The fuzzy S^2 structure of M2-M5 systems in ABJM

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(w/ H. Nastase and S. Ramgoolam, arXiv:0903.3966 and in progress)

Motivation

Important developments over last year relating to effective action for multiple M2-branes. Following leads from Bagger-Lambert and Gustavsson involving 3-algebras, ABJM wrote a CS action in 3d with $\mathrm{U}(N) \times \mathrm{U}(\bar{N})$ gauge fields, coupled to bifundamental matter [Aharony-Bergman-Jafferis-Maldacena]

$$\begin{aligned} S = \quad & \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \mathrm{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right. \\ & -\mathrm{Tr} D_\mu C_I^\dagger D^\mu C^I + \frac{4\pi^2}{3k^2} \mathrm{Tr} \left(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K \right. \\ & \left. \left. + 4 C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6 C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger \right) \right] \end{aligned}$$

- $\mathcal{N} = 6$ superconformal, $\mathrm{SU}(4)$ R-symmetry
- Can use $\lambda = N/k$ as 't Hooft coupling

- For $\lambda \ll 1$ gauge theory weakly coupled
- For $\lambda \gg 1$ string theory dual in terms of near horizon limit of N M2 branes on a $\mathbb{C}^4/\mathbb{Z}_k$ singularity.
- For small k M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$
- The orbifold acts on the circle of the Hopf fibration $S^1 \hookrightarrow S^7 \xrightarrow{\pi} \mathbb{CP}^3$
- For large k Type IIA on $\mathrm{AdS}_4 \times \mathbb{CP}^3 \rightarrow (\mathrm{AdS}_4/\mathrm{CFT}_3)$

Even though M2-brane physics emerge at strong coupling this is significant progress!

But what about the M5-brane?

M5-brane potentially emerges from M2-brane through generalisation of Myers effect:

In the presence of external $p+4$-form flux Dp-branes polarise into Dp+2-branes with worldvolume scalars obeying

$$[X_i, X_j] = 2i\epsilon_{ijk}X_k$$

This is defining equation for fuzzy S^2 of radius $R^2 \propto \frac{1}{N}\mathrm{Tr}X_iX_i \sim N^2$

At large-N this approaches the classical sphere with the noncommuting matrices becoming Euclidean coordinates on S^2

$$\frac{X_i}{N} \to x_i$$

This idea can be realised in M-theory:

Require that M2s blow up into a fuzzy three-sphere with the matriceal scalars becoming fuzzy directions on M5:

- As a vacuum of a mass-deformed theory of M2s (akin to $\mathcal{N} = 1^*$) [Bena]
- Set of M2-M5 geometries with $\mathrm{SO}(4) \times \mathrm{SO}(4)$ symmetry and 16 supercharges also constructed [Bena-Warner, Lin-Lunin-Maldacena]
- Via an M2$\perp$M5 intersection or 'fuzzy funnel' (see BIon) [Basu-Harvey]

$\rightarrow$ Revisit these constructions within ABJM

Interesting mass-deformation of ABJM found by GRvV: Split complex scalars into $C^I = (R^\alpha, Q^{\dot\alpha})$ and introduce potential [Gomis-Rodríguez Gómez-van Raamsdonk-Verlinde, Hosomichi-Lee^3-Park]

$$V = |M^\alpha|^2 + |N^\alpha|^2$$

where

$$\begin{aligned} M^\alpha =& \mu Q^\alpha + \frac{2\pi}{k}(2Q^{[\alpha}Q^\dagger_\beta Q^{\beta]} + R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta + 2Q^\beta R^\dagger_\beta R^\alpha) \\ N^\alpha =& -\mu R^\alpha + \frac{2\pi}{k}(2R^{[\alpha}R^\dagger_\beta R^{\beta]} + Q^\beta Q^\dagger_\beta R^\alpha - R^\alpha Q^\dagger_\beta Q^\beta + 2R^\beta Q^\dagger_\beta Q^\alpha) \end{aligned}$$

- Breaks conformal invariance and R-symmetry
  $\mathrm{SU}(4) \to \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$
- Preserves $\mathcal{N} = 6$ supersymmetry

GRvV also found set of classical vacua for $Q^{\dot\alpha} = 0$. Need to solve

$$R^\alpha = \frac{2\pi}{k\mu}(R^\alpha R^\dagger_\beta R^\beta - R^\beta R^\dagger_\beta R^\alpha)$$

which leads to

$$R^\alpha = f G^\alpha \qquad \text{for} \qquad f^2 = \frac{k\mu}{2\pi}$$

where

$$G^1 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \sqrt{N-2} & \\ & & & & \sqrt{N-1} \end{pmatrix} \qquad G^2 = \begin{pmatrix} 0 & \sqrt{N-1} & & & \\ & 0 & \ddots & & \\ & & \ddots & \sqrt{2} & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

