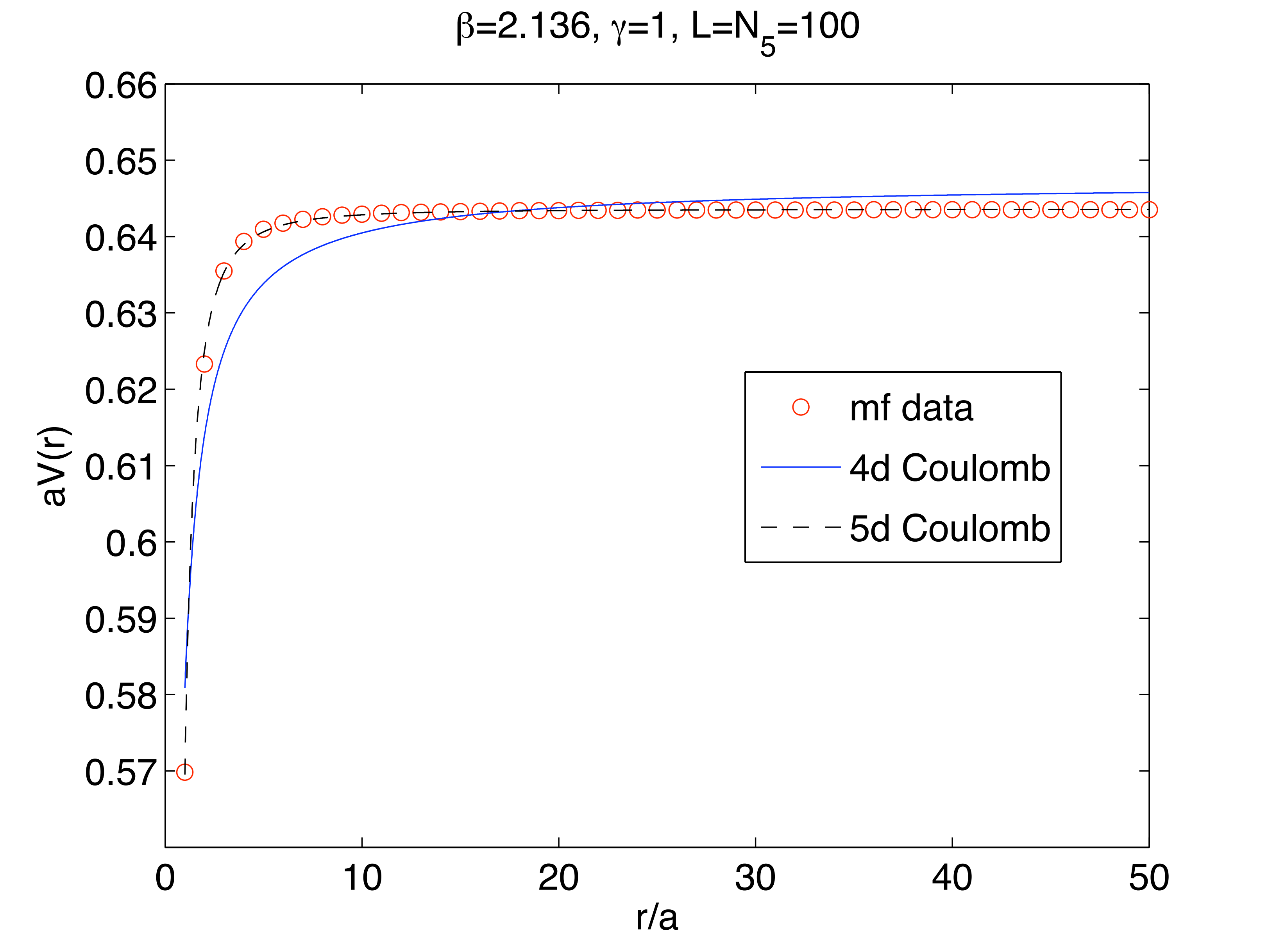




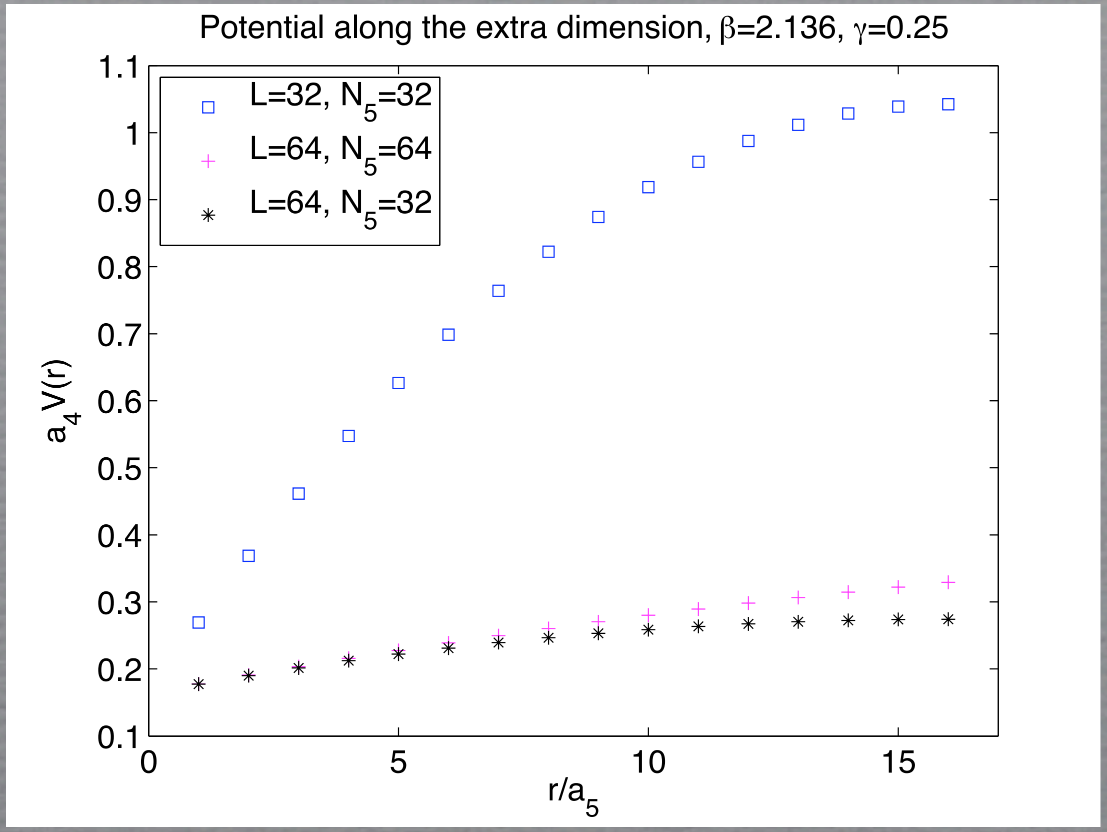
The physical charge at fixed $\rho = \frac{m_W}{m_H}$

The Static Potential

The static potential on the isotropic lattice:

