Cecile DeWitt-Morette

The Power of Functional Integration

Functional integration is underutilized

Two challenges within reach

- Jackiw's Problem
  Fermion charge fractionalization in the presence of a topological defect.
  Reference: R.W. Jackiw "Topology and fractional charge: where to find it in a functional integral" (Proceedings Dresden, 2007.)

- Quantum fluctuations of the DeSitter metric;
  Brownian motion on symmetric spaces.
  Reference: T. Damour and A. A. Starobinski (on the drawing board)

A much greater challenge

Quantum Gravity

Ref. P. Cartier and C.DW: Functional Integration: Action and Symmetries (CUP 2006)

A definition:

$$\int_a^b dx := b - a$$

Global Properties: Domain and domain

Domain of integration: a function space $\mathcal{F}$

$$f \in \mathcal{F}$$

f a variable of integration

f : domain $\rightarrow \mathbb{M}^D$ (Riemannian, multiply connected, symmetric, etc...)

f may have some analytic properties (Sobolev, Poisson, etc...)

f may satisfy some boundary conditions

Two examples of well studied Domains $\mathcal{F}$

- Contractible space of paths
- Spaces of Poisson paths

Given $\mathcal{F}$, define volume elements $\mathcal{D}f$ in $\mathcal{F}$
(e.g. a gaussian volume element)

Characterize functionals F of f integrable w.r.t. $\mathcal{D}f$

A differential property:

The Koszul formula gives the transformation property of a volume element ω under a group of transformations generated by a vector field X

$$\mathcal{L}_X \omega = \mathbf{Div}_\omega(X) \cdot \omega$$

Divergence and Gradient

$$(X|\nabla F)_\omega = -(\mathbf{Div}_\omega(X)|F)_\omega$$

$\mathbf{Div}_\omega$ Divergence wrt ω
∇ Gradient
$(|)_\omega$ Global scalar product

Given a differential equation on $\mathbb{M}^D$, parametrized by *z:*

$$
\begin{aligned}
x \quad &: \quad \mathbb{T} \to \mathbb{M}^D \\
dx(t,z) \quad &:= \quad \sum_A X_{(A)}\left(x(t,z)\right) dz^A(t)
\end{aligned}
$$

then

$$x(t,z) = \mathbf{x}_0 \cdot \Sigma(t,z) \qquad \mathbf{x}_0 \in \mathbb{M}^D$$

$\sum$ is a right action on $\mathbb{M}^D$ defined by the vector fields $\{X_{(A)}\}_A$ on $\mathbb{M}^D$

A basic theorem

Definitions:

$$\mathcal{F} := \mathcal{P}_0\mathbb{R}^D \quad , \quad s \in \{1, i\} \quad , \quad z : \mathbb{T} \to \mathbb{R}^D$$

$$Q(z) := \int_{\mathbb{T}} dt\, h_{AB}\dot{z}^A(t)\dot{z}^B(t) < \infty$$

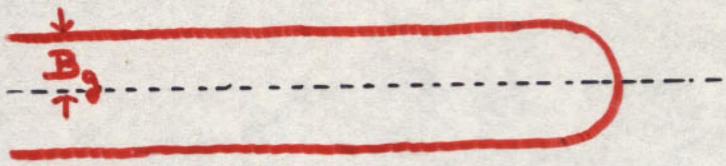
$$\begin{aligned}\Psi(t, \mathbf{x}_0) &:= \int_{\mathcal{F}} \mathcal{D}_{s,Q}z\ \exp\left(-\frac{\pi}{s}Q(z)\right)\phi(\mathbf{x}_0 \cdot \Sigma(t, z)) \\ &\equiv \int_{\mathcal{F}} \mathcal{D}\Gamma(z)\,\phi(\mathbf{x}_0 \cdot \Sigma(t, z))\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial\Psi}{\partial t} &= \frac{s}{4\pi}h^{AB}\mathcal{L}_{X_{(A)}}\mathcal{L}_{X_{(B)}}\Psi \\ \Psi(t_0, \mathbf{x}) &= \phi(\mathbf{x})\end{aligned}$$

Glory Scattering

$$d\sigma_{cl}(\Omega) = B(\theta)\,\frac{dB}{d\theta}\,\frac{d\Omega}{\sin\theta} \qquad \theta \neq \pi, 0$$



$$d\sigma_{WKB}(\Omega) = 4\pi^2\lambda^{-1}B^2(\theta)\,\frac{dB}{d\theta}\,J^2_{2s}(2\pi\lambda^{-1}B_g\sin\theta)\,d\Omega, \qquad \theta\approx\pi$$

Example: gravitational wave scattering by a Schwarzschild black hole.

$$B(\theta) = M\,(3\sqrt{3} + 3.48\exp(\theta-\pi)) \qquad \theta\approx\pi$$