(for $f^2 = \frac{k}{2\pi s}$ the above is 'funnel' solution in undeformed ABJM, where s the direction along which M2s extend away from the M5)

The G^αs should encode all information about geometry. GRvV also notice that

$$R^\alpha R^\dagger_\alpha = X^1X^1 + X^2X^2 + X^3X^3 + X^4X^4 \equiv R^2$$

Looks like S^3. Extrapolating to $k = 1$ seem to get the M5-brane. At finite N the 'three-sphere' is fuzzy.
[Terashima, Hanaki-Lin]

However, the solution has $G^1 = G^\dagger_1$. Immediately reduces

$$G^\alpha G^\dagger_\alpha = X^1X^1 + X^2X^2 + X^3X^3 = N - 1$$

Q: where is the (fuzzy) S^3? [Nastase, CP, Ramgoolam]

Outline

- G^{α} matrix algebra
- Fuzzy S^2 realisation in terms of bifundamentals
- Small fluctuation analysis
- D4/M5 interpretation
- Summary and Open questions

G^α matrix algebra

Fuzzy sphere: discretisation of S^n while retaining $\mathrm{SO}(n+1)$ isometry.

Immediately note that 'usual' $\mathrm{SO}(4)$-covariant fuzzy S^3 of Guralnik-Ramgoolam cannot be at work: solution has only got $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry.

Investigate algebra of G^αs: First construct bilinears $J^\alpha_\beta = G^\alpha G^\dagger_\beta$. These obey U(2) algebra

$$[J^\alpha_\beta, J^\mu_\nu] = \delta^\mu_\beta J^\alpha_\nu - \delta^\alpha_\nu J^\mu_\beta$$

Further defining $J_i = (\sigma_i^T)_\beta^\alpha J_\alpha^\beta$ picks out traceless combinations and leads to SU(2) algebra

$$[J_i, J_j] = 2i\epsilon_{ijk} J_k$$

Similar things hold for $\bar{J}_\beta^\alpha = G_\beta^\dagger G^\alpha$

$$[\bar{J}_i, \bar{J}_j] = 2i\epsilon_{ijk} \bar{J}_k$$

Looks like there are two independent $\mathrm{SU}(2)$s.

Fields in the adjoint of $\mathrm{U}(N)$ admit a decomposition in terms of J_is.

This is an expansion in terms of fuzzy spherical harmonics

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm}(J_i)$$

with

$$\hat{Y}_{lm}(J_i) = \sum_{i} \alpha_{lm}^{(i_1 \ldots i_l)} J_{i_1} \ldots J_{i_l}$$

Once again, there is another set of fuzzy spherical harmonics for $\mathrm{U}(\bar{N})$ adjoint fields, in an expansion in terms of $\bar{J}_i$s.

However, the algebra for odd products of G^α or $G^\dagger_\alpha$ will combine the two. One finds that

$$
\begin{aligned}
J_i G^\alpha - G^\alpha \bar{J}_i &= (\sigma_i^T)^\alpha_\beta G^\beta \\
G^\dagger_\alpha J_i - \bar{J}_i G^\dagger_\alpha &= G^\dagger_\beta (\sigma_i^T)^\beta_\alpha
\end{aligned}
$$

and only a diagonal $\mathrm{SU}(2)$ survives.

This is a sign that the G^αs describe a single S^2. The large-N limit helps clarify this.

It is known that at large-N limit the J_is and $\bar{J}_i$s approach the commutative Euclidean coordinates on the 'classical' sphere

$$
\frac{J_i}{\sqrt{N^2-1}} \to x_i \qquad \text{and} \qquad \frac{\bar{J}_i}{\sqrt{(N^2-1)^2-1}} \to \bar{x}_i
$$

One can also define some classical object corresponding to

$$\frac{G^\alpha}{\sqrt{N}} \to g^\alpha \qquad \text{and} \qquad \frac{G^\dagger_\alpha}{\sqrt{N}} \to g^\dagger_\alpha$$

Then notice that for the defining equation of J_i

$$\frac{J_i}{N} = \frac{1}{N}(\sigma_i^T)^\alpha_\beta G^\beta G^\dagger_\alpha \to x_i = (\sigma_i^T)^\alpha_\beta g^\beta g^\dagger_\alpha$$

This looks like first Hopf map $S^3 \xrightarrow{\pi} S^2$. Defined modulo a $\mathrm{U}(1)$ phase. In order to match the degrees of freedom we need to consider the objects defined by extracting the phase. These can be identified with Killing spinors on S^2.