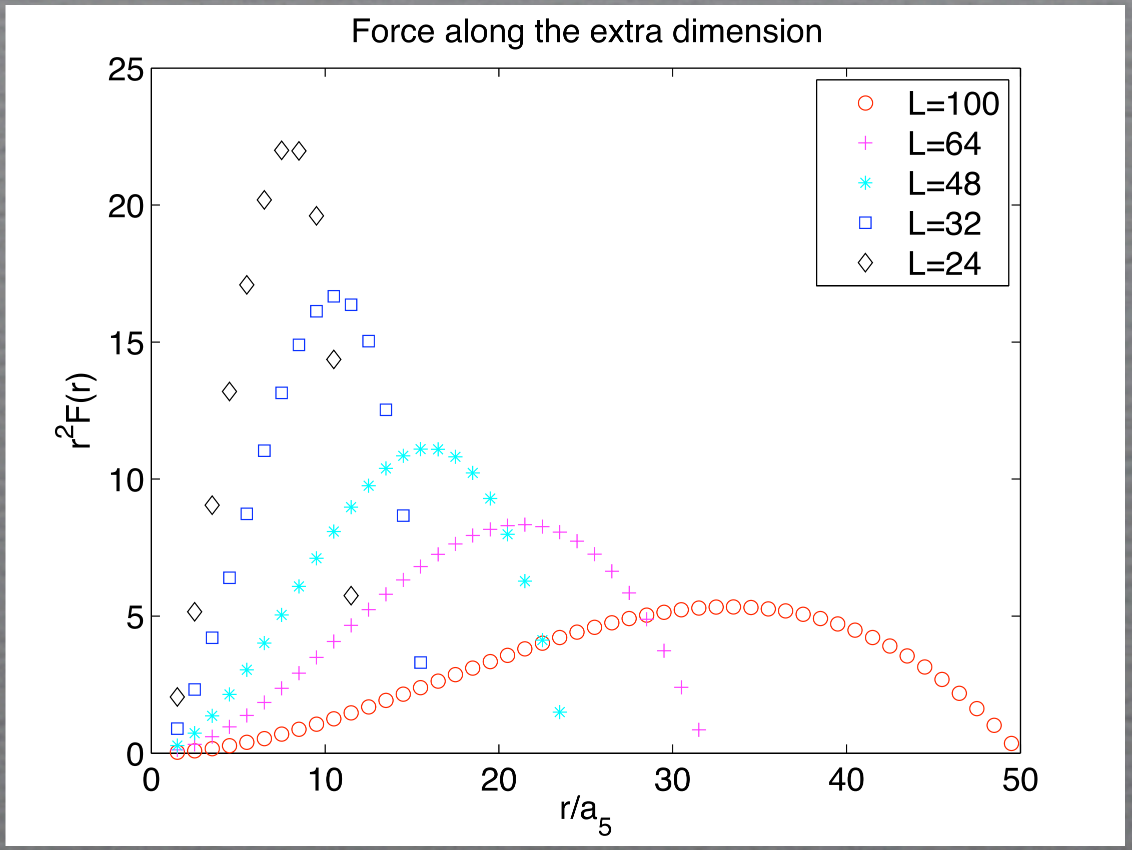


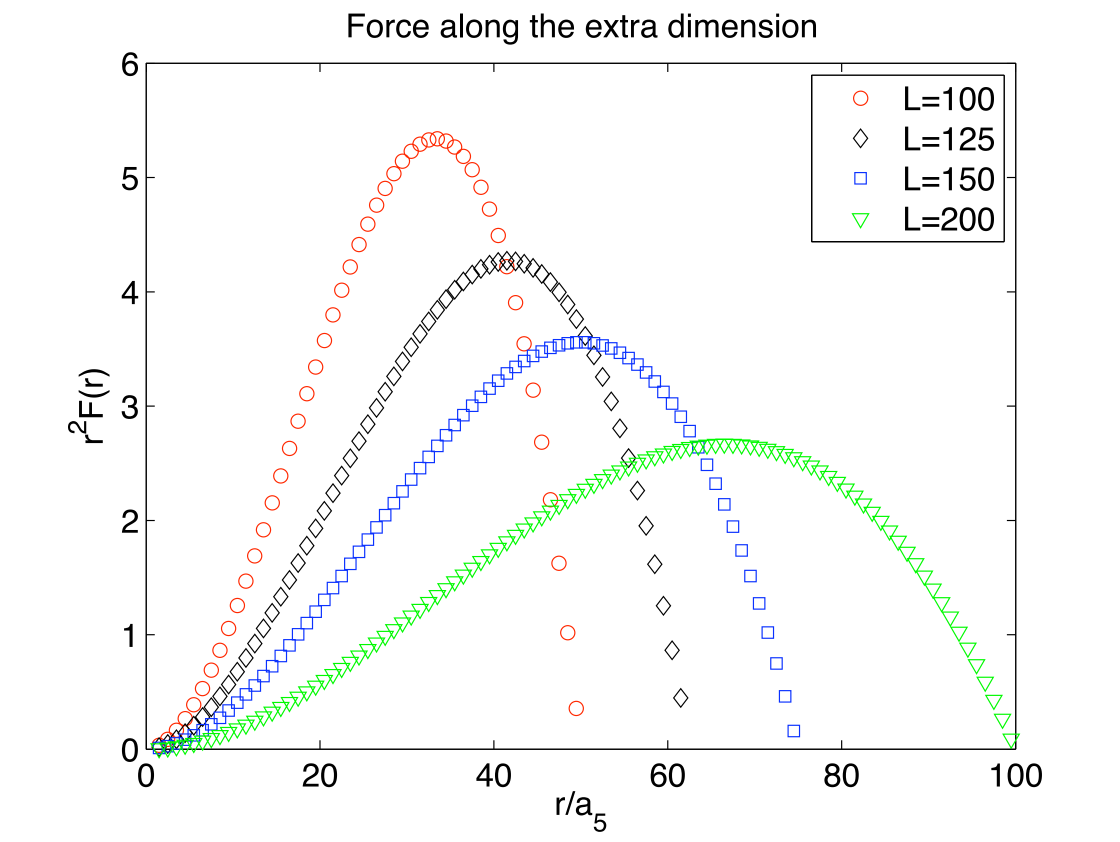
The static potential (Wilson loop) in the d-compact phase along the extra dimension:



