Asymptotically free 4-fermi theory in 4-dims at the z=3 Lifshitz-like fixed point

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with Mandal and Wadia, arXiv:0905.2928

Motivation

A system at a Lifshitz-like fixed point labeled by the exponent z is characterized by an anisotropic scaling symmetry, i.e. different space-time directions scale differently.

<u>Example</u>: Free scalar field theory at $z = 2$ in 4-dim:

$$\int dt\, d^3x \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x^2 \phi)^2 \right)$$

Here, under the scaling $x \to \lambda x$, t scales as $t \to \lambda^2 t$.

An immediate observation is that such theories violate Lorentz invariance. So why should we be interested?

Motivation

Field theory – Particle Physics

- New examples of ultraviolet complete field theories; 4-fermi theory in 4-dims at the $z = 3$ fixed point provides ultraviolet completion of the familiar Lorentz-invariant (low-energy effective) 4-fermi field theories, like the NJL model.
- In this model, fermion mass is generated dynamically and a composite scalar field arises as a collective excitation of fermions around the broken symmetry vacuum. Such a scenario raises the possibility of eliminating the Higgs field and the associated hierarchy problem.
- Of course, the main issue then is to ensure that a phenomenologically viable Lorentz invariance emerges at low energies.......

Motivation

String Theory

- 2-dim string theory has a nonperturbative formulation in terms of a $z = 2$ system of fermions!
- In recent studies, examples of flows between fixed-points with different values of z have been constructed in string theory:
  - Kachru, Liu and Mulligan, arXiv:0808.1725 - example of $z = 2$ theory at the boundary flowing to $z = 1$ theory (in the dual geometry) in the IR bulk.
  - Azeyanagi, Li and Takayanagi, arXiv:0905.0688 - example of a $z = 1$ theory at the UV boundary flowing to an anisotropic scale invariant theory in the IR bulk. These are constructions in Type IIB theory.

Motivation

Condensed Matter Physics

Many strongly correlated fermion systems exhibit Lifshitz type multi-critical points.

<u>Examples:</u>

- space-like anisotropic fixed points appear in realistic magnetic substances, e.g. MnP. These substances are modeled by an axial next-to-nearest-neighbour Ising model. A competition between the ferromagnetic nearest-neighbour and antiferromagnetic next-to-nearest-neighbour interactions (along a single lattice axis) produces a modulated phase, in addition to the usual ferromagnetic and paramagnetic ones.

Motivation

- quantum dimer models, e.g. Rokhsar-Kivelson model, which is believed to be in the universality class of the $z = 2$ scalar field model. A more general (euclidean) Lagrangian that reproduces properties of a class of quantum dimer models is

$$L = \frac{1}{2}(\partial_\tau h)^2 + \frac{1}{2}\rho_2(\nabla h)^2 + \frac{1}{2}\rho_4(\nabla^2 h)^2 + \lambda \cos(2\pi h)$$

These models may explain certain features of high T_c superconductivity.

Outline

- 4-fermi model at $z = 3$; relevant and marginal deformations

- exact solution in the limit of a large number of species
  - with only marginal couplings present
  - with marginal and relevant couplings switched on, including the coupling which induces flow to the Lorentz invariant $z = 1$ fixed point

- anomalies

- application to particle physics

The Model

- The basic degrees of freedom of the model are:

  2N species of fermions $\psi_{ai}(t, \vec{x})$, $a = 1, 2; i = 1, ..., N$,

  which belong to the fundamental representation of $SU(N)$ and transform under the flavour group $U(1)_1 \times U(1)_2$:

$$\psi_{ai} \to e^{i\alpha_a}\psi_{ai}, \; a = 1, 2$$

- Each of these fermions is an $SU(2)_s$ spinor, where $SU(2)_s$ is the double cover of the spatial rotation group $SO(3)$.

  The generators of rotations on fermions are the Pauli matrics $\{\vec{\sigma}\}$.

The Model

An action which is consistent with the above symmetries is:

$$S = \int d^3\vec{x}\, dt \left[\psi^\dagger{}_{1i} \left(i\partial_t + i\vec{\partial}.\vec{\sigma}\, \partial^2 \right) \psi_{1i} + \psi^\dagger{}_{2i} \left(i\partial_t - i\vec{\partial}.\vec{\sigma}\, \partial^2 \right) \psi_{2i} \right.$$
$$\left. + g^2\, \psi^\dagger{}_{1i}\psi_{2i}\psi^\dagger{}_{2j}\psi_{1j} \right]$$

- Note the sign flip of the spatial derivative term between the two flavours $a = 1$ and $a = 2$; this ensures that the Lagrangian is invariant under the parity operation $\psi_{1i}(t, \vec{x}) \to \psi_{2i}(t, -\vec{x})$.
- We will study the dynamics of this action in the large N limit, holding the 'tHooft coupling $\lambda = g^2 N$ fixed.

The Model

- According to $z = 3$ scaling dimensions, $[x] = -1, [t] = -3$. It follows that $[\psi] = 3/2$. In this case, all the three terms appearing in the above action are of dimension 6 and hence marginal.
- Recall that in the usual context of a $3 + 1$ dimensional Lorentz invariant theory, any interaction involving four fermions represents an irrelevant operator and so must be understood as a low energy effective interaction.
- Here the marginality of the interaction leads one to hope that the theory is perhaps uv-complete. This is indeed the case since the four-fermi coupling turns out to be asymptotically free.

The Model

A more general $z = 3$ action with all relevant and marginal couplings, which is consistent with all the symmetries, is:

$$\begin{aligned} S &= \int d^3\vec{x}\, dt \Big[{\psi^\dagger}_{1i} \left(i\partial_t - i\vec{\partial}.\vec{\sigma} \left((-i\partial)^2 + g_1 \right) + g_2(-i\partial)^2 \right) \psi_{1i} \\ & + {\psi^\dagger}_{2i} \left(i\partial_t + i\vec{\partial}.\vec{\sigma} \left((-i\partial)^2 + g_1 \right) + g_2(-i\partial)^2 \right) \psi_{2i} \\ & + g_3 \left({\psi^\dagger}_{1i}\psi_{1i} + {\psi^\dagger}_{2i}\psi_{2i} \right) + g_4^2 \left(\left({\psi^\dagger}_{1i}\psi_{1i} \right)^2 + \left({\psi^\dagger}_{21}\psi_{2i} \right)^2 \right) \\ & + g_5^2 \left({\psi^\dagger}_{1i}\psi_{1i}{\psi^\dagger}_{2j}\psi_{2j} \right) + g_6^2 \left({\psi^\dagger}_{1i}\psi_{2i}{\psi^\dagger}_{2j}\psi_{1j} \right) \Big] , \end{aligned}$$

The earlier action corresponds to putting all the couplings $g_1, ..., g_5 = 0$ and setting $g_6 = g$.

Large-N Solution

- One can eliminate the 4-fermi interaction using a standard Gaussian trick. This introduces a complex scalar field and gives the following action, which is equivalent to the 4-fermi action:

$$S = \int d^3\vec{x}\, dt \left[{\psi^\dagger}_{1i} \left(i\partial_t + i\vec{\partial}.\vec{\sigma}\partial^2 \right) \psi_{1i} + {\psi^\dagger}_{2i} \left(i\partial_t - i\vec{\partial}.\vec{\sigma}\partial^2 \right) \psi_{2i} \right.$$
$$\left. + \phi^* {\psi^\dagger}_{1i}\psi_{2i} + \phi {\psi^\dagger}_{2i}\psi_{1i} - \frac{1}{g^2}|\phi|^2 \right]$$

- The scalar field ϕ is an $SU(N)$-singlet and is charged under the axial $U(1)$ parametrized by $\exp[i(\alpha_1 - \alpha_2)]$.

Large-N Solution

- Since the action is now quadratic in fermions, one can integrate them out, leading to the following effective action for the boson:

$$S_{\text{eff}}[\phi] = -iN\,\mathrm{Tr}\ln\tilde{D} - \frac{1}{g^2}\int|\phi|^2$$

- The operator $\tilde{D}$ can be written in terms of Dirac gamma matrices γ^0, γ^i:

$$\tilde{D} = \gamma^0 D, \qquad D = i\gamma^0\partial_t + i\gamma^i\partial_i(i\partial)^2 + (\phi^* P_L + \phi P_R)$$

- Although we find it expedient to use the Dirac gamma matrices, the operator D is NOT the Dirac operator. For instance, the coefficient of γ^i has three powers of momenta, as appropriate for a $z = 3$ theory.

Large-N Solution

- In the large N limit, the classical equation of motion $\delta S_{\rm eff}/\delta\phi = 0$ is exact, leading to

$$i\int \frac{dk_0 d^3k}{(2\pi)^4}\frac{1}{k_0^2 - |\vec{k}|^6 - |\phi|^2 + i\epsilon} = \frac{1}{2\lambda}$$

- This gap equation determines only the absolute value of ϕ. The phase of ϕ can be identified with the Nambu-Goldstone mode of the symmetry breaking $U(1)\times U(1)\to U(1)$.
- The left-hand-side of the gap equation is logarithmically divergent by $z = 3$ power counting - both numerator and denominator have dimension 6.

Large-N Solution

- Rotating the contour to Euclidean signature $k_0 \to ik_\tau$ and doing the angular integration gives

$$\frac{4\pi}{(2\pi)^4} \int dk_\tau \; k^2 dk \; \frac{1}{k_\tau^2 + k^6 + |\phi|^2} = \frac{1}{2\lambda}$$

- Change to the variable $k_1 = k^3$ and extend the range of k_1-integral to the entire real line (possible because the integrand has $k_1 \leftrightarrow -k_1$ symmetry):

$$\int \frac{dk_\tau \; dk_1}{(2\pi^2)} \; \frac{1}{k_\tau^2 + k_1^2 + |\phi|^2} = \frac{3\pi}{\lambda}$$

  - This is gap equation for the Gross-Neveu model!
  - True for any spatial dimension d with $z = d$. GN model is just the first, $z = d = 1$, in the series.

Large-N Solution

- The integral is logarithmically divergent - need a cut-off Λ. As in the GN model, gap equation determines the dynamically generated (cut-off independent) scale m ($m^3 = |\phi|$) in terms of the cut-off dependent coupling $\lambda(\Lambda)$:

$$\frac{4\pi^2}{\lambda(\Lambda)} = \ln(\frac{\Lambda^2}{m^2}) \qquad \Longrightarrow \qquad m = \Lambda e^{-\frac{2\pi^2}{\lambda(\Lambda)}}$$

- This is equivalent to a statement for the beta-function for λ:

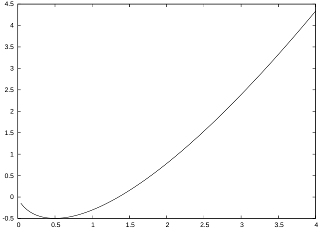
$$\beta(\lambda) = \Lambda \frac{d\lambda}{d\Lambda} = -\frac{\lambda^2}{2\pi^2}$$

- Condensate generated for arbitrarily weak coupling - contrast with the critical coupling required in the usual relativistically invariant NJL model.

Large-N Solution

- Effective potential for the homogeneous mode of ϕ is:

$$V_{\text{eff}}(\phi) = N|\phi|^2 \left(\frac{1}{\lambda(\Lambda)} - \frac{1}{12\pi^2} \ln(\frac{\Lambda^6}{|\phi|^2}) - \frac{1}{12\pi^2} \right)$$



At the minimum, $|\phi| = m^3 = \Lambda^3 \exp[-6\pi^2/\lambda]$

- The treatment of the effective potential and the RG flow presented above is exact in the strict $N = \infty$ limit.

Large-N Solution

- $V_{\text{eff}}(\phi)$ should be cut-off independent. Define a (cut-off independent) running coupling:

$$\frac{1}{\lambda(\mu)} = \frac{1}{\lambda(\Lambda)} - \frac{1}{2\pi^2}\ln(\frac{\Lambda}{\mu}),$$

with beta-function

$$\beta(\lambda(\mu)) = \mu\frac{d\lambda(\mu)}{d\mu} = -\frac{\lambda(\mu)^2}{2\pi^2}$$

Here μ is an arbitrary scale. Then,

$$V_{\text{eff}}(\phi) = N|\phi|^2\left(\frac{1}{\lambda(\mu)} - \frac{1}{12\pi^2}\ln(\frac{\mu^6}{|\phi|^2}) - \frac{1}{12\pi^2}\right)$$

Large-N Solution

- That there is a running coupling in the large-N solution can also be seen by considering fluctuations of ϕ. Parametrize these as

$$\phi(t,\vec{x}) = \left(m^3 + \sqrt{\frac{\lambda}{2N}}\sigma(t,\vec{x})\right) e^{\sqrt{\frac{\lambda}{N}}\pi(t,\vec{x})}$$

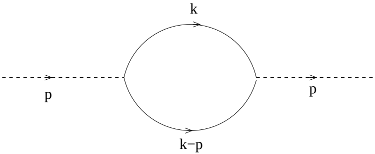
- The Nambu-Goldstone mode is the phase $\pi(t,\vec{x})$. It can be absorbed into fermions by an axial $U(1)$ rotation (but reappears through the fermion kinetic terms).

- The interaction of $\sigma(t,\vec{x})$ with fermions is

$$\sqrt{\frac{\lambda}{2N}}\sigma({\psi^\dagger}_{1i}\psi_{2i} + {\psi^\dagger}_{2i}\psi_{1i}) - \frac{1}{2}\sigma^2 - \sqrt{\frac{2N}{\lambda}}\sigma$$

Large-N Solution

- The only diagrams that survive in the leading order in $1/N$ contribute to the renormalization of the 2-point function of σ:



- This leads to a renormalization of the Yukawa coupling of σ and a running coupling, consistent with what we obtained from requiring $V_{\mathrm{eff}}(\phi)$ to be cut-off independent.

Marginal deformations

- Full set of marginal couplings, consistent with symmetries, is

$$g_4^2\left(\left(\psi^\dagger{}_{1i}\psi_{1i}\right)^2+\left(\psi^\dagger{}_{2i}\psi_{2i}\right)^2\right)$$
$$+g_5^2\left(\psi^\dagger{}_{1i}\psi_{1i}\psi^\dagger{}_{2j}\psi_{2j}\right)+g_6^2\left(\psi^\dagger{}_{1i}\psi_{2i}\psi^\dagger{}_{2j}\psi_{1j}\right)$$

- The full system can also be analysed exactly in the large-N limit. Other possible fermion condensates ($<\psi^\dagger{}_{ai}\psi_{ai}>$, $a=1,\ 2$) vanish $\Longrightarrow$ vacuum solution remains unchanged.
- Additional deformation by the relevant operator $\left(\psi^\dagger{}_{1i}\psi_{1i}+\psi^\dagger{}_{2i}\psi_{2i}\right)$ leads to nonzero values of $<\psi^\dagger{}_{ai}\psi_{ai}>$. However, at leading order in large N, g_4 and g_5 remain exactly marginal and hence the vacuum state remains unchanged.

Relevant deformations

- This set consists of the two terms

$$g_1\left(\psi^{\dagger}{}_{1i}(-i\vec{\partial}.\vec{\sigma})\psi_{1i}+\psi^{\dagger}{}_{2i}(i\vec{\partial}.\vec{\sigma})\psi_{2i}\right)+g_2\psi^{\dagger}{}_{ai}(-i\partial)^2\psi_{ai}$$

- The operator multiplying g_1 is the usual spatial derivative term of the Dirac action. At $z=3$, g_1 is a relevant coupling with dimension 2. This deformation causes flow to the Lorentz invariant $z=1$ fixed point.
- The other relevant coupling, g_2, has dimension 1. It can be argued that this term does not affect physics, so we may set $g_2=0$.

Restoration of Lorentz invariance

- The sign of g_1 could matter – similar deformations in other cases lead to different phases for different signs. We will restrict to the positive sign and set $g_1 = M^2$.
- M is the energy scale at which Lorentz symmetry is restored. Fermion dispersion relation, which is exact in large N, reads:

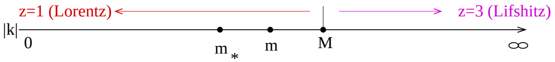
  $$k_0^2 - k^2(k^2 + M^2)^2 - |\phi|^2 = 0$$

  For $k << M$, this is approximately $k_0^2 - k^2M^4 - |\phi|^2 = 0$
- Now, rescale energy $k_0 = k_0' M^2$ (equivalently rescale time, $k_0/M^2 = \frac{i}{M^2}\frac{\partial}{\partial t} = i\frac{\partial}{\partial t'} = k_0'$). The new dispersion relation is the standard Lorentz-invariant mass-shell condition:

  $$(k_0')^2 = (k^2 + m_*{}^2), \qquad m_* = |\phi|/M^2$$

Restoration of Lorentz invariance

- Phenomenological consistency requires the Lorentz-invariant mass scale $m_* = |\phi|/M^2$ to be much smaller than M. This implies $|\phi| \equiv m^3 << M^3$. This results in the hierarchy of scales



- To leading order in $1/N$, the scale M does not get renormalized in the present model. In other models, or in the absence of a large-N argument, one will need to worry about the renormalization of M.

Restoration of Lorentz invariance

- Renormalization of M would result in a scale-dependent notion of time in the low-energy theory. Equivalently, this would lead to a scale-dependent velocity of light and one should then worry about consistency with observations (Iengo, Russo and Serone, arXiv:0906.3477).
- A related issue is the possibility of different renormalizations of the Lorentz symmetry restoration scale for different fields in the theory, leading to a fine-tuning problem.

Relevant deformation and gap equation

- The presence of a nonzero M affects the gap equation, which needs to be solved again for a vacuum solution. The modified gap equation is:

$$\frac{4\pi}{(2\pi)^4}\int dk_\tau\, k^2 dk\, \frac{1}{k_\tau^2 + k^2(k^2+M^2)^2 + |\phi|^2} = \frac{1}{2\lambda}$$

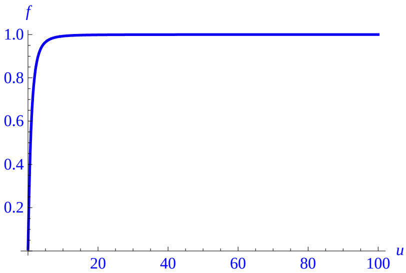
- The divergence structure of the above integral remains unchanged by the deformation:

$$\int dk_\tau k^2 dk \left[\frac{1}{k_\tau^2 + k^2(k^2+M^2)^2 + |\phi|^2} - \frac{1}{k_\tau^2 + k^6 + |\phi|^2}\right] + \int dk_\tau\, k^2 dk\, \frac{1}{k_\tau^2 + k^6 + |\phi|^2}$$

Relevant deformation and gap equation

- Using a cut-off, as before, gives

$$\frac{4\pi^2}{\lambda} = \ln(\frac{\Lambda^2}{m^2}) - \int_0^{\frac{M^2}{m^2}} \frac{du}{u} f(u)$$



Relevant deformation and gap equation

- Since M does not get renormalized in leading order at large-N, λ must be given the same cut-off dependence as before $\Longrightarrow$ a running coupling $\lambda(\mu)$, as before.
- Then, the gap equation for $\mu = M$ can be rewritten as

$$\frac{4\pi^2}{\lambda(M)} = \ln(\frac{M^2}{m^2}) - \int_0^{\frac{M^2}{m^2}} \frac{du}{u} f(u)$$

- Note that a solution exists only for $M > m \exp[\int_0^{\frac{M^2}{m^2}} \frac{du}{u} f(u)]$, since otherwise the rhs is negative.

Relevant deformation and gap equation

- For large M/m, we get

$$\frac{8\pi^2}{\lambda(M)} = \ln(\frac{M^2}{m_*^2})/\frac{M^2}{m_*^2}$$

where $m_* = m^3/M^2$.

- This should be compared with the gap equation in the relativistic theory:

$$1 - \frac{8\pi^2}{Ng^2\Lambda^2} = \ln(\frac{\Lambda^2}{m^2})/\frac{\Lambda^2}{m^2}$$

Λ is the cut-off required to regularize the quadratically divergent integral.

Anomalies

- Let us couple the fermions to a $U(1)$ gauge field. In a useful Dirac notation, this coupling can be written as

$$S = \int d^3x\, dt\, \left(\bar{\Psi}_i i \not{\mathbf{D}} \Psi_i + \mathrm{four} - \mathrm{fermi\ terms}\right)$$

where

$$\not{\mathbf{D}} = \gamma^\mu \mathbf{D}_\mu,\ \mathbf{D}_t = D_t,\ \mathbf{D}_i = -D_i(\vec{D})^2,\ D_\mu = \partial_\mu + ieA_\mu$$

- As in the Lorentz-invariant case, this theory has global axial $U(1)$ symmetry

$$\delta\Psi_i = i\alpha(x)\gamma^5\Psi_i,\ \delta\bar{\Psi}_i = \bar{\Psi}_i\, i\alpha(x)\gamma^5$$

Is the axial current conserved? Note that the current has a complicated expression in terms of fields.

Anomalies

- Use Fujikawa's argument and heat kernel method of regularization to compute the anomaly:

$$\partial_\mu J^{\mu 5} = 2\mathrm{Tr}\left(\gamma^5 \exp[i\not{D}^2/\Lambda^6]\right)$$

- Recall that in the usual Lorentz-invariant case, the next step is to write

$$(\not{D})^2 = -\mathbf{D}_\mu \mathbf{D}^\mu + \frac{i}{2}\Sigma^{\mu\nu}[\mathbf{D}_\mu, \mathbf{D}_\nu]$$

Anomalies

- A calculation gives

$$\int \frac{d^4k}{(2\pi)^4} \exp\left[-(k_\tau^2 + |\vec{k}|^6)/\Lambda^6\right] \epsilon^{ijk} \left(\frac{F_{0i}|\vec{k}|^2}{\Lambda^6}\right)\left(\frac{F_{jk}|\vec{k}|^4}{\Lambda^6}\right)$$

- The final answer is exactly as in the Lorentz-invariant case:

$$\partial_\mu J^{\mu 5} = -\frac{e^2 N}{16\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$$

Application to particle physics

- We will now consider a simple extension of the system to describe Higgs mechanism in which the Higgs field is a composite object.
- The degrees of freedom of the extended model are:

$$\psi_{i\alpha}, \qquad \chi_i, \qquad \alpha = 1, 2$$

$\psi_{i\alpha}$ transforms as the fundamental repn. of $SU(2)$ and χ_i as a singlet. The index α is gauged.

- The 4-fermi interaction

$$(\psi^{\dagger}{}_{i\alpha}\chi_i)(\chi_i^{\dagger}\psi_{i\alpha})$$

leads to breaking of the $SU(2)$ symmetry.

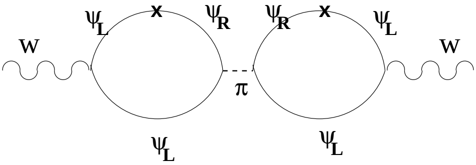
Application to particle physics

- The bilinear order parameter $< {\psi^\dagger}_{i\alpha} \chi_i >= \phi_\alpha$ acts as (a composite) Higgs field because its $SU(2)$ phase is eaten up by the gauge fields to which the $\psi_{i\alpha}$ couple.
- Parametrize ϕ_α as follows:

$$\phi(t,x) = \exp(i\vec{\pi}(t,x).\vec{\tau})\rho(t,x)$$

The key point is that gauge field masses arise from their gauge-invariant interactions with the ψ's because of the exchange of the would-be Nambu-Goldstone bosons, the "pions", $\pi(t,x)$, a well-known mechanism originally discovered in the context of Meissner effect.

Application to particle physics



- In addition, we have the usual quarks and leptons, appropriately coupled to the above fermions:

$$(\psi^{\dagger}{}_{i\alpha}\chi_i)(q_2^{\dagger}q_{1\alpha})$$

which is equivalent, after symmetry breaking, to the standard $q_2^{\dagger}q_{1\alpha}\phi_\alpha$ Yukawa coupling.

Application to particle physics

- After mass generation and Higgs mechanism, at low energies, the $\psi_{i\alpha}$ and χ_i combine to give massive fermions. It may be possible to arrange their masses to be sufficiently high, consistent with phenomenological constraints.
- Actually two χ_i's are needed to give mass to both components of $\psi_{i\alpha}$. Assuming that the pair $\chi_{ia},\ a = 1,\ 2$ transforms as a doublet of another $SU(2)$, one can arrange the four-fermi interactions to be invariant under this "custodial" $SU(2)$.

Summary

- The ultraviolet properties of a field theory at an arbitrary z Lifshitz-like fixed point can be quite different from its $z = 1$ counterpart.
- Fermions at $z = 3$ in 4-dims have an asymptotically free four-fermi coupling. Hence, one could say that this theory provides UV completion of relativistic effective four-fermi theories at low energies.
- Our example gives rise to mass generation and, in the appropriate case, a composite Higgs field. This eliminates the need for a Higgs potential, thus avoiding the hierarchy problem.

Summary

- The key issue for a successful application to phenomenology is that of restoration of Lorentz invariance at low energies. This seems hard to achieve without introducing new fine tuning problems, but further investigation is needed to clarify the situation.
- Applications to condensed matter physics seem to be more feasible at present. Combined with AdS/CFT, this could provide a powerful tool for solving problems in this area.