A Wavefunction for Stringy Universes

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String dualities give us profound insights into the nature of *Space*.

Perhaps the most dramatic of these is that Space itself emerges "holographically" in special quantum mechanical systems.

E.g. Gauge Theory/Gravity duality based on the AdS/CFT correspondence

Here, gravitons arise as collective excitations in a strongly coupled gauge theory, but together with them we find additional macroscopic dimensions on which they propagate.

$\mathcal{N} = 4$ SUSY YM in 4 dimensions $\rightarrow$ Type IIB String Theory on $AdS_5 \times S^5$

Many other surprising phenomena arise just when we try to probe features of Space and geometry at short distances, distances of order the string scale l_s or the Planck length l_p.

These lead to important properties and consequences such as

- UV finiteness
- The stringy spacetime uncertainty principle: $\Delta x \Delta t \sim l_s^2$
- T-duality
- Resolution of orbifold and conifold singularities
- Smooth topology changing transitions in CY compactifications ...

illustrating how String Theory can provide concrete answers to many of the puzzles one has to face in trying to quantize Einstein's theory of general relativity ...

However, almost all of string dualities have been developed in time-independent set-ups, on (asymptotically) flat or Anti de-Sitter backgrounds.

Thanks to supersymmetry, we can obtain a lot of information about the strong coupling limit of the theory in such backgrounds.

Moreover we can develop a rigorous mathematical framework for studying *scattering problems* where particles propagate in from infinity, scatter off each other, and then propagate out to infinity again $\rightarrow$ Flat space S-matrix, CFT correlators for AdS spaces.

It is important to understand the implications of such a rich structure on the nature of *Time*
try to extend the web of string dualities to time-dependent, cosmological backgrounds.

In trying to provide a string theoretic framework for studying cosmological backgrounds we face many new challenges

Some of these are

- Spacetime supersymmetry is necessarily broken
- When the cosmological evolution is extrapolated back in time, we are often driven to an initial singularity. Such a spacelike singularity is harder to understand
- In the case of cosmology, we are interested in measurements that are carried out in the interior of spacetime. To define the corresponding observables, a second quantized, string field theory approach seems more appropriate
- Particle production, back-reaction, finite temperature . . .

To address some of the difficult issues it could proved useful to embed the Hartle-Hawking no-boundary proposal for a wavefunction description the universe in string theory.

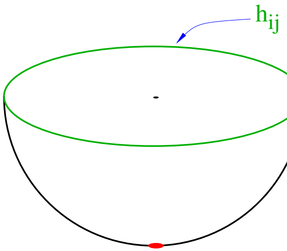
The Hartle-Hawking proposal pertains in particular to spatially closed universes, in the presence of a positive cosmological constant Λ.
For simplicity, we take the spatial slices to be of spherical topology S^3. On such a slice we can specify a three-metric h_{ij} and the matter field configurations ϕ_0.

The Hartle-Hawking wavefunction is given by a *formal* Euclidean path integral

$$\Psi(h_{ij}, \phi_0) = \int [dg][d\phi] e^{-S_E(g,\phi)}$$

where we sum over *compact* Euclidean four-geometries with an S^3 boundary, and on which the induced metric is h_{ij}.
The topology of the four-manifold is that of a four-dimensional ball B_4.
The wavefunction specified in this way is a special solution to the Wheeler-De Witt equation.

We can interpret the Hartle-Hawking wavefunction as giving the amplitude for the three-geometry h_{ij} to arise from a single point → "the amplitude for the universe to appear from nothing".



"Nothing" means no classical spacetime.

The norm of the wavefunction is given by the full Euclidean path integral on a compact four-manifold of spherical topology S^4.

It can be computed in the semi-classical approximation by evaluating the Euclidean action for a given solution to the classical equations of motion.
Suppose the scalar fields are frozen at a minimum of their potential.

The classical solution is the round four-sphere of radius $R \sim \Lambda^{-1/2}$ and metric

$$ds^2 = R^2(d\theta^2 + \sin^2\theta \; d\Omega_3^2)$$

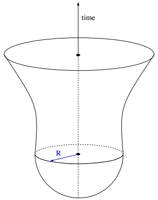
giving $\longrightarrow$

$$||\Psi_{HH}||^2 \sim e^{\frac{3\pi}{G\Lambda}}$$

Rotating the gravitational instanton to Lorentzian signature $\theta = \pi/2 + it$, we obtain a de Sitter cosmology:

$$ds^2 = R^2(-dt^2 + \cosh^2 t \ d\Omega^2)$$

Imagine cutting the sphere along the equator $\theta = \pi/2$ and gluing smoothly one half of it to half of de-Sitter space.



Then the HH wavefunction defines an initial state for subsequent Lorentzian evolution.

In the semiclassical approximation, the norm of the HH wavefunction is also given by the de Sitter entropy $S = 3\pi/G\Lambda$

$$||\Psi_{HH}||^2 \sim e^S$$

showing that it can be given a statistical interpretation ...

It is tempting to give to it a probabilistic interpretation, where universes with a small positive cosmological constant are favored. However these are nearly *empty* universes, unlike our own with a hot big bang history.

There are various proposals to remedy this problem, some of which require going beyond the semi-classical approximation, incorporating quantum corrections.
However the underlying field theory is non-renormalizable $\rightarrow$ need to impose a UV cut-off.

Some of the difficulties can be already seen in the mini-superspace approximation. The Euclidean action is given by

$$S_E = \frac{1}{2} \int dt(-\alpha\dot{\alpha} - \alpha + \lambda\alpha^3)$$

This is unbounded from below. It can become large and negative because of rapid oscillations of the scale factor α. This is an ultraviolet instability ...

One way to deal with the ultraviolet ambiguities is to embed the calculation in a string theoretic framework, where we expect the ultraviolet divergences to be absent.
Unfortunately there are no known classical de Sitter solutions in string theory to begin with.
Therefore, we seek other cosmological backgrounds which are exact solutions to string theory and for which we can generalize the Hartle–Hawking computation.

One motivation for exploring this idea is the following:

It is likely that the early universe arises as a *compact space of microscopic size*.

If all spatial directions are compact, then the moduli fields of string theory cannot be frozen.

We should expect to find a wavefunction on moduli space, whose amplitude peaks at the values of moduli satisfying the classical equations of motion.

This would allow us to compare the relative probabilities of different string compactifications.

The important property of de Sitter space from our perspective is that the Euclidean continuation is a smooth, *compact* manifold, a four-sphere.

In the string theoretic context, we look for cosmological backgrounds, which admit a continuation to a Euclidean background described by a *compact worldsheet CFT.*

If the corresponding Euclidean string theory is tachyon free, there is a finite, calculable quantity, namely the string partition function Z_{string},
which at the perturbative level can be computed as usual as a sum of CFT vacuum amplitudes over compact worldsheets of all topologies.
We propose for the cosmological wavefunction

$$||\Psi_{cosm.}||^2 = e^{Z_{string}} \leftarrow \text{normalizable}$$

The only known string theory example which satisfies these criteria is based on the parafermionic $SU(2)_{|k|}/U(1)$ coset model, which can be realized as a gauged $SU(2)$ WZW model at level $|k|$.

Euclidean background: $SU(2)_{|k|}/U(1) \times K$, where K is an internal, compact CFT

This admits a continuation to a cosmological background described by a two-dimensional CFT of the form $SL(2,R)_{-|k|}/U(1) \times K$ [C. Kounnas, D. Lust] which can be also realized as a solution to superstring theory.

When we fix the internal CFT K, the level $|k|$ is determined by the central charge condition $c_{tot} = 15$ ($\hat{c} = 10$).

The non-trivial time-dependence of the cosmology necessarily breaks space-time supersymmetry.

As in the de Sitter case, the full Euclidean path integral can be interpreted as a thermal ensemble, giving rise to an effective temperature that breaks supersymmetry.

So there is another requirement for finiteness of the norm: The effective temperature of the model must be below the Hagedorn temperature.

The Cosmological Background

It is based on an $SL(2,R)/U(1)$ gauged WZW model at negative level k.

The sigma-model metric is given by

$$ds^2 = |k|\alpha' \frac{dudv}{1-uv} = |k|\alpha' \frac{-dT^2 + dX^2}{1+T^2-X^2}$$

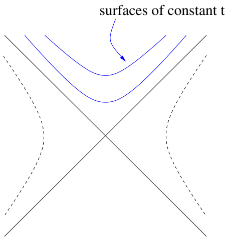
where $u = -T + X$ and $v = T + X$.

There is also a non-trivial dilaton

$$e^{2\Phi} = \frac{e^{2\Phi_0}}{1-uv}$$

The geometry consists of a singularity-free light-cone region, and there are time-like curvature singularities in the regions outside the light-cone horizons.

The singularities occur at $X = \pm\sqrt{1 + T^2}$, where the dilaton field is also singular.



If we perform a double analytic continuation, we obtain Witten's 2d black hole.

This is equivalent to changing the sign of k.

At the singularities the sigma-model geometric description breaks down. As we will see later there is a well defined CFT prescription to describe them.

For now think of them as spatial boundaries.
Their proper distance is finite with respect to the string frame metric: $L = \pi(|k|\alpha')^{1/2}$.
So with respect to stringy probes, the cosmology is spatially closed.

The cosmological region of interest is the future part of the lightcone region.
It is an expanding, asymptotically flat geometry with the string coupling vanishing at late times:

$$ds^2 = |k|\alpha' \frac{-dt^2 + t^2 dx^2}{1+t^2}, \qquad e^{2\Phi} = \frac{e^{2\Phi_0}}{1+t^2}$$

Asymtotically we get a timelike linear dilaton background.

The cosmological observer never encounters the singularities, as these are hidden behind the visible horizons at $T = \pm X$.
However signals from the singularities can propagate into the lightcone region, and therefore influence its future evolution.

When $|k|$ is small the early universe region $t \sim 0$ is highly curved, with curvature of order the string scale.
In this sense, this background is similar to a big-bang cosmology.

The central charge of the superconformal $SL(2,R)/U(1)$ model at negative level k, is given by

$$c = 3 - \frac{6}{|k|+2}, \quad \hat{c} = 2 - \frac{4}{|k|+2}$$

To obtain a 4d model we add two large (however compact) free super-coordinates together with a compact, superconformal system of central charge $\delta\hat{c} = 6 + 4/(|k|+2)$.

The metric in Einstein frame is given by

$$ds_E^2 = |k|\alpha'(-dt^2 + t^2 dx^2) + (1 + t^2)(R_y^2 dy^2 + R_z^2 dz^2).$$

This is an anisotropic cosmology which at late times however, and for large $R_y \sim R_z$, asymptotes to an isotropic flat Friedman-Robertson cosmology.

The cosmological region is non-compact, and when $R_{y,z}$ are large it has the desired four-dimensional interpretation. This is so *irrespective of how small the level k is.*

In contrast, for the internal, $\hat{c} = 6 + 4/(|k| + 2)$ system, the naive six-dimensional interpretation, valid for large level $|k|$, is not valid for small values of $|k|$.
In fact for $|k| = 2$, the system can be taken to be a *seven*-dimensional torus.

We can rotate to Euclidean signature by setting $T \to -iT_E$. The Euclidean continuation is a disk of unit coordinate radius parameterized by a complex coordinate $Z = \rho e^{i\phi}$ such that $|Z|^2 \leq 1$.

The Euclidean metric and dilaton are given by

$$ds^2 = |k|\alpha' \frac{d\rho^2 + \rho^2 d\phi^2}{1 - \rho^2}, \qquad e^{2\Phi} = \frac{e^{2\Phi_0}}{1 - \rho^2}$$

The singularity now occurs at the boundary circle $\rho = 1$.

The radial distance of the center to the boundary of the disk is finite, but the circumference of the boundary circle at $\rho = 1$ is infinite. Geometrically the space looks like a bell.

This Euclidean background corresponds to a well defined worldsheet conformal field theory based on an $SU(2)/U(1)$ gauged WZW model at level $|k|$. The central charge remains the same after the continuation.

The worldsheet CFT is perfectly well behaved at $\rho = 1$.
In fact near this region the gauged WZW action is given to leading order by

$$S = \frac{-|k|}{2\pi} \int d^2 z \phi F_{z\bar{z}}$$

This is a simple topological theory, which shows that an alternative, non-singular description of the theory can be given including the region near $\rho = 1$.

Notice that worldsheet instanton configurations for which $\int F_{z\bar{z}} = 2\pi i n$ break the $U(1)$ symmetry corresponding to shifts of the angle ϕ to a discrete symmetry $Z_{|k|}$.

We argue now that the non-singular description of the theory is an almost-geometrical one in terms of a “T-fold”.

To obtain it, we perform a T-duality along the angular direction ϕ.

The resulting sigma model is based on the metric and dilaton

$$ds'^2 = \frac{\alpha'}{1-\rho'^2}\left(|k|d\rho'^2 + \frac{\rho'^2}{|k|}d\phi'^2\right), \quad e^{2\Phi'} = \frac{e^{2\Phi_0}}{|k|(1-\rho'^2)}$$

$$\rho' = (1-\rho^2)^{\frac{1}{2}}$$

Note that the transformation on ρ exchanges the boundary of the disk and its center.

The T-dual description is weakly coupled near $\rho = 1$ or $\rho' = 0$.

The only curvature singularity there is a benign orbifold singularity.

In fact, we can identify the T-dual as a $Z_{|k|}$ orbifold of the original model (at coupling $g_s(0)/\sqrt{k}$).

The level $|k|$ parafermionic theory and its $Z_{|k|}$ orbifold give two models with identical spectrum and partition function.

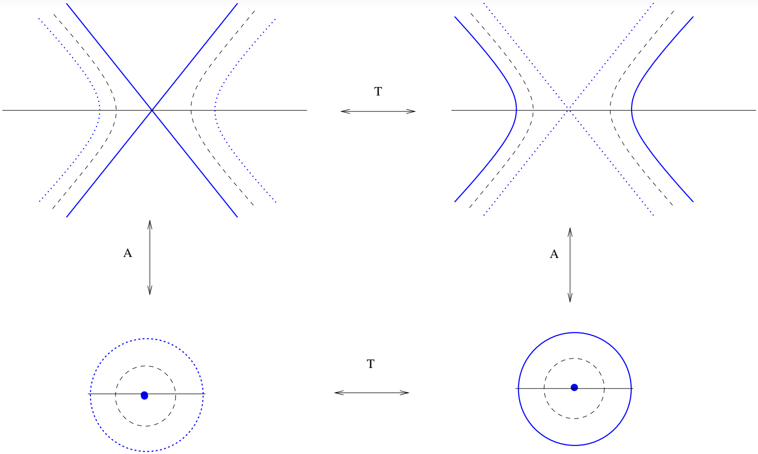
By gluing the two T-duals along a non-singular circle (e.g. at $\rho = \rho' = 1/\sqrt{2}$) we obtain a compact T-fold. This has no boundaries or singularities.
Notice that this gluing is non-geometrical as it involves a T-duality transformation on the fields.

One aspect of the model that is rendered manifest by the T-fold description is the breaking of the $U(1)$ rotation symmetry to a discrete $Z_{|k|}$ symmetry, due to the $Z_{|k|}$ orbifolding in one patch.

In the case of the cosmology as well, we can obtain a regular T-fold description as the target space of the conformal field theory.

T-duality interchanges the light-cone and the singularities.
We must glue the T-duals along a hyperbola in between the lightcone and the singularities.

The gluings are shown in the following figure.

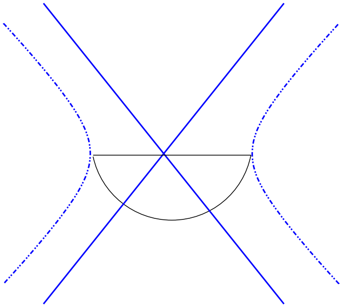


The resulting almost-geometrical description is very much like 2d de-Sitter space, which we can think of as a hyperboloid embedded in three-dimensional space.

Next we define a wavefunction for this cosmology.

We consider a time-reversal symmetric space-like slice of the cosmological T-fold.

In the past of the space-like slice, we glue half of the T-fold associated to the $SU(2)/U(1)$ coset conformal field theory.



We define the wave-function for this universe by performing a “half T-fold” Euclidean path integral over *all* target space fields with specified values on the boundary:

$$\Psi[h_\partial, \phi_\partial, \dots] = \int [dg][d\phi] \dots e^{-S(g,\phi,\dots)}$$

The path integral can in principle be performed off-shell, in a second-quantized string field theory context, where we may also express it as an integral over a single string field Φ.
No other boundary condition needs to be specified.

Notice that we are considering fluctuations around a classical string background, a solution to the string field theory equations of motion.

Of course wave-function thus defined is hard to compute, but we can understand some of its global properties by calculating its norm.

The norm of the wavefunction is given by the full Euclidean path integral and can be computed in a first quantized formalism.
It is given by a sum of CFT vacuum amplitudes over all closed worldsheet topologies, including disconnected diagrams.

In fact it is equal to the following exponential of a sum of connected diagrams

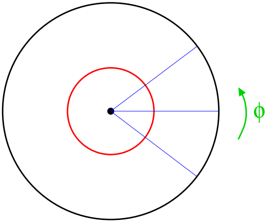
$$||\Psi||^2 = \exp(Z_{string})$$

where the function Z_{string} is the total string partition function

$$Z_{string} = \frac{1}{g_s^2} Z_{S^2} + Z_{T^2} + g_s^2 Z_{genus=2} + \sum_{g=3}^{\infty} g_s^{2g-2} Z_{genus=g}$$

A natural way to perform the Euclidean path integral is as follows.

Divide the space into angular wedges spanning an overall angle equal to 2π.
Then we can evaluate the path integral in terms of the generator of angular rotations, taking ϕ as Euclidean time.



A fermion going m times around the origin must pick up a phase $(-1)^m$.

Since the Euclidean time direction is compact (and the associated cycle is contractible), the space-time fermionic fields have to be taken anti-periodic, *and the full path integral can be interpreted as a thermal ensemble.*

As in the de Sitter case, the effective proper temperature is set by the curvature of space.
For large k, this is given by $T \sim 1/(2\pi\sqrt{|k|\alpha'})$ (also from semi-classical considerations such as particle production in the cosmology).

For small level $|k|$, we need a string calculation to deduce the proper temperature of the system.

We can construct explicitly low level $|k|$ models for which the effective temperature is below the Hagedorn temperature.

Let us give an argument why this is the case. Consider a $\hat{c} = 10$ superconformal model of the form

$$\frac{SU(2)_{|k|}}{U(1)} \quad \times \quad K$$

(the parafermionic factor has central charge $\hat{c} = 2 - 4/(|k| + 2)$).
The model has a Hagedorn transition at the fermionic radius $R_H = \sqrt{2\alpha'}$

As we can see from the spectrum of primaries, the $U(1)$ current associated to rotations in the disk is at level $|k| + 2$, and we can associate to it a radius

$$R = \sqrt{(|k| + 2)\alpha'}$$

This radius sets the temperature of the model. It is below the Hagedorn temperature for any $|k| \geq 0$ $\rightarrow$

When we are about to hit the Hagedorn temperature at $|k| = 0$, the cosmology disappears from the string theory background!
Let me now summarize some of our results for specific examples.

We choose the level $|k| = 2$, and we take the internal conformal field theory K to be:