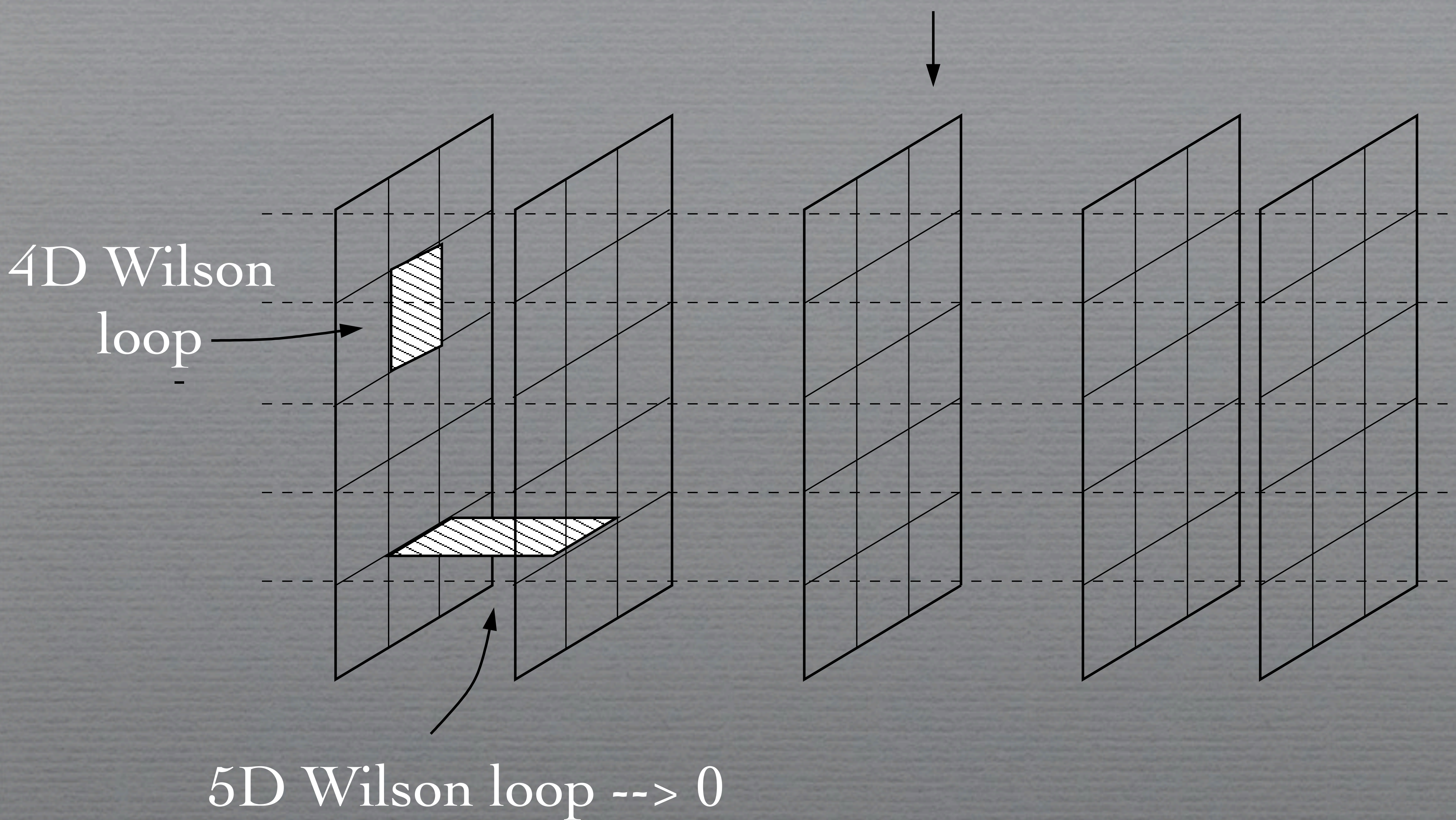
It doesn't scale with L:

The force vanishes in the infinite volume limit



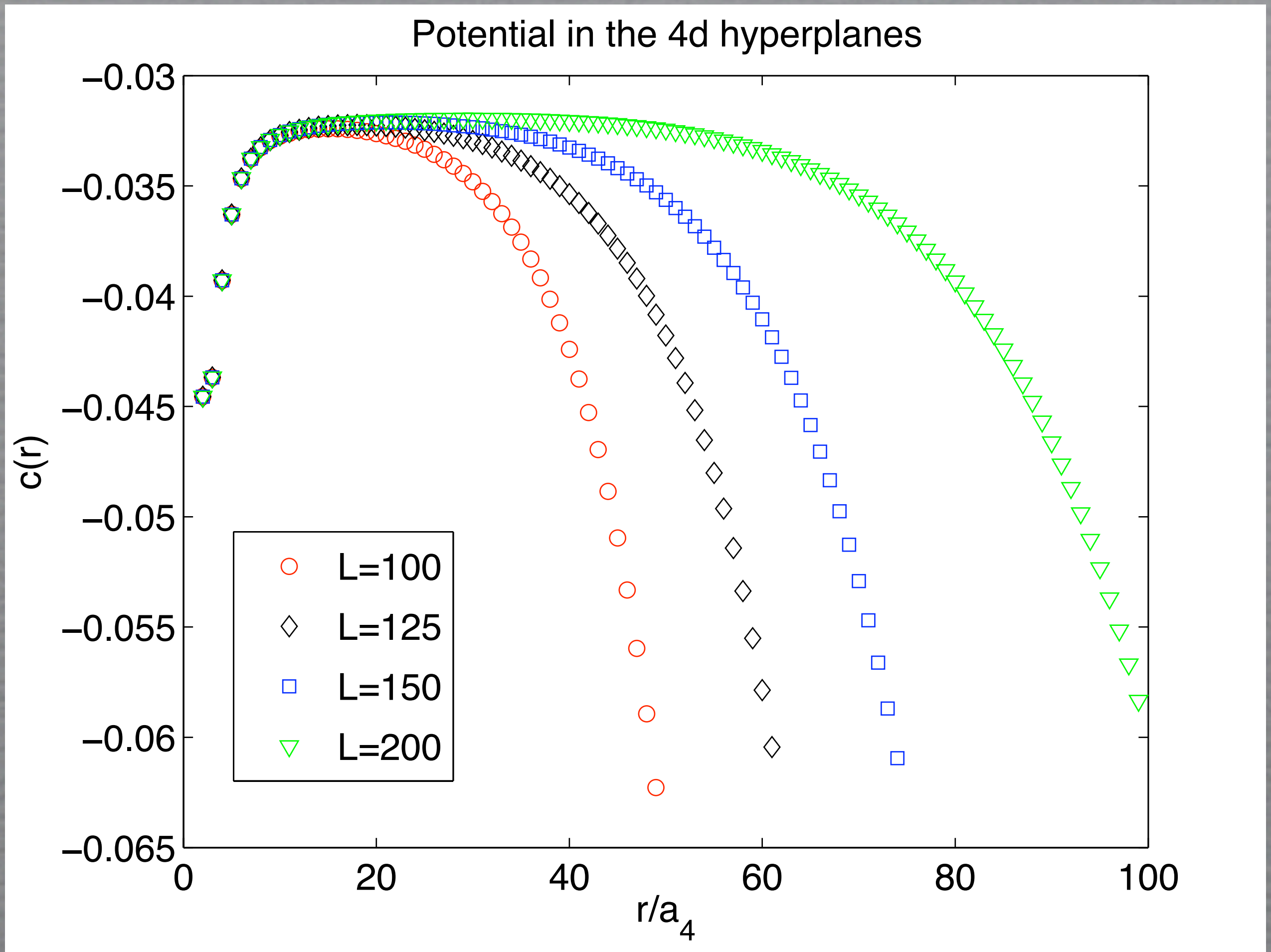


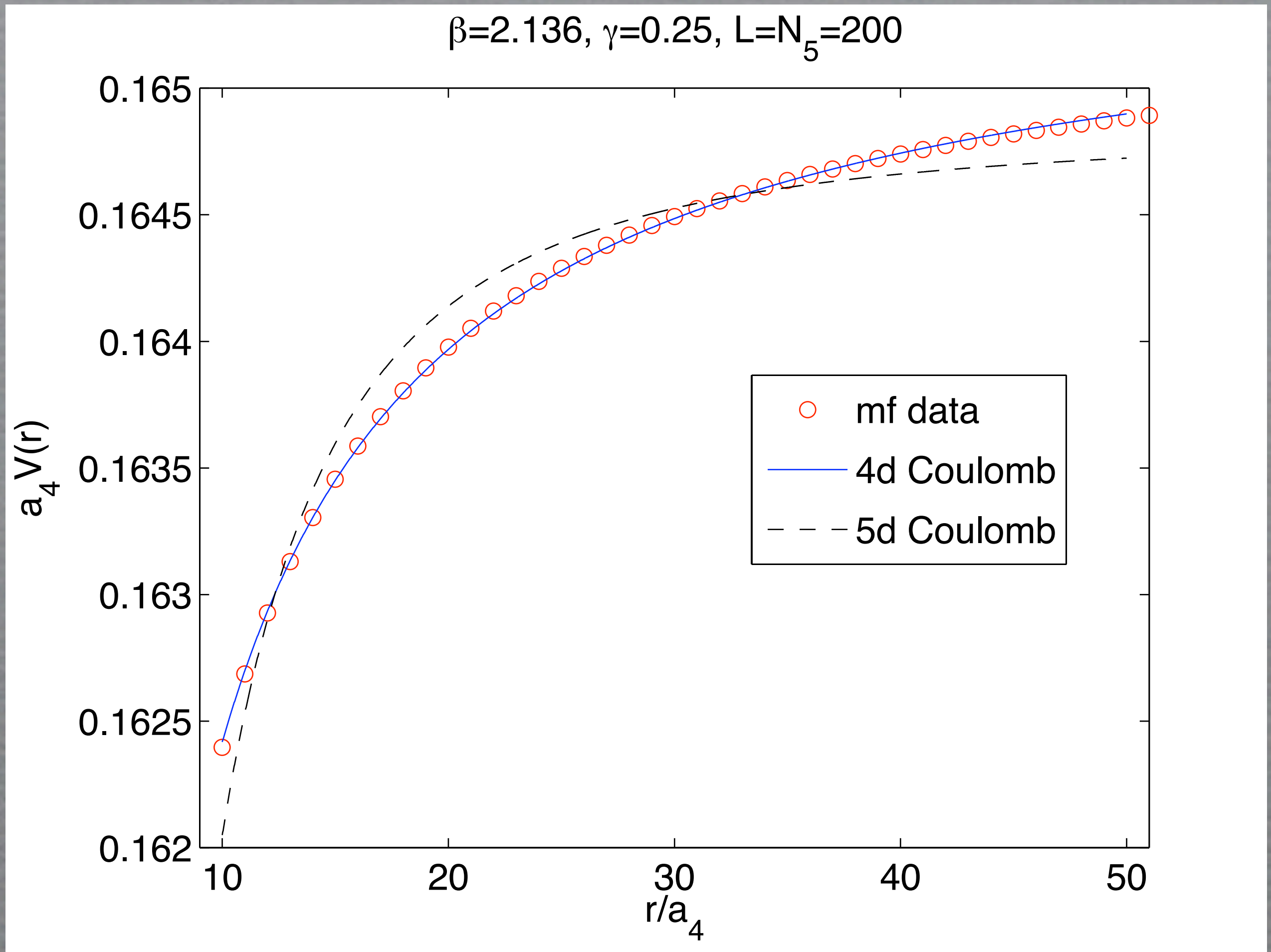
We are looking at an array of non-interacting 4D hyperplanes



The short distance static potential in the d-compact phase along the 4d hyperplanes:

β_4 large and β_5 small



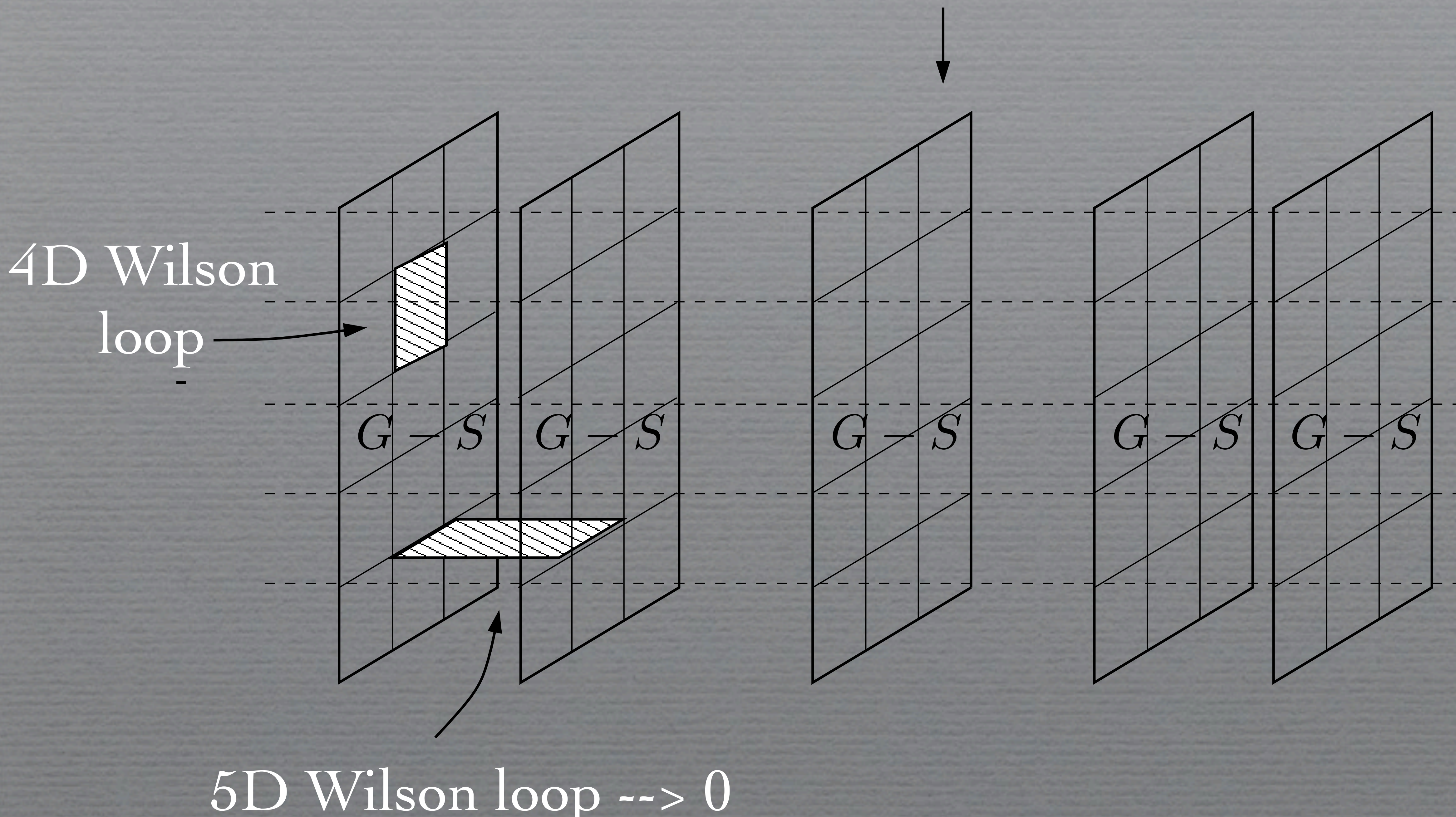


In the stability region r=10...50 the potential is clearly 4d Coulomb, large Yukawa excluded

$\longrightarrow$

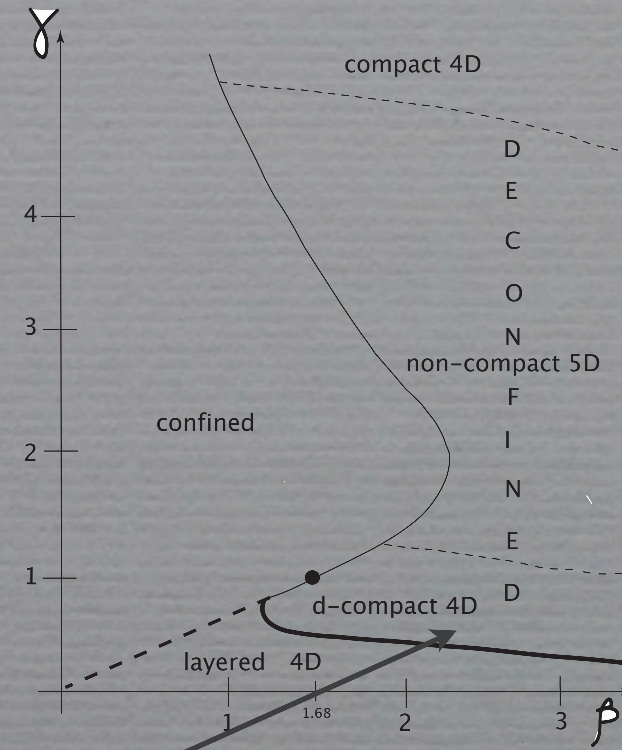
dimensional reduction in the d-compact phase

