(2,2) and (4,4) Sigma Models and Complex Geometry

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Outline

- Background
- (Generalized) Complex geometry
- (2,2) superspace
- Duality
- Bi-hermitean geometry
- New vector multiplets
- Additional supersymmetry

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$$\phi^i : \Sigma \to \mathcal{T}$$

$$S = \int_\Sigma d\phi^i g_{ij}(\phi) \star d\phi^j$$

$$\nabla^2 \phi^i := \partial^2 \phi^i + \partial\phi^j \Gamma_{jk}{}^i \partial\phi^k = 0$$

Supersymmetry:

$$\phi(x) \to \phi(x, \theta)$$

$$S \to S = \int dx D^2 \bar{D}^2 \, K(\phi, \bar{\phi})$$

$$= \int dx (\partial\phi \, g_{\phi\bar{\phi}}(\phi, \bar{\phi}) \bar{\partial}\bar{\phi} + \ldots)$$

where

$$g_{\phi\bar{\phi}}(\phi,\bar{\phi}) = \partial_\phi\partial_{\bar{\phi}}K(\phi,\bar{\phi})$$

$\Longleftrightarrow$ $\mathcal{T}$ carries Kähler Geometry

$$\boxed{\text{Susy } \sigma \text{ models} \Longleftrightarrow \text{Geometry of } \mathcal{T}}$$

(1,1) analysis by Gates Hull and Roček in d=2:

N=(2,2): bi-hermitean geometry

N=(4,4): bi-hypercomplex

Complex Geometry

Complex structure: $J : TM \circlearrowleft \qquad J^2 = -1$

Nijenhuis: $\mathcal{N}(J) = 0 \quad \Longleftrightarrow \quad \pi_{\mp}[\pi_{\pm}u, \pi_{\pm}v] = 0$

Hermitean Metric: $\quad J^t g J = g$

Kähler: $\quad \nabla J = 0, \qquad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z, \bar{z})$

Hyperkähler: $J^A, A = 1, 2, 3 \quad J^A J^B = -\delta^{AB} + \epsilon^{ABC} J^C$

Generalized Complex Geometry

Complex structure: $\mathcal{J} : TM \oplus T^*M \circlearrowleft \qquad \mathcal{J}^2 = -1$

"Nijenhuis": $\mathcal{N}_C(\mathcal{J}) = 0 \quad \Longleftrightarrow \quad \Pi_\mp[\Pi_\pm u, \Pi_\pm v]_C = 0$

Hermitean Metric: $\quad \mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I}$

Generalized Kähler:

$$\exists \ \ (\mathcal{J}_1, \mathcal{J}_2) \ ; \ [\mathcal{J}_1, \mathcal{J}_2] = 0$$

$$\mathcal{G} = -\mathcal{J}_1 \mathcal{J}_2 \ , \ \ \mathcal{G}^2 = 1$$

Kähler:

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad \mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

The (2,2)-D-algebra:

$$\{D_\pm, \bar{D}_\pm\} = 2i\partial_{\substack{+\\=}}$$

Reduction to (1,1):

$$\mathbb{D}_\pm := \frac{1}{\sqrt{2}}\left(D_\pm + \bar{D}_\pm\right)$$
$$\mathbb{Q}_\pm := \frac{i}{\sqrt{2}}\left(D_\pm - \bar{D}_\pm\right)$$

The (1,1)-D-algebra:

$$\mathbb{D}_\pm^2 = i\partial_{\substack{+\\=}}$$

Chiral fields ϕ:

$$\bar{D}_\pm \phi = 0 \Rightarrow D_\pm \bar{\phi} = 0$$

Twisted chiral fields χ:

$$\bar{D}_+ \chi = D_- \chi = 0 \Rightarrow D_+ \bar{\chi} = \bar{D}_- \bar{\chi} = 0$$

Left/Right semi-chiral fields $\mathbb{X}_{L/R}$:

$$\bar{D}_+ \mathbb{X}_L = 0 \Rightarrow D_+ \bar{\mathbb{X}}_L = 0$$

$$\bar{D}_- \mathbb{X}_R = 0 \Rightarrow D_- \bar{\mathbb{X}}_R = 0$$

These are all the fields needed.

