Holographic Renormalization for Improved Holographic QCD and other backgrounds

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Holography as a tool

- Holography provides us with a powerful tool to study strongly coupled gauge theories.
- We would like to extend the holographic techniques to a wider range of physically important QFTs, beyond the much studied supersymmetric and relativistic examples.
- Ultimately, a full holographic duality must involve some QFT on one side and a complete theory of quantum gravity, such string theory, on the other.
- Nevertheless, there has been remarkable evidence that holographic models capture some universal properties of strongly coupled QFTs, such as transport coefficients in quark-gluon plasmas or QCD glueball spectra that are not too far from lattice computations.
- This gives us hope that we can build holographic models as effective descriptions of strongly coupled QFTs, which will provide us with an analytic tool to address long standing questions that cannot be easily answered with other present techniques.

- There have been recent efforts to construct and study holographic models for:
  - QCD
  - Quark-gluon plasmas
  - Superfluidity and high temperature superconductivity
  - Non-relativistic (Lifshitz) fixed points arising as quantum critical points
  - Strongly correlated atoms
- Most “bottom up” holographic models assume that the strongly coupled QFT in question admits some limit, such as large-N and large ’t Hooft coupling in the case of gauge theories, where the holographic model is well approximated by some (super)gravity theory.
- So, I will consider only holographic models within supergravity in this talk, at least as a starting point.

What is holography after all?

- Having restricted myself to the class of holographic models within supergravity, I would like to turn the question around and ask:

  Is there a systematic way to understand which supergravity theories and/or backgrounds can serve as holographic models, or what makes a supergravity theory/background a good holographic model?

- To address this question we need to make some simple and reasonable assumtions about the holographic dictionary, based on our experience with the Ads/CFT correspondence and other holographic dualities we understand relatively well.

The "axioms" of holography

1. The holographic model we are seeking is supposed to describe a local QFT, at least over a range of energy scales.
2. The observable quantities of the local QFT, as captured by correlation functions of gauge invariant operators are obtained from the supergravity theory via the standard relation

$$W[J^\alpha] = -S_{\mathrm{on-shell}}[\phi^\alpha; \phi \sim J^\alpha].$$

   where $W[J^\alpha]$ indicates the generating functional of connected correlation functions of a local gauge-invariant operator $\mathcal{O}_\alpha$, J^α is the source, and $\phi \sim J^\alpha$ indicates that the source enters in the supergravity description as some form of a boundary condition.
3. Any long-distance divergences of the on-shell supergravity action should be removable with a *finite* number of *local* in the sources counterterms, i.e.

$$W_{\mathrm{ren}}[J^\alpha] = -S_{\mathrm{on-shell}}[\phi^\alpha; \phi \sim J^\alpha] - S_{\mathrm{local}}[J^\alpha].$$

Holographic Renormalization

- To address the last point we need a generalization of Holographic Renormalization, that is a systematic analysis of the long-distance properties of supergravities in non-compact spacetimes.
- There is a natural way to formulate this problem:
  - Since spacetime is non-compact, there is a notion of asymptotic infinity where spacetime locally looks like
    $$\mathcal{M} \sim \mathbb{R} \times \Sigma_r,$$
    where r is a radial coordinate along which asymptotic infinity is approached, and Σ_r is a constant r slice.
  - This facilitates a Hamiltonian analysis of the supergravity theory in a $d+1$ decomposition, where the bulk supergravity fields are traded for the induced fields on the d-dimensional slices Σ_r and r serves as the Hamiltonian "time".
  - Hamilton-Jacobi theory gives directly the on-shell action
  - This provides us with a clear-cut first criterion for which supergravity theories/backgrounds can serve as legitimate holographic duals of local QFTs:

    Such theories must admit a radial Hamiltonian formulation and the corresponding Hamilton-Jacobi equation must admit solutions of the form
    $$\mathcal{S}_r = \mathcal{S}_{\text{local}} + \mathcal{S}_{\text{finite}},$$
    where $\mathcal{S}_{\text{local}}$ is local in the induced fields on Σ_r and $\mathcal{S}_{\text{finite}}$ admits a well-defined non-zero limit as $r \to \infty$.

Toy example...

- To illustrate how this criterion imposes constraints on the possible supergravity theories and backgrounds, let us consider a free massive scalar field in a pure gravitational background

$$ds^2_{d+1} = dr^2 + \gamma_{ij} dx^i dx^j = dr^2 + e^{2A(r)} d\vec{x}^2.$$

- The equation of motion following from the Lagrangian

$$L = \frac{1}{2} \int d^d x \sqrt{\gamma} \left(\dot{\varphi}^2 + \gamma^{ij} \partial_i \varphi \partial_j \varphi + m^2 \varphi^2 \right),$$

  is

$$\left(\partial_r^2 + d\dot{A}\partial_r - p^2 e^{-2A} - m^2 \right) \tilde{\varphi} = 0.$$

- The corresponding on-shell action is given by

$$\mathcal{S}_r = \frac{1}{2} \int_{\Sigma_r} d^d x \sqrt{\gamma} \varphi \dot{\varphi}.$$

- From Hamilton-Jacobi theory and the fact that the e.o.m. is linear we know we can write

$$\dot{\varphi} = f(A; p^2) \varphi,$$

  where the function $f(A; p^2)$ satisfies the equation

$$(f' + df)\dot{A} + f^2 - p^2 e^{-2A} - m^2 = 0.$$

- The function $A(r)$, or equivalently the function $\omega(A) \equiv \dot{A}$, specifies the background, while $f(A;p^2)$ determines the on-shell action:

$$\mathcal{S}_r = \frac{1}{2}\int_{\Sigma_r} d^d x \sqrt{\gamma}\varphi f(A;p^2)\varphi.$$

- For backgrounds that satisfy the holographic "axioms" it should be possible to find solutions to the above equation for f that take the form

$$f = \sum_{k=0}^{n} f_k(A)p^{2k} + \tilde{f}, \quad n < \infty.$$

- In order for $\tilde{f}$ to give a finite contribution to the on-shell action we must also have

$$\tilde{f}(A;p^2) \sim e^{-\int dA(d+2f_0/\omega)}, \text{ as } A \to \infty.$$

- f_0 satilfies
$$(f_0' + df_0)\omega(A) + f_0^2 - m^2 = 0,$$
while f_k, $k > 0$, satisfy *linear* equations:
$$\omega(A) f_k' + (d\omega(A) + 2f_0) f_k = \mathcal{R}_k, \quad k > 0,$$
where
$$\mathcal{R}_1 = e^{-2A}, \quad \mathcal{R}_k = -\sum_{l=1}^{k-1} f_l f_{k-l}, \quad k > 1.$$
- The homogeneous solution of the linear equations can be discarded as they are finite:
$$f_k^{\text{hom}} = c_k e^{-\int dA(d+2f_0/\omega)}, \text{ as } A \to \infty.$$
- The inhomogeneous solutions are given by
$$f_k = U(A) \int^A \frac{d\bar{A}}{U(\bar{A})} \mathcal{R}_k(\bar{A}), \quad U(A) \equiv e^{-\int dA(d+2f_0/\omega)}.$$
- The condition that $n < \infty$ requires that there exists n s.t.
$$f_{n+1} = o\left(e^{-\int dA(d+2f_0/\omega)}\right).$$
- This essentially restricts the background geometry to be asymptotically AdS: $A \sim r$

Improved Holographic QCD

Improved Holographic QCD (IHQCD) [Gursoy, Kiritsis, Nitti] was put forward as an effective 5-dimensional holographic model for low energy QCD, but allows for the logarithmic running of the gauge coupling in the UV. The above approach lets us decide if it satisfies the holographic "axioms" and, if it does, to construct the local counterterms.

- IHQCD is described by the action

$$S=-\frac{1}{2\kappa^2}\left(\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\left(R-\partial_\mu\varphi\partial^\mu\varphi-Z(\varphi)\partial_\mu\chi\partial^\mu\chi+V(\varphi)\right)+G.H.\right),$$

  where $(2\kappa^2)^{-1}=(16\pi G_5)^{-1}=M_{pl}^3N_c^2$, $\varphi=\xi\log\lambda$ is the dilaton, dual to the operator $\mathrm{Tr}\,F^2$, and χ is the axion, dual to $\mathrm{Tr}\,F\wedge F$.
- $V(\lambda)$ and $Z(\lambda)$ completely determine the bulk action. They are respectively related to the exact beta functions of the 't Hooft coupling and the instanton angle in the gauge theory.
- Asymptotic freedom requires that they admit expansions around $\lambda=0$ of the form

$$Z(\lambda)=(M_{pl}^3N_c^2)^{-1}\sum_{n=0}^{\infty}Z_n\lambda^n,\qquad V(\lambda)=\frac{12}{\ell^2}\left(1+\sum_{n=1}^{\infty}V_n\lambda^n\right),$$

  where ℓ is the radius of the AdS corresponding to the UV (free) fixed point of the gauge theory. These expansions correspond to the purturbative expansions of these beta functions.
- Z_n are $\mathcal{O}(N_c^0)$, since the axion is in the RR sector of the string theory and so its kinetic term is $\mathcal{O}(1/N_c^2)$ relative to the rest of the terms in the action.

Radial Hamiltonian analysis

- To proceed with the Hamiltonian analysis of this action, we start with the standard ADM-like decomposition of the metric

  $$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j,$$

  where N and N_i are respectively the lapse and shift functions, and γ_{ij} is the induced metric on the hypersurfaces Σ_r of constant radial coordinate r.
- The Lagrangian then takes the form

  $$\begin{aligned} 2\kappa^2 L \quad = \quad & -\int_{\Sigma_r} d^d x \sqrt{\gamma} N \left(R[\gamma] + K^2 - K^i_j K^j_i \right) \\ & + \int_{\Sigma_r} d^d x \sqrt{\gamma} N \left\{ \frac{1}{N^2}(\dot{\varphi}^2 + Z(\varphi)\dot{\chi}^2) - \frac{2N^i}{N^2}(\dot{\varphi}\partial_i\varphi + Z(\varphi)\dot{\chi}\partial_i\chi) \right. \\ & \left. + \left(\gamma^{ij} + \frac{N^i N^j}{N^2} \right) (\partial_i\varphi\partial_j\varphi + Z(\varphi)\partial_i\chi\partial_j\chi) - V(\varphi) \right\}. \end{aligned}$$

  where K_{ij} is the extrinsic curvature of the hypersurface Σ_r and $K \equiv K^i_i$.

- The canonical momenta conjugate to γ_{ij}, φ and χ are now obtained respectively as

$$\pi^{ij} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(K\gamma^{ij} - K^{ij}),$$
$$\pi_\varphi = \frac{1}{\kappa^2 N}\sqrt{\gamma}\left(\dot{\varphi} - N^i\partial_i\varphi\right),$$
$$\pi_\chi = \frac{1}{\kappa^2 N}\sqrt{\gamma}Z(\varphi)\left(\dot{\chi} - N^i\partial_i\chi\right).$$

- The Hamiltonian is given by

$$H = \int_{\Sigma_r} d^d x\left(\pi^{ij}\dot{\gamma}_{ij} + \pi_\varphi\dot{\varphi} + \pi_\chi\dot{\chi}\right) - L = \int_{\Sigma_r} d^d x\left(N\mathcal{H} + N_i\mathcal{H}^i\right),$$

where

$$\begin{aligned}
\mathcal{H} &= 2\kappa^2\gamma^{-\frac{1}{2}}\left(\pi^i_j\pi^j_i - \frac{1}{d-1}\pi^2 + \frac{1}{4}\pi_\varphi^2 + \frac{1}{4}Z^{-1}(\varphi)\pi_\chi^2\right) \\
&\quad + \frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi - Z(\varphi)\partial_i\chi\partial^i\chi + V(\varphi)\right), \\
\mathcal{H}^i &= -2D_j\pi^{ij} + \pi_\varphi\partial^i\varphi + \pi_\chi\partial^i\chi.
\end{aligned}$$

The Hamilton-Jacobi equation

- Hamilton-Jacobi theory now gives the canonical momenta as derivatives of the on-shell action:

$$\pi^{ij} = \frac{\delta \mathcal{S}_r}{\delta \gamma_{ij}}, \quad \pi_\varphi = \frac{\delta \mathcal{S}_r}{\delta \varphi}, \quad \pi_\chi = \frac{\delta \mathcal{S}_r}{\delta \chi},$$

  where γ_{ij}, ϕ^α should be understood as “boundary conditions” on Σ_r.

- This follows from the identity

$$\pi^{ij}\delta\gamma_{ij} + \pi_\varphi \delta\varphi + \pi_\chi \delta\chi = \delta\mathcal{L} + \partial_i v^i(\delta\gamma, \delta\varphi, \delta\chi),$$

  for some local $v^i(\delta\gamma, \delta\phi)$, where $\mathcal{S}_r = \int_{\Sigma_r} d^dx \mathcal{L}(\gamma, \phi)$.

- Inserting these momenta into the Hamiltonian constraint $\mathcal{H} = 0$ leads to the Hamilton-Jacobi equation

$$2\kappa^2 \gamma^{-\frac{1}{2}} \left(\left(\gamma_{ik}\gamma_{jl} - \tfrac{1}{d-1}\gamma_{ij}\gamma_{kl}\right) \tfrac{\delta \mathcal{S}_r}{\delta \gamma_{ij}} \tfrac{\delta \mathcal{S}_r}{\delta \gamma_{kl}} + \tfrac{1}{4}\left(\tfrac{\delta \mathcal{S}_r}{\delta \varphi}\right)^2 + \tfrac{1}{4} Z^{-1}(\varphi) \left(\tfrac{\delta \mathcal{S}_r}{\delta \chi}\right)^2 \right)$$
$$+ \tfrac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi - Z(\varphi)\partial_i\chi\partial^i\chi + V(\varphi)\right) = 0$$

Long-distance solution of the H-J equation

- The aim now is to solve this equation for long distances, or for $r \to \infty$. In this limit the on-shell action takes the form

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} U(\varphi, \chi),$$

  for some function $U(\varphi, \chi)$. Note that this is a non-trivial statement about the asymptotic form of the induced fields. It implies, for example, that

$$\gamma_{ij} \sim e^{2A} g_{(0)ij}(x), \quad A = -\frac{1}{d-1} \int^{\varphi} \frac{d\bar{\varphi}}{U'(\bar{\varphi})} U(\bar{\varphi}),$$

  where $g_{(0)ij}(x)$ is an arbitrary metric independent of the radial coordinate.
- We can systematically compute corrections to this action as egenfunctions of the operator

$$\delta_\gamma = \int d^d x 2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}},$$

  namely

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(2)} + \mathcal{S}_{(4)} + \cdots, \quad \delta_\gamma \mathcal{S}_{(2n)} = (d-2n)\mathcal{S}_{(2n)}.$$

- Applying the above identity to the variation δ_γ we obtain

$$2\pi_{(2n)} = (d-2n)\mathcal{L}_{(2n)} + \partial_i v^i{}_{(n)}.$$

  Since $\mathcal{L}$ is defined up to a total derivative, we can absorb the last term in $\mathcal{S}_{(2n)}$ such that

$$2\pi_{(2n)} = (d-2n)\mathcal{L}_{(2n)}.$$

- Inserting the above expansion of the on-shell action in the Hamiltonian constraint and matching terms of equal δ_γ eigenvalue we obtain

$$(\partial_\varphi U)^2 + Z^{-1}(\varphi)(\partial_\chi U)^2 - \frac{1}{d-1}U^2 + V(\varphi) = 0,$$

$$U'(\varphi)\frac{\delta}{\delta\varphi}\int d^d x \mathcal{L}_{(2n)} - \left(\frac{d-2n}{d-1}\right)U(\varphi)\mathcal{L}_{(2n)} = \mathcal{R}_{(2n)}, \quad n>0,$$

where

$$\begin{aligned}
\mathcal{R}_{(2)} &= -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi - Z(\varphi)\partial_i\chi\partial^i\chi\right), \\
\mathcal{R}_{(2n)} &= -2\kappa^2\gamma^{-\frac{1}{2}}\sum_{m=1}^{n-1}\left(\pi_{(2m)}{}^i_j\pi_{(2(n-m))}{}^j_i - \frac{1}{d-1}\pi_{(2m)}\pi_{(2(n-m))}\right. \\
&\qquad \left. + \frac{1}{4}\pi_{\varphi(2m)}\pi_{\varphi(2(n-m))} + \frac{1}{4}Z^{-1}(\varphi)\pi_{\chi(2m)}\pi_{\chi(2(n-m))}\right), \quad n>1.
\end{aligned}$$

- The linear equation for $\mathcal{L}_{(2n)}$, $n>0$, admits the homogeneous solution

$$\mathcal{L}^{hom}_{(2n)} = F_{(2n)}[\gamma]\exp\left(\left(\frac{d-2n}{d-1}\right)\int^\varphi \frac{d\bar\varphi}{U'(\bar\varphi)}U(\bar\varphi)\right) = F_{(2n)}[\gamma]e^{-(d-2n)A},$$

where $F_{(2n)}[\gamma]$ is a covariant function of the induced metric of weight $d-2n$, and $\mathcal{N}_\varphi$ is the number of scalars that U depends on. As in the toy model, the homogeneous solution contributes only to *finite* local terms and can therefore be discarded.

- We are therefore only interested in the inhomogeneous solution:

$$\mathcal{L}_{(2n)} = e^{-(d-2n)A(\varphi)} \int^{\varphi} \frac{d\bar{\varphi}}{U'(\bar{\varphi})} e^{(d-2n)A(\bar{\varphi})} \mathcal{R}_{(2n)}(\bar{\varphi}).$$

- Evaluating this integral is straightforward if $\mathcal{R}_{(2n)}$ does not involve derivatives of the dilaton, φ, but it requires some caution when it does. In that case, one needs to use the freedom to add total derivatives to $\mathcal{L}_{(2n)}$ to write the integrand as

$$\frac{\delta\varphi}{U'(\varphi)} e^{(d-2n)A(\varphi)} \mathcal{R}_{(2n)}(\varphi) = \delta_{\varphi} F_{(2n)} + e^{(d-2n)A(\varphi)} \partial_i {v_{(2n)}}^i(\delta\varphi), \tag{1}$$

where $F_{(2n)}$ and v^i are to be determined. Then,

$$\mathcal{L}_{(2n)} = e^{-(d-2n)A(\varphi)} F_{(2n)}. \tag{2}$$

- These integrals can be evaluated systematically to obtain the local divergent part of the on-shell action.
- This then leads to the full expression for the covariant counterterms for $d = 4$:

$$
\begin{aligned}
S_{ct} \quad = \quad & -\frac{1}{8\pi G_5}\int_{\Sigma_r} d^4x\sqrt{\gamma}\lambda\left\{U(\lambda) - \frac{1}{2}\xi^2 e^{-2A}\int^{\lambda}\frac{d\bar{\lambda}}{\bar{\lambda}^2 U'(\bar{\lambda})}e^{2A(\bar{\lambda})}R\right. \\
& +\xi^2\frac{U}{U'}\Xi'\lambda^{-2}\partial^i\lambda\partial_i\lambda + \frac{1}{2}\xi^2 e^{-2A}\int^{\lambda}\frac{d\bar{\lambda}}{\bar{\lambda}^2 U'(\bar{\lambda})}e^{2A(\bar{\lambda})}Z(\bar{\lambda})\partial^i\chi\partial_i\chi \\
& +\log e^{-2r}\frac{\ell^3}{16}\left[R_{ij}R^{ij} - \frac{1}{3}R^2 - \frac{Z_0}{M_{pl}^3 N_c^2}\left(R^{ij}\partial_i\chi\partial_j\chi - \frac{1}{6}R\partial^i\chi\partial_i\chi\right.\right. \\
& \left. -6b_0^{-2}\lambda^{-4}\partial_i\lambda\partial^i\lambda\partial_j\chi\partial^j\chi + 6b_0^{-1}\lambda^{-2}\partial_i\lambda\partial_j\chi D^iD^j\chi + D^iD^j\chi D_iD_j\chi\right) \\
& \left.\left. -\frac{2}{3}\frac{Z_0^2}{(M_{pl}^3 N_c^2)^2}\partial_i\chi\partial^i\chi\partial_j\chi\partial^j\chi\right]\right\},
\end{aligned}
$$

where

$$
\Xi(\lambda) = -\frac{1}{2}\xi^2 e^{-(d-2)A}\int^{\lambda}\frac{d\bar{\lambda}}{\bar{\lambda}^2 U'(\bar{\lambda})}e^{(d-2)A(\bar{\lambda})},
$$

and b_0 is related to the logarithmic asymptotics of the dilaton via the relation

$$
U(\lambda) = -\frac{d-1}{\ell} - \frac{\xi^2 b_0}{\ell}\lambda + \mathcal{O}(\lambda^2).
$$

Yet another example...

- Kovtun and Nickel have proposed the following supergravity action as a candidate for a holographic dual to non-relativistic backgrounds with Schrödinger symmetry:

$$S = \frac{1}{2\kappa^2}\int d^{d+3}x\sqrt{-g}\left(R - \frac{a}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{-a\phi}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu - V(\phi)\right),$$

  where

$$V(\phi) = (\Lambda + \Lambda')e^{a\phi} + (\Lambda - \Lambda')e^{b\phi},$$

$$\Lambda = -\frac{1}{2}(d+1)(d+2), \quad \Lambda' = \frac{1}{2}(d+2)(d+3), \quad m^2 = 2(d+2), \quad a = (d+2)b = \frac{2(d+2)}{d+1}.$$

- This action admits the non-relativistc vacuum solution

$$ds^2_{d+3} = dr^2 + e^{2r}\left(d\vec{x}^2 - 2dx^+dx^-\right) - \beta^2 e^{4r}(dx^+)^2, \quad A = e^{2r}dx^+,$$

  as well as black holes with the same asymptotics.
- Such non-relativistic solutions were recently put forward as holographic duals of non-relativistic systems, e.g. strongly correlated fermionic atoms [Son; Balasubramanian, McGreevy] .
- Using the approach discussed above, one can easily show that this action satisfies the “axioms” of holography and the local counterterms can be obtained explicitly.

Summary & Conclusions

- Holographic techniques provide a powerful analytic tool to address long standing problems in strongly coupled QFTs.
- In our quest to extend the range of applicability of these techniques to ever more physically relevant systems, it is important to have a systematic way of classifying supergravity theories/solutions that can potentially serve as holographic duals to local quantum field theories.
- Holographic Renormalization can be understood as the analysis of supergravity (String Theory) at long distances and provides a general and systematic way of addressing this question.