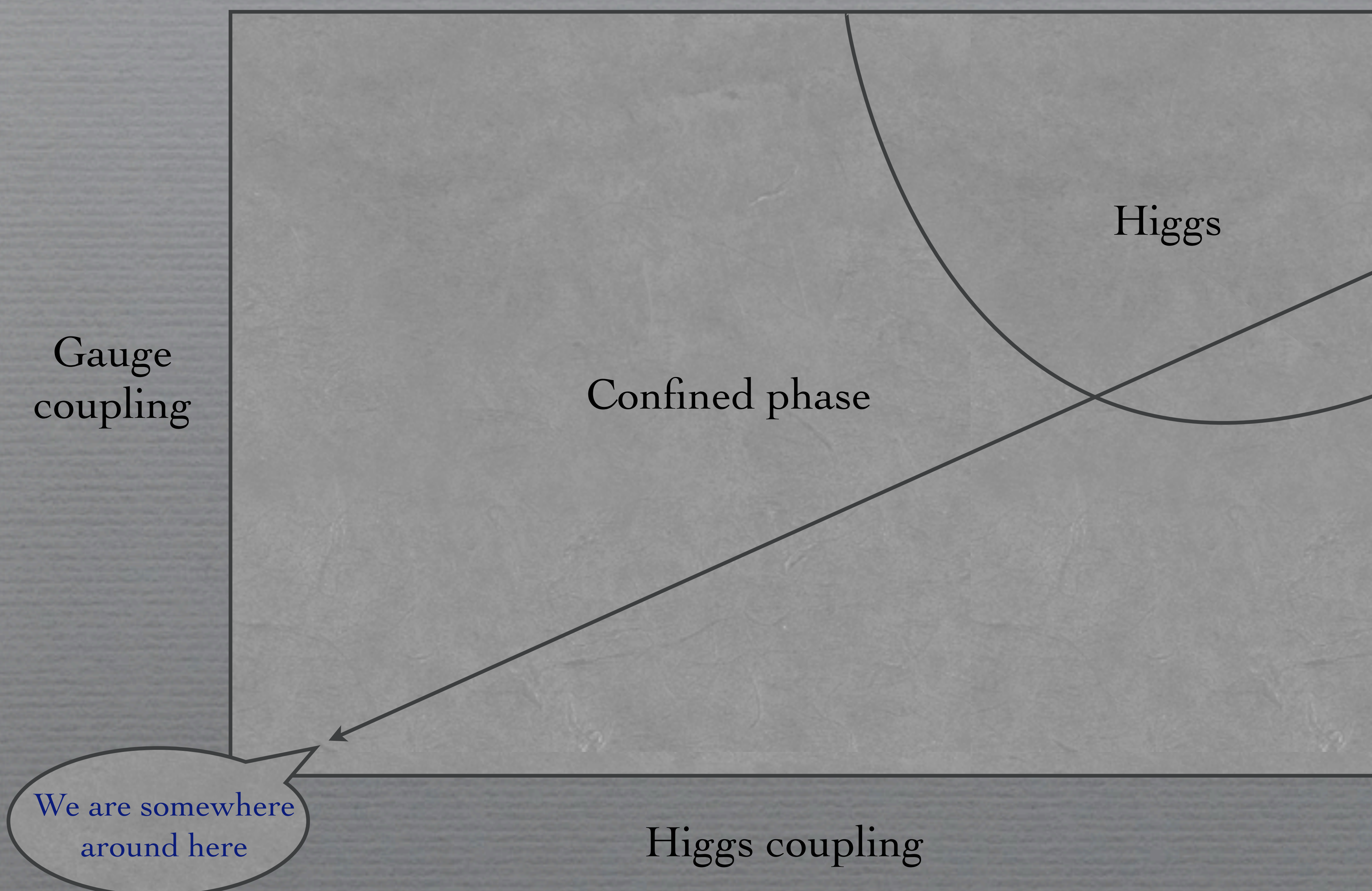
We are looking at an array of non-interacting 4D hyperplanes where gauge interactions are localized.
4D Georgi-Glashow on each of the branes



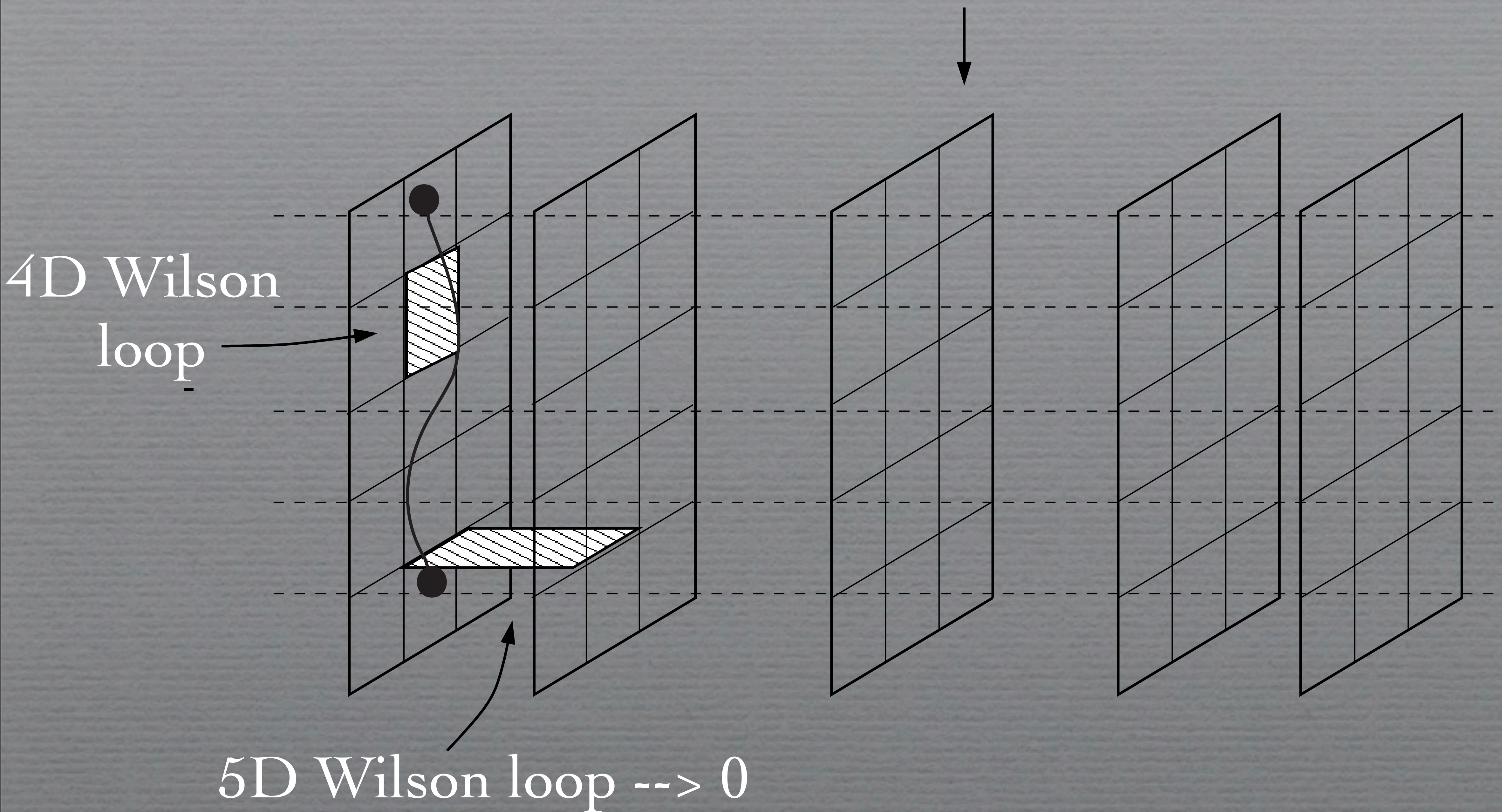
The localization is “perfect” in the infinite L limit