Complex linear fields Σ_ϕ:

$$\bar{D}_+\bar{D}_-\Sigma_\phi = 0 \Rightarrow D_+D_-\bar{\Sigma}_\phi = 0$$

Dual to chiral fields

Complex twisted linear fields Σ_χ:

$$\bar{D}_+D_-\Sigma_\chi = 0 \Rightarrow D_+\bar{D}_-\bar{\Sigma}_\chi = 0$$

Dual to twisted chiral fields

N=(1,1) content

Define:

$$J := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Chiral fields:

$$\Phi := \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix} \Rightarrow \mathbb{Q}_{\pm}\Phi = JD_{\pm}\Phi$$

Twisted chiral fields:

$$\chi := \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} \Rightarrow \mathbb{Q}_{\pm}\chi = \pm JD_{\pm}\chi$$

Read off the non-manifest second susy by projecting to the θ_2 independent part.

Semi-chiral fields:

$$X_{L/R} := \mathbb{X}_{L/R}|, \quad \psi_{L-/R+} := \mathbb{Q}_{\mp}\mathbb{X}_{L/R}|$$

$$\mathbf{X}_{L/R} := \begin{pmatrix} X_{L/R} \\ \bar{X}_{L/R} \end{pmatrix}, \quad \Psi_{L-/R+} := \begin{pmatrix} \psi_{L-/R+} \\ \bar{\psi}_{L-/R+} \end{pmatrix}$$

$$\mathbb{Q}_{+}\mathbf{X}_{L} = JD_{+}\mathbf{X}_{L}, \quad \mathbb{Q}_{-}\mathbf{X}_{R} = JD_{-}\mathbf{X}_{R}$$

and

$$\mathbb{Q}_{+}\Psi_{L-} = JD_{+}\Psi_{L-}, \quad \mathbb{Q}_{-}\Psi_{L-} = -i\partial_{=}\mathbf{X}_{L}$$

$$\mathbb{Q}_{-}\Psi_{R+} = JD_{-}\Psi_{R+}, \quad \mathbb{Q}_{+}\Psi_{R+} = -i\partial_{+\!\!+}\mathbf{X}_{R}$$

The Ψ's are auxiliary fermions.

Duality

Back to the chiral complex linear duality:

$$\mathcal{S} = \int d^2xD^2\bar{D}^2K(\Sigma,\bar{\Sigma})$$
$$\to \tilde{\mathcal{S}} = \int d^2xD^2\bar{D}^2\left(K(S,\bar{S}) - \phi S - \bar{\phi}\bar{S}\right)$$

$$\delta_\phi\tilde{\mathcal{S}} = \delta_{\bar{\phi}}\tilde{\mathcal{S}} = 0 \Rightarrow \bar{D}_+\bar{D}_-S = 0, D_+D_-S = 0$$
$$\Rightarrow S = \Sigma, \bar{S} = \bar{\Sigma}, \quad \tilde{\mathcal{S}} \to \mathcal{S}$$

$$\delta_S\tilde{\mathcal{S}} = \delta_{\bar{S}}\tilde{\mathcal{S}} = 0 \iff K_S = \phi, K_{\bar{S}} = \bar{\phi}$$
$$\Rightarrow K - \phi S - \bar{\phi}\bar{S} = \hat{K}(\phi,\bar{\phi})$$

Relation to GKG

★

$$
\begin{aligned}
\mathcal{S} &= \int d^2xD^2\bar{D}^2K(\phi,\bar{\phi},\chi,\bar{\chi},\mathbb{X}_{L/R},\bar{\mathbb{X}}_{L/R}) \\
&\to \int d^2x\left(\partial_{+\!\!+}\varphi^i(g_{ij}+B_{ij})\partial_{=}\varphi^j+.....\right).
\end{aligned}
$$

In $(1,1)$:

$$\delta_\Psi\mathcal{S}=0:,\quad \Rightarrow (J_\pm,g,H=dB)\ ,$$

$$J_\pm^2=-1,\qquad N(J)=0,\quad [J_+,J_-]\neq 0\ ,$$

$$J_\pm^t gJ_\pm=g,\quad H=d_+^c\omega_+=-d_-^c\omega_-$$

A complete description of GKG.

New vector multiplets

$$K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_{L/R}, \bar{\mathbb{X}}_{L/R})$$

(Abelian) Isometries:

$$k_\phi = i(\partial_\phi - \partial_{\bar{\phi}})$$

$$k_{\phi\chi} = i(\partial_\phi - \partial_{\bar{\phi}} - \partial_\chi + \partial_{\bar{\chi}})$$

$$k_{LR} = i(\partial_L - \partial_{\bar{L}} - \partial_R + \partial_{\bar{R}})$$

The corresponding gauged Lagrangians:

$$K_\phi(\phi + \bar{\phi} + V^\phi, x)$$

$$K_{\phi\chi}(\phi + \bar{\phi} + V^\phi, \chi + \bar{\chi} + V^\chi, i(\phi - \bar{\phi} + \chi - \bar{\chi}) + V', x)$$

$$K_{\mathbb{X}}(\mathbb{X}_L + \bar{\mathbb{X}}_L + \mathbb{V}^L, \mathbb{X}_R + \bar{\mathbb{X}}_R + \mathbb{V}^R, i(\mathbb{X}_L - \bar{\mathbb{X}}_L + \mathbb{X}_R - \bar{\mathbb{X}}_R) + \mathbb{V}', x)$$

with gauge transformations for the vectors;

$$\delta V^\phi = i(\bar{\Lambda} - \Lambda)$$

$$\delta V^\chi = i(\bar{\tilde{\Lambda}} - \tilde{\Lambda})$$

$$\delta V' = \bar{\Lambda} + \Lambda + \bar{\tilde{\Lambda}} + \tilde{\Lambda}$$

$$\delta \mathbb{V}^{L/R} = i(\bar{\Lambda}_{L/R} - \Lambda_{L/R})$$

$$\delta \mathbb{V}' = \bar{\Lambda}_L + \Lambda_L + \bar{\Lambda}_R + \Lambda_R$$

The invariant field strengths are the usual ones

$$W = iD_-\bar{D}_+V^\phi, \quad \bar{W} = i\bar{D}_-D_+V^\phi$$

$$\tilde{W} = i\bar{D}_-\bar{D}_+V^\chi, \quad \bar{\tilde{W}} = iD_-D_+V^\chi$$

and the new

$$\mathbb{F} = \frac{1}{2}\bar{D}_+\bar{D}_-(\mathbb{V}' + i(\mathbb{V}^L + \mathbb{V}^R))$$

$$\tilde{\mathbb{F}} = \frac{1}{2}\bar{D}_+D_-(\mathbb{V}' + i(\mathbb{V}^L - \mathbb{V}^R))$$

$$\mathbb{G}_+ = \frac{1}{2}\bar{D}_+(V' + i(V^\phi + V\chi)) := \frac{1}{2}\bar{D}_+V$$

$$\mathbb{G}_- = \frac{1}{2}\bar{D}_-(V' + i(V^\phi - V^\chi)) := \frac{1}{2}\bar{D}_-\tilde{V}$$

Reduction to (1,1). Non-abelian extensions. Applied to T-duality.

T-duality

$$K_{\phi\chi}(\phi + \bar{\phi} + V^{\phi}, \chi + \bar{\chi} + V^{\chi}, i(\phi - \bar{\phi} + \chi - \bar{\chi}) + V')$$

$$-\frac{1}{2}\mathbb{X}_L V - \frac{1}{2}\bar{\mathbb{X}}_L \bar{V} - \frac{1}{2}\mathbb{X}_R \tilde{V} - \frac{1}{2}\bar{\mathbb{X}}_R \bar{\tilde{V}}$$

$\delta\mathbb{X}_{L,R}$:

$\Rightarrow$ V and $\tilde{V}$ pure gauge. Plug back to find $K_{\phi\chi}(\phi, \bar{\phi}, \chi, \bar{\chi})$

$\delta V, \delta\tilde{V}$:

$\Rightarrow$ $\partial_V K_{\phi\chi} = \mathbb{X}_L$ etc. Solve to give $V(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$,......

Plug back to find $\hat{K}(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{L,R})$

A similar relation starting from the gauged semi-chiral action also displays the duality between (twisted) chiral and semi-chiral models.

Additional supersymmetry

The chiral sector:

$$K \to K(\phi, \bar{\phi})$$

$$\delta\phi^a = \bar{\epsilon}^\alpha \bar{D}_\alpha \Omega^a(\phi, \bar{\phi}), \quad \delta\bar{\phi}^{\bar{a}} = \epsilon^\alpha D_\alpha \bar{\Omega}^{\bar{a}}(\phi, \bar{\phi})$$

On-shell algebra.

$$J^{(3)i}{}_j = \begin{pmatrix} i\delta^a_b & 0 \\ 0 & -i\delta^{\bar{a}}_{\bar{b}} \end{pmatrix},$$

$$J^{(1)i}{}_j = \begin{pmatrix} 0 & \Omega^a_{\bar{b}} \\ \bar{\Omega}^{\bar{a}}_b & 0 \end{pmatrix}, \quad J^{(2)i}{}_j = \begin{pmatrix} 0 & -i\Omega^a_{\bar{b}} \\ i\bar{\Omega}^{\bar{a}}_b & 0 \end{pmatrix}$$

The semi-sector:

$$K \to K(\mathbb{X}_L, \mathbb{X}_R, \bar{\mathbb{X}}_L, \bar{\mathbb{X}}_R)$$

The general situation not known at the $(2,2)$ level.

Linear tf:

$$\delta\mathbb{X}_L = i\bar{\epsilon}^+\bar{\mathbb{D}}_+(\bar{\mathbb{X}}_L + \mathbb{X}_R + \tfrac{1}{\kappa}\bar{\mathbb{X}}_R) + i\kappa\bar{\epsilon}^-\bar{\mathbb{D}}_-\mathbb{X}_L + i\kappa^{-1}\epsilon^-\mathbb{D}_-\mathbb{X}_L,$$

$$\begin{aligned}\delta\mathbb{X}_R = {} & i\bar{\epsilon}^-\bar{\mathbb{D}}_-(\bar{\mathbb{X}}_R - (|\kappa|^2 - 1)\mathbb{X}_L + \tfrac{|\kappa|^2-1}{\bar{\kappa}}\bar{\mathbb{X}}_L) - i\bar{\kappa}\bar{\epsilon}^+\bar{\mathbb{D}}_+\mathbb{X}_R \\ & -i\bar{\kappa}^{-1}\epsilon^+\mathbb{D}_+\mathbb{X}_R,\end{aligned}$$

Invariance:

$$\begin{aligned} K_{1\bar{1}} - K_{12} - \kappa K_{\bar{1}2} &= 0, \\ (|\kappa|^2 - 1)K_{2\bar{2}} + K_{12} - \bar{\kappa}K_{1\bar{2}} &= 0. \end{aligned}$$

$$
\begin{gathered}
\{J_+, J_-\} = 2c\;, \\
\Longleftrightarrow \\
(1+c)|K_{12}|^2 + (1-c)|K_{1\bar{2}}|^2 = 2K_{1\bar{1}}K_{2\bar{2}}.
\end{gathered}
$$

Using the invariance condition we find:

$$
\boxed{c = -\frac{|\kappa|^2+1}{|\kappa|^2-1}}
$$

Since $c^2 > 1$ is a constant we can form the following two local product structures:

$$
\begin{aligned}
S &:= \frac{1}{\sqrt{c^2-1}}(J_- + cJ_+), \qquad & S^2 = 1\;, \\
T &:= \frac{1}{2\sqrt{c^2-1}}[J_+, J_-], \qquad & T^2 = 1\;,
\end{aligned}
$$

such that the commutator algebra of (J_+, S, T) is $SL(2,\mathbb{R})$.

The structures (J_+, S, T) preserve a metric of signature $(2,2)$ and this geometry of the target space is called neutral hypercomplex.

When $c^2 < 1$, the corresponding construction yields a triple of complex structures, the metric is positive definite and the geometry hyperkähler.

The general case is presently under investigation, i.e., 2d-dimensions and non-linear transformations. We do not expect that it will give a constant c, but it seems to have other interesting geometric properties related to Yano f-structures $f^3 + f = 0$.

Conclusions

• A complete description of GKG uses chiral, twisted chiral and semi-chiral superfields.
• The generalized Kähler potential doubles as a (non-linear) potential for the metric and B-field and as a generating function of symplectomorphisms.
• New vector-multiplets are available for gauging an important class of isometries.
• T-duality and quotients may be discussed in terms of these multiplets.
• Global issues (bi-holomorphic gerbes...) may be addressed.
• Additional supersymmetries, when examined at the $(2,2)$ level, lead to interesting new structures on the target space.