The G^αs should then be thought of as fuzzy Killing spinors on S^2. [Nastase, CP – in progress]

Another interesting picture is the following: Embed everything into $2N \times 2N$ super-matrices

$$\mathbf{J}_i = \begin{pmatrix} J_i & 0 \\ 0 & \bar{J}_i \end{pmatrix} \qquad \text{and} \qquad \mathbf{J}_\alpha = \begin{pmatrix} 0 & \sqrt{N} G_\alpha \\ -\sqrt{N} G_\alpha^\dagger & 0 \end{pmatrix}$$

Then we can summarise the relations

$$\begin{aligned} [\mathbf{J_i}, \mathbf{J_j}] &= 2i\epsilon_{ijk}\mathbf{J_k} \\ [\mathbf{J_i}, \mathbf{J}_\alpha] &= (\sigma_i^T)_{\alpha\beta}\mathbf{J}^\beta \\ \{\mathbf{J}_\alpha, \mathbf{J}_\beta\} &= -(\sigma_i^T)_{\alpha\beta}\mathbf{J_i} \end{aligned}$$

This is an $\mathrm{OSp}(1|2)$ super-algebra, capturing the isometries of the fuzzy supersphere. In the large-N limit they become the coordinates on the classical supersphere. [Grosse-Reiter, Hasebe-Kimura]

The (fuzzy) Killing spinors and hence the bifundamental matter fields live on the spinor bundle over S^2.

The bifundamental fields also admit harmonic decomposition under the $\mathrm{SU}(2)$, which can either be given in terms of (fuzzy) spinor spherical harmonics

or

One can define new fields by 'extracting' the G^α part and having an expansion in terms of usual (fuzzy) spherical harmonics.

$$R^\alpha = R^\alpha_\beta G^\beta \qquad \text{with} \qquad R^\alpha_\beta = \sum_{l,m} (a_{lm})^\alpha_\beta \hat{Y}_{lm}(J_i)$$

This construction will apply more generally to gauge theories with bifundamental matter admitting fuzzy sphere solutions. [Maldacena-Martelli]

One can also have vacua in terms of reducible representations

$$G^{\alpha} = G_1^{\alpha} \oplus G_2^{\alpha} \oplus \dots$$

These should have interpretation in terms of concentric higher dimensional branes wrapping the S^2.

Small Fluctuation Analysis

Next look at (bosonic) quadratic fluctuation action around irreducible GRvV vacuum. Fluctuation parameter is $\frac{1}{f^2} = \frac{2\pi}{\mu k}$

$$R^\alpha = fG^\alpha + r^\alpha \ , \qquad Q^{\dot\alpha} = q^{\dot\alpha} \ , \qquad A_\mu = a_\mu \ , \qquad \hat{A}_\mu = \hat{a}_\mu$$

The radius of the sphere can once again be defined as $R^2 = \frac{2}{N}\mathrm{Tr} R^\alpha R^\dagger_\alpha = 8\pi^2 f^2 N l_p^3$.

The r^α further decompose into trace and traceless parts

$$r^\alpha = r\delta^\alpha_\beta G^\beta + s_i \frac{1}{2}(\sigma_i^T)^\alpha_\beta G^\beta \ , \qquad r^\dagger_\alpha = G^\dagger_\alpha r + G^\dagger_\beta s_i \frac{1}{2}(\sigma_i^T)^\beta_\alpha$$

Use Matrix Theory technology to convert into fields on smooth (classical) S^2 at large N.

[Iso-Kimura-Tanaka-Wakatsuki, CP-Ramgoolam-Toumbas]

At large N the mode s_i can be further decomposed into radial and angular components on S^2

$$s_i = x_i \phi + K_i^a A_a$$

with K_i^a a set of Killing vectors on the sphere such that $x_i K_i^a = 0$.

For the 'transverse' scalars write similarly

$$q^{\dot{\alpha}} = Q_{\alpha}^{\dot{\alpha}} G^{\alpha}$$

Then use standard dictionary between large N fuzzy sphere matrices and functions on the sphere

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm}(J_i) \to a(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x_i)$$

$$\frac{1}{N} \mathrm{Tr} \to \int d\Omega$$

$$\frac{J_i}{N} \to x_i$$

$$[J_i, \hat{a}] \to -2iK_i^a \partial_a a(\theta, \phi)$$

How about gauge fields? Expanding around the GRvV solution also triggers version of Higgs mechanism present in ABJM (CS-matter) theories.

When a single scalar gets a large, trivial vev the diagonal subgroup of the two non-dynamical CS gauge fields become dynamical. Matter gets promoted from bifundamental to adjoint. CS-matter $\rightarrow$ YM [Mukhi-CP, Aharony-Bergman-Jafferis-Maldacena]

In M-theory corresponds to moving all M2-branes far away from the $\mathbb{C}^4/\mathbb{Z}_k$ singularity.

However, for the case at hand things significantly more complicated: half fields get a vev and that has nontrivial matrix structure (proportional to G^α)

To cut a (very) long story short:

- The calculation involves combination of large-N Matrix Theory techniques + Higgsing
- Through the above: $(r, s_i) \rightarrow (r + \phi, A_a) \equiv (\Phi, A_a)$
- Non-dynamical $(A_\mu, \hat{A}_\mu)$ become a dynamical $\tilde{A}_\mu = A_\mu + \hat{A}_\mu$
- The combination $(r - \phi)$ played role of Goldstone mode and does not appear
- ‘Transverse’ scalars in action can be written as real scalars X^I, $I = 1, ..., 4$.

The final answer for the bosonic sector is:

$$S^B_{\text{flucts}} = \frac{1}{g^2_{YM}} \int d^3x d^2\sigma \sqrt{h} \Big[-\frac{1}{4} F_{AB}F^{AB} - \frac{1}{2}\partial_A \Phi \partial^A \Phi - \frac{1}{2}\partial_A X^I \partial^A X^I \\ - \frac{3}{4}\mu^2 X^I X^I - \frac{\mu^2}{2}\Phi^2 + \frac{\mu}{2}\, \omega^{ab} F_{ab}\Phi \Big]$$

where $A = \{\mu, a\}$ and the metric $g_{AB} = (\eta_{\mu\nu}, h_{ab})$ with the size of the S^2 set by μ^{-1}. Finally $g^2_{YM} = g_s l_s$.

This is the action one expects for fluctuations around a single spherical D4 in non-trivial background with μ dependence.

Result very reminiscent to similar calculation in the BMN matrix model [Maldacena-Sheikh Jabbari-van Raamsdonk]

D4/M5 interpretation

Thus: D4-brane - no M5-brane! Since the fluctuation action is valid at weak coupling this makes sense. However, S^2 structure and realisation is now explicit.

Interesting related question involves D4 compactification and susy. Usually D-branes with (partly) curved worldvolumes realise a twisted version of susy, although in some special cases (e.g. giant gravitons in AdS) that's not necessary.
[Bershadsky-Sadov-Vafa, Maldacena-Núñez, Andrews-Dorey]

This should be obvious at the level of the action once we include the fermionic sector.

At strong coupling $\rightarrow$ M5-brane

How does this tally with initial geometric description in terms of G^αs?

- Relation between J_i and G^α hints to the Hopf fibration
  $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$
- The D4 is wrapping the S^2 base of the above bundle. At large N this base is smooth. At finite N it becomes fuzzy.
- At weak coupling (large k) the S^1 fibre has shrunk
- At strong coupling (small k) the S^1 fibre (M-theory direction) unfolds and one ends up with an M5-brane wrapping an $S^3/\mathbb{Z}_k$
- Due to large degree of susy this interpretation ought to be similar for all sphere sizes, controlled by $f^2 = \frac{\mu k}{2\pi}$.

Summary...

- Analysed ground state solutions for mass-deformed ABJM theory
- Found that the algebra obeyed by G^α's gives rise to fuzzy S^2
- In fact G^α correspond to fuzzy Killing spinors on S^2 and lead to realisation in terms of bifundamentals
- Classical GRvV solutions do not describe (fuzzy) S^3; instead describe S^2 base of Hopf fibration
- S^1 fibre coincides with M-theory direction and is beyond perturbative regime

- If M5-brane is to emerge at small k, as expected, that would only be seen by non-perturbative effects
- Fluctuation analysis around irreducible vacuum adds credibility to this picture
- Starting from M5 wrapped on S^3 the D4 on S^2 is obtained by a double-dimensional reduction
- Gravity duals obtained by a $\mathbb{Z}_k$ orbifold of the appropriate M-theory LLM geometries
  [Auzzi-Kumar]

... & open questions

- Analyse the fermion sector of the fluctuation action
- Clarify D4-brane picture - background - susy
- Repeat fluctuation analysis for reducible vacua
- Reducible vacua of mass-deformed ABJM describe multiple, concentric spherical M5-branes at strong coupling
- Finite N effects
- Use all of the above in a clever way to get information about M5-branes in M-theory.