Remember: 4D Georgi-Glashow model
+ ~~KK~~



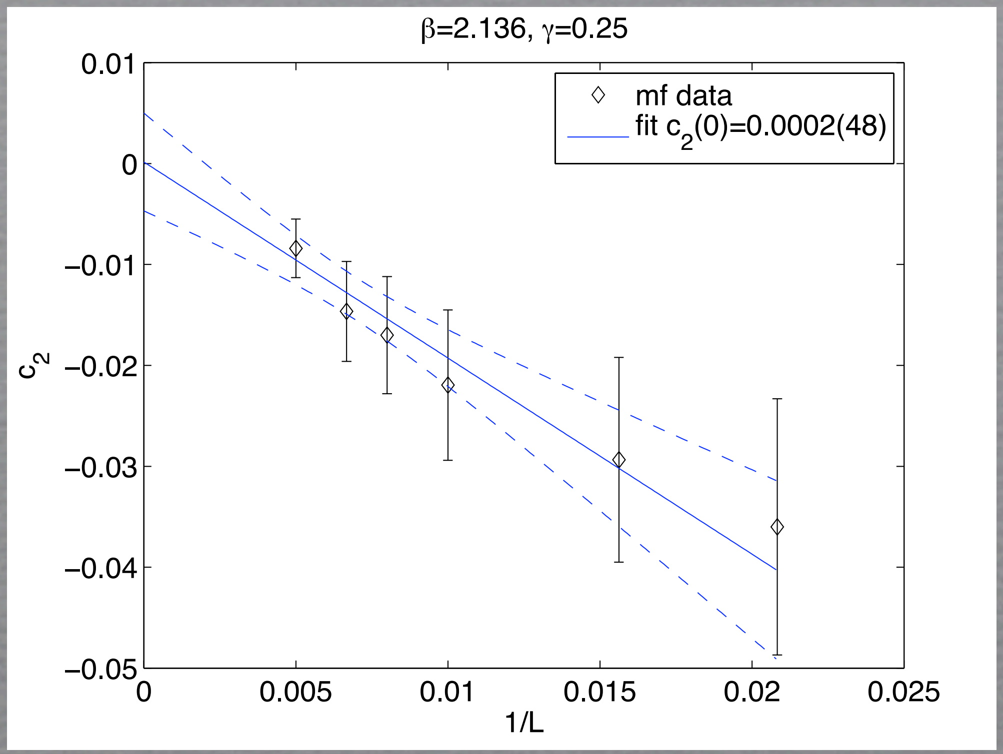


We are looking at an array of non-interacting 4D hyperplanes where gauge interactions are localized.
4D G-S on each of the branes.
The 4D Wilson loop at large r must describe a string

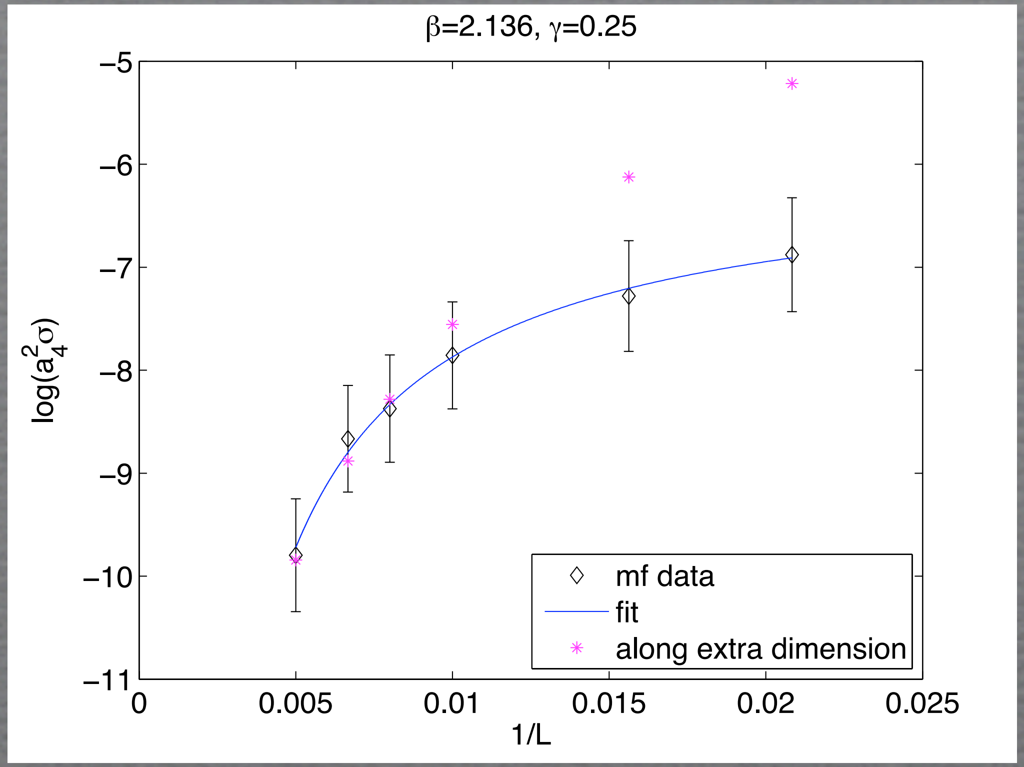


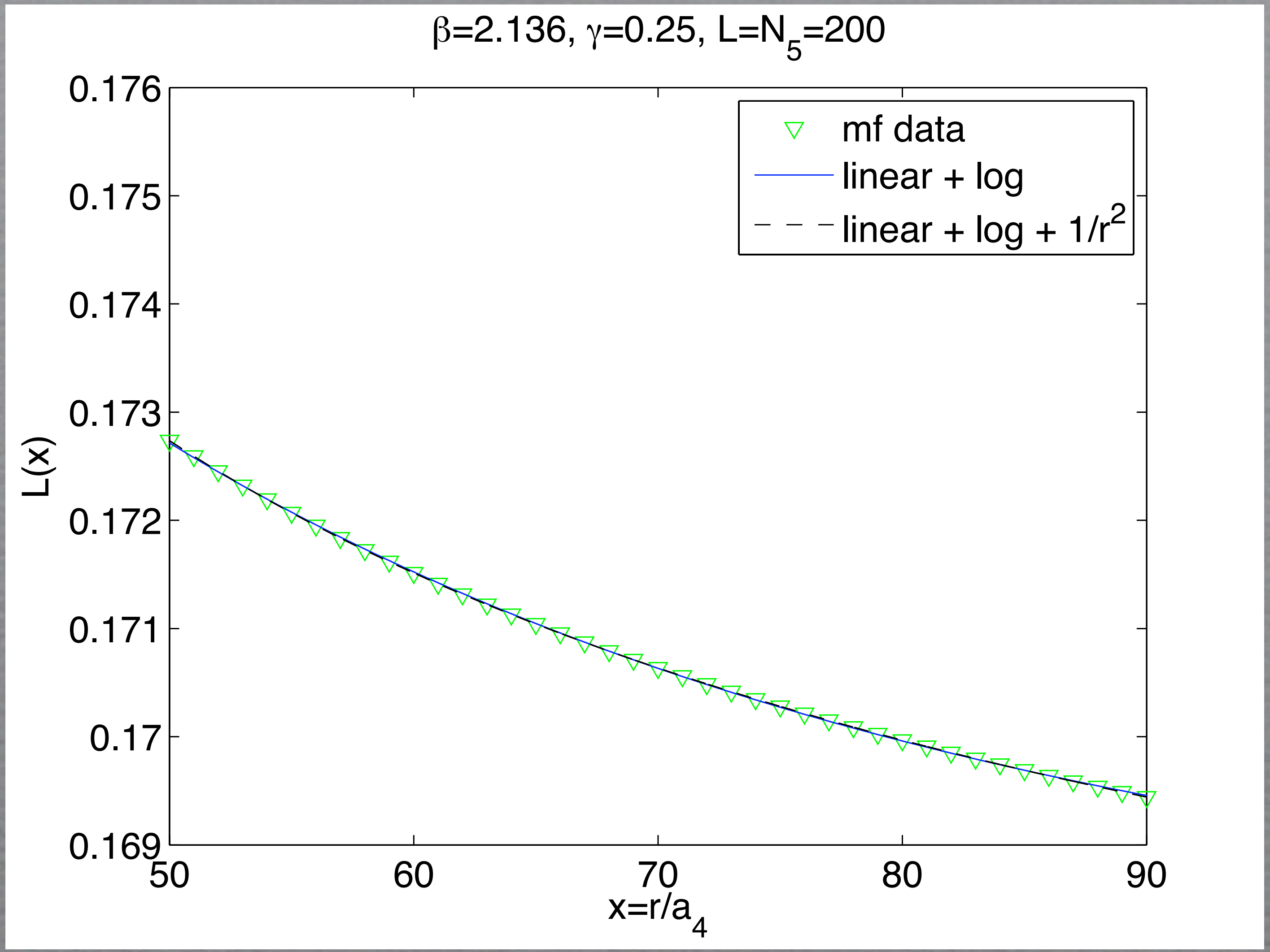
The long distance static potential in the d-compact phase along the 4d hyperplanes:

$$c_2 \sim 1/L$$



$$\overline{\sigma} = a_4^2 \sigma \sim c e^{-kL}$$





d=5 Luescher term

$$L(x) = a_4 \left[V(r) + \pi/(8r) \right]$$

and fit to $\overline{\sigma} x + \overline{\mu} + c_2 \ln(x)$

Conclusions

1. (Part of) the non-perturbative regime of 4d gauge theories can be probed analytically by the mean-field expansion in 5d

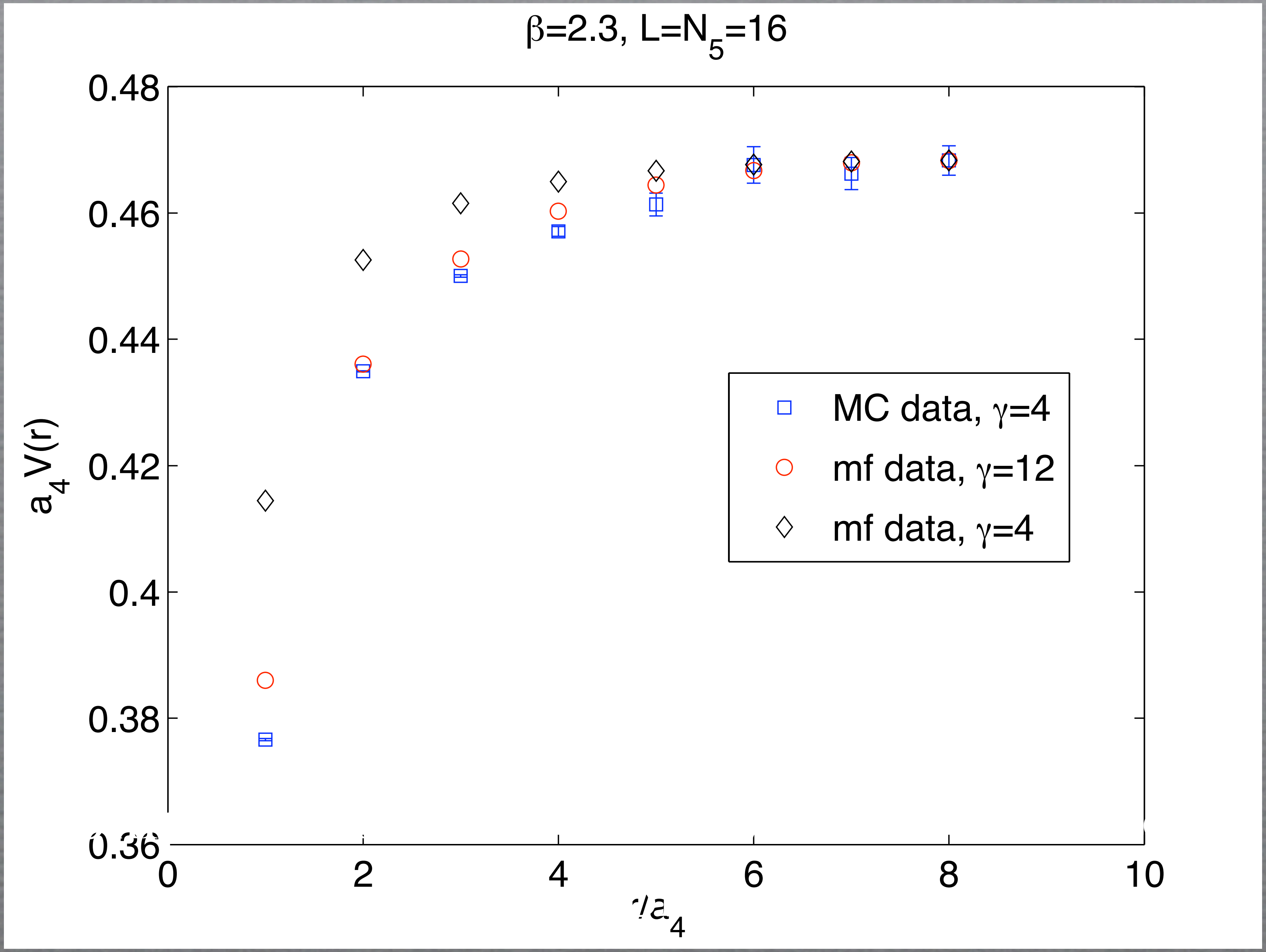
2. The phase diagram has a 2nd order phase transition where the system reduces dimensionally to an array of non-interacting "3-branes" and where the continuum limit can be taken

3. In a dimensionally reduced phase there may be traces of confinement and of the associated effective string

I would like to thank the

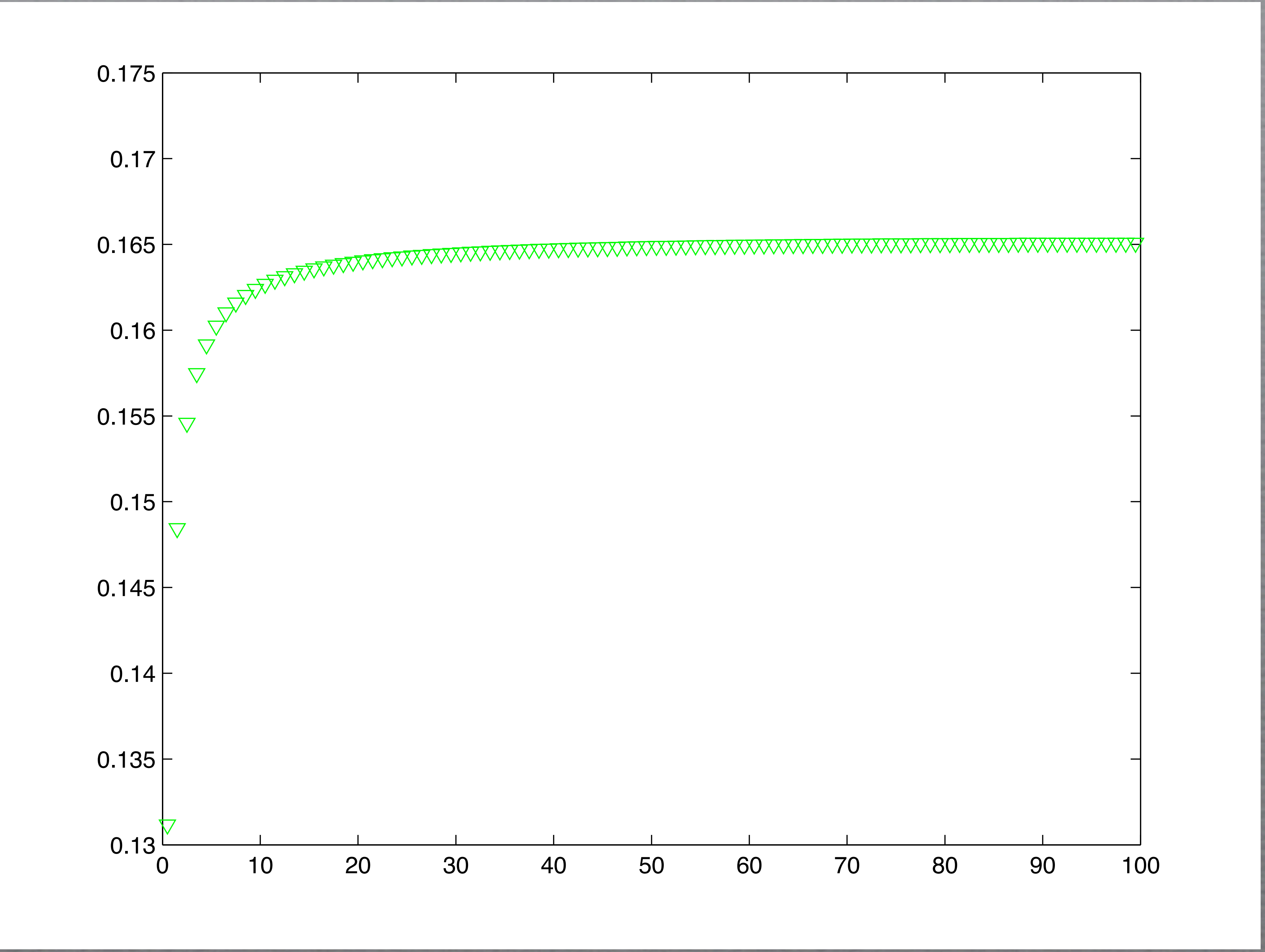
Alexander von Humboldt Foundation for finacial support

Comparison Mean-Field vs Monte Carlo: (sample)



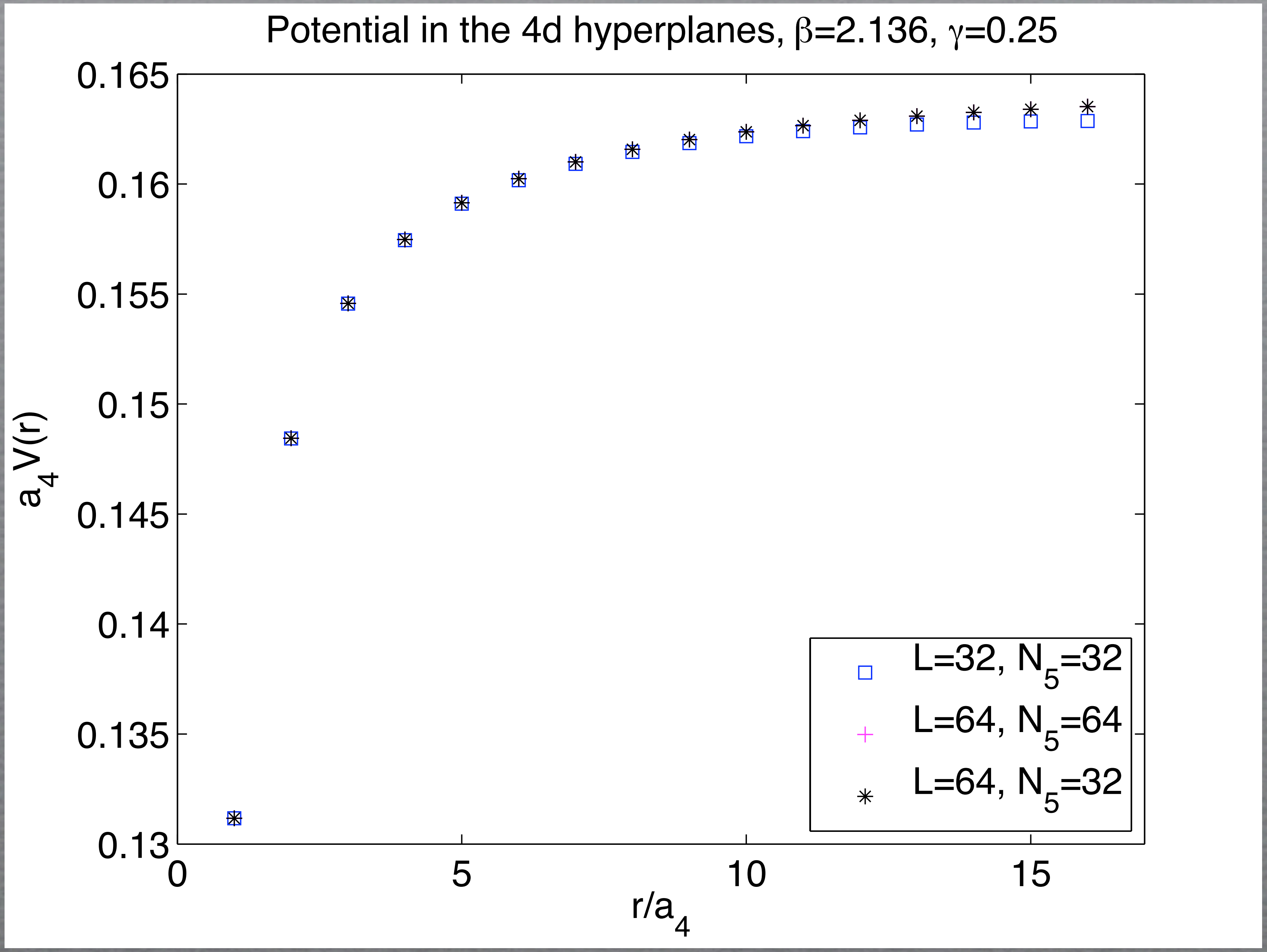
MC data generated by M. Luz

The raw data...



0.175
0.17
0.165
0.16
0.155
0.15
0.145
0.14
0.135
0.13
0
10
20
30
40
50
60
70
80
90
100

Yes, it scales with the lattice size...



No, it can not be 5D Coulomb or Yukawa: