Galilean Conformal Algebras

and AdS/CFT

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Based on

Galilean Conformal Algebra:

- Arjun Bagchi and Rajesh Gopakumar, (arXiv: 0902.1385).
- Arjun Bagchi, Ipsita Mandal (arXiv: 0903.4524).
- M. Alishahiha et.al. (arXiv: 0903.3953), D. Martelli and Y. Tachikawa (arXiv: 0903.5184), C. Duval and P. Horvathy (arXiv: 0904.1531).

Outline of the talk

- Non-Relativistic Limits
  - Schrodinger Symmetry
- Galilean Conformal Symmetry
  - Contraction of relativistic conformal symmetry
- Gravity Dual to the GCA
  - Newton-Cartan Geometries
- Summary

1 Non-Relativistic Limit: Schrodinger Symmetry

Non-Relativistic Limits

- Useful limit of AdS/CFT which might help understand a larger class of systems (real world?).
- Two kinds of nonrelativistic limits of relativistic systems.
- For massive systems, can consider the limit where the rest energy $\gg$ kinetic energy.
- Replace $\partial_0 \to -im_0 + \partial_t$; $m_0 \to \frac{m}{\epsilon^2}$; $x_i \to \epsilon x_i$ (with $\epsilon \to 0$).
- Then Klein Gordon equation reduces to Schrodinger equation

$$(\partial_0^2 - \partial_i^2 + m_0^2)\phi = 0 \to (i\partial_t + \frac{1}{2m}\partial_i^2)\phi = 0.$$

- The parameter $\epsilon \sim \frac{v}{c} \to 0$ signifies taking the nonrelativistic limit.
- What are the symmetries in this limit?

Schrodinger Symmetry

- The set of symmetries of the free particle Schrodinger equation, $(i\partial_t + \frac{1}{2m}\partial_i^2)\phi = 0$ is called the Schrodinger Symmetry $Sch(3,1)$.

- It is sometimes referred to as a non-relativistic analogue of conformal symmetry.

- This symmetry also believed to arise in interacting systems like those of fermions whose scattering length becomes infinite.

- Realised in cold atom systems with coupling tuned between BEC and BCS transition.

- $Sch(3,1)$ contains all the usual Galilean symmetries $G(3,1)$:

$$\begin{aligned}
[J_{ij}, J_{rs}] &= so(3) \\
[J_{ij}, B_r] &= -(B_i\delta_{jr} - B_j\delta_{ir}) \\
[J_{ij}, P_r] &= -(P_i\delta_{jr} - P_j\delta_{ir}), \quad [J_{ij}, H] = 0 \\
[B_i, B_j] &= 0, \quad [P_i, P_j] = 0, \quad [B_i, P_j] = m\delta_{ij} \\
[H, P_i] &= 0, \quad [H, B_i] = -P_i.
\end{aligned} \tag{1}$$

Schrodinger Symmetry *continued*

- But also has *two* more generators $\tilde{K}$, $\tilde{D}$
- $\tilde{D}$ is a dilatation operator which scales time and space *differently*

$$x_i \to \lambda x_i, \qquad t \to \lambda^2 t. \tag{2}$$

- $\tilde{K}$ is like a time component of special conformal transformations.

$$x_i \to \frac{x_i}{(1+\mu t)}, \qquad t \to \frac{t}{(1+\mu t)}. \tag{3}$$

- No analogue of the spatial components K_i of special conformal transformations.
- Thus smaller group compared to the relativistic conformal group: 12 parameters (+ central mass term) as opposed to 15 parameters for $SO(4,2)$.

2 Galilean Conformal Symmetry

Galilean Conformal Symmetry

- Second kind of non-relativistic limit appropriate for (massless) conformal field theories.
- Here the starting symmetry group is bigger e.g. $SO(4,2)$.
- Taking the non-relativistic limit now means taking a *group contraction* of this group.
- Generalisation of the process by which one recovers the Galilean group $G(d,1)$ from the Poincare group $ISO(d,1)$.
- Take $t \to t$ and $x_i \to \epsilon x_i$ and scale $\epsilon \to 0$.
- Thus $v_i \sim \epsilon \Rightarrow$ non-relativistic limit.
- Poincare generators reduce to the Galilean generators (after appropriate rescaling)

$$\begin{aligned} H &= -\partial_t, \quad P_i = \partial_i \\ B_i &= t\partial_i, \quad M_{ij} = -(x_i\partial_j - x_j\partial_i). \end{aligned} \tag{4}$$

Galilean Conformal symmetry *continued*

- Now extend this scaling to all the extra generators of the relativistic conformal group $SO(4,2)$ i.e (D, K_0, K_i) (Lukierski et.al.; Gomis, Gomis and Kamimura)

- Gives the contracted vector fields

$$\begin{aligned} D &= -(x_i\partial_i + t\partial_t) \\ K &= -(2tx_i\partial_i + t^2\partial_t) \\ K_i &= t^2\partial_i \end{aligned} \tag{5}$$

- Note the dilatation generator D is the *same* as in the relativistic theory. $x_i \to \lambda x_i, t \to \lambda t$ ($z = 1$ scaling: See Henkel).

- Therefore different from $\tilde{D} = -(2t\partial_t + x_i\partial_i)$.

- Spatial conformal transformation generators K_i *are present* and generate constant acceleration transformations. $x_i \to x_i + \frac{1}{2}a_i t^2$.

- The temporal special conformal generator different from $\tilde{K} = -(tx_i\partial_i + t^2\partial_t)$

Galilean Conformal Symmetry *continued*

- The algebra of these generators (together with that of the Galilean Algebra) is quite different from the Schrodinger group.

- Note we now have 15 generators as opposed to 12 in $Sch(3,1)$.

- Galilean central mass term in $[B_i, P_j]$ not admissible here – "massless non-relativistic system".

- The Galilean conformal symmetries are actually realised on the Euler equations of fluid mechanics. (With K acting trivially) (Bhattacharya, Minwalla and Wadia).

$$\partial_t v_i(x_i,t) + v_j \partial_j v_i(x_i,t) = -\partial_i p(x_i,t). \tag{6}$$

- In fact, these equations admit a much larger symmetry. Under arbitrary boosts $x_i \to x_i + b_i(t)$.

Galilean Conformal Symmetry *continued*

- Algebra of the contracted conformal group: Define

$$\begin{aligned} L^{(-1)} &= H, \qquad L^{(0)} = D, \qquad L^{(+1)} = K, \\ M_i^{(-1)} &= P_i, \qquad M_i^{(0)} = B_i, \qquad M_i^{(+1)} = K_i. \end{aligned} \tag{7}$$

- Then

$$\begin{aligned} [J_{ij}, L^{(n)}] &= 0, \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)} \\ [J_{ij}, M_k^{(m)}] &= -(M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik}), \qquad [M_i^{(m)}, M_j^{(n)}] = 0, \\ [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)}. \end{aligned} \tag{8}$$

- Note the $SL(2,R)$ algebra in the last line. *Different* from that in the Schrodinger group.

Non-Relativistic Conformal Symmetries - Correlation Functions

- We can use the Schrodinger/Galilean Conformal symmetry to constrain two and three point functions.

- In the Schrodinger case, define Quasi-Primary Operators which obey $[B_i, \mathcal{O}] = [\tilde{K}, \mathcal{O}] = 0$.

- These primary operators are labelled by their eigenvalue under $\tilde{D}$ and m.

$$G_{12}(\Delta x_i, \Delta t) = C_{12}\delta_{h_1,h_2}\delta_{m_1,m_2}(\Delta t)^{-h} \exp \frac{m(\Delta x_i)^2}{2\Delta t}.$$

$$\begin{aligned} G_{123}(\vec{x}^{(a)}, t^{(a)}) \quad = \quad & C_{123}\delta_{m_1+m_2+m_3,0}(t_{12})^{\frac{h_3-h_1-h_2}{2}}(t_{23})^{\frac{h_1-h_2-h_3}{2}}(t_{31})^{\frac{h_2-h_3-h_1}{2}} \\ & \exp\left(\frac{m_1(x_{13})^2}{2t_{13}} + \frac{m_2(x_{23})^2}{2t_{23}}\right) f\left(\frac{[x_{13}t_{23} - x_{23}t_{13}]^2}{t_{12}t_{23}t_{31}}\right). \end{aligned} \tag{9}$$

Non-Relativistic Conformal Symmetries - Correlation Functions *continued*

- In the GCA case, the situation is closely parallel to the relativistic conformal case. Two and three point functions are essentially fixed.

- Define Quasi-Primary Operators which obey $[K_i, \mathcal{O}] = [K, \mathcal{O}] = 0$.

- These primary operators are labelled by their eigenvalues (h, ξ_i) under $D = L^{(0)}$ and $B_i = M_0^{(i)}$.

$$G_{12}(\Delta x_i, \Delta t) = C_{12}\delta_{h_1,h_2}\delta_{\xi_1+\xi_2,0}(\Delta t)^{-2h}\exp\left(\frac{2\xi^i \Delta x_i}{\Delta t}\right).$$

$$\begin{aligned} G_{123}(\vec{x}^{(a)}, t^{(a)}) &= C_{123}\delta_{\xi_1^i+\xi_2^i+\xi_3^i,0}(t_{12})^{h_3-h_1-h_2}(t_{23})^{h_1-h_2-h_3}(t_{31})^{h_2-h_3-h_1} \\ &\quad \exp\left(\frac{2\xi_1^i x_{23}^i}{t_{23}} + \frac{2\xi_2^i x_{31}}{t_{31}} + \frac{2\xi_3^i x_{12}}{t_{12}}\right). \end{aligned} \tag{10}$$

Extended Galilean Conformal Symmetry

- A remarkable feature of this algebra is that it admits a very natural extension to an infinite dimensional $SO(d)$ Current Algebra.

- Define the vector fields for arbitrary integer n

$$\begin{aligned} L^{(n)} &= -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t \\ M_i^{(n)} &= t^{n+1}\partial_i \\ J_{ij}^{(n)} &= -t^n(x_i\partial_j - x_j\partial_i) \end{aligned} \tag{11}$$

- They obey exactly the same commutation relations as the ones for $m, n = 0, \pm 1$.

$$\begin{aligned} [L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} \qquad [L^{(m)}, J_a^{(n)}] = nJ_a^{(m+n)} \\ [J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)} \end{aligned} \tag{12}$$

- The Virasoro and Kac-Moody algebra of the vector fields is, of course, without the central extension.

Extended Galilean Conformal Symmetry *continued*

- The $M_i^{(n)}$ act as generators of generalised time dependent but spatially homogeneous accelerations

$$x_i \to x_i + b_i(t). \tag{13}$$

- This is same symmetry possessed by the Euler equations.
- Similarly, the $J_{ij}^{(n)} \equiv J_a^{(n)}$ are generators of arbitrary time dependent rotations

$$x_i \to R_{ij}(t) x_j \tag{14}$$

- These two together generate what is sometimes called the Coriolis group: the biggest group of "isometries" of "flat" Galilean spacetime.
- $L^{(n)}$ seem to be generators of a conformal "isometry" of Galilean spacetime.

$$t \to f(t), \qquad x_i \to \frac{df}{dt} x_i \tag{15}$$

- All these generators together describe, in fact, the natural set of conformal isometries of Galilean spacetime.

3 Gravity Duals

Gravity Duals

- What can we say about the spacetime geometries dual to a system with non-relativistic Conformal Symmetry (either Schrodinger or Galilean Conformal)?

- We would like to have a spacetime which has these symmetries as isometries (in an appropriate sense).

- The corresponding spacetimes are somewhat unfamiliar.

- In the case of the Schrodinger Symmetry, it is a spacetime in six dimensions, with an identification along a null direction.

- $$ds^2 = \frac{2dx^+dx^- - dx^i dx^i - dz^2}{z^2} + 2\frac{(dx^+)^2}{z^4}. \tag{16}$$

- The first piece is the metric of AdS_6 while the second is a deformation.

- The null direction x^- is identified with a period proportional to the mass (or particle number density).

The Gravity Dual *continued*

- In the case of the Galilean Conformal Algebra, the spacetime is five dimensional, obtained by taking a Galilean scaling limit of the AdS_5 geometry.
- However, the metric degenerates on taking the Galilean limit – spatial interval scales to zero.
- There is a surviving AdS_2 piece and a degenerate Euclidean metric on the remaining R^3 directions.
- Might look like a singular limit. However, analogous to taking a Newtonian limit of Einstein's equations.
- Well-defined geometric theory of newtonian gravitation - Newton-Cartan theory.
- Spacetime with a non-metric dynamical connection.

  In our case, can be viewed as connection of an R^3 fibre bundle over AdS_2.
- Has the right asymptotic symmetries – the infinite dimensional extension of the Galilean conformal algebra.

The Bulk Dual *Symmetries*

- AdS_{d+2} in Poincare coordinates:

$$ds^2 = R^2 \frac{dt^2 - dz^2 - dx_i^2}{z^2} \tag{17}$$

- In radially infalling coordinates for null geodesics $(t' = t + z, z' = z)$

$$ds^2 = \frac{R^2}{z'^2}(-2dt'dz' + dt'^2 - dx_i^2) = \frac{R^2}{z'^2}(-dt'(2dz' - dt') - dx_i^2). \tag{18}$$

- Take the generators of the AdS_{d+2} isometries and perform the contraction by taking $t', z^{'} \to \epsilon^r, x_i \to \epsilon^{r+1} x_i$. Metric degenerates as expected.

- Contracted Killing vectors given by

$$\begin{aligned} P_i &= -\partial_i, \qquad B_i = -(t' - z')\partial_i, \qquad K_i = -(t'^2 - 2t'z')\partial_i \\ H &= \partial_{t'}, \quad D = t'\partial_{t'} + z'\partial_{z'} + x_i\partial_i, \quad K = t'^2\partial_{t'} + 2(t' - z')(z'\partial_{z'} + x_i\partial_i). \end{aligned} \tag{19}$$

The Bulk Dual *Symmetries*

★ More compactly (for $m, n = 0, \pm 1, l = 0$).

$$\begin{aligned} L^{(n)} &= t'^{n+1}\partial_{t'} + (n+1)(t'^{n} - nzt'^{n-1})(x_i\partial_i + z'\partial_{z'}) \\ M_i^{(m)} &= -(t'^{m+1} - (m+1)zt'^{m})\partial_i \\ J_{ij}^{(l)} &= -t'^{n}(x_i\partial_j - x_j p_i) \end{aligned} \tag{20}$$

★ Reduces at the boundary ($z = 0$) to the generators of the contracted conformal algebra. And satisfies the *same* algebra.

★ In fact, these bulk vector fields (for arbitrary m, n, l) reduce to that of the extended Kac-Moody algebra at the boundary.

★ What is the role of these vector fields in the bulk?

★ The Virasoro generators act as the generators of asymptotic symmetries of the AdS_2.

★ The others act only on the R^3 (like Galilean "isometries" on the boundary).

To Summarise

Summary

☆ Gauge-Gravity dualities can be generalised to a non-relativistic setting.

☆ Galilean conformal symmetry relevant to "massless" non-relativistic systems.

☆ Identify the sector in e.g. $\mathcal{N} = 4$ SYM described by the GCA.

☆ Do the ward identities of the full GCA constrain correlators in this sector more than expected otherwise?

☆ Are there real life systems which are described by the GCA?

☆ Need to develop a better understanding of the gravity duals which involve novel features such as the Newtonian limit.

☆ Spell out the bulk-boundary dictionary, as a parametric limit of the relativistic case.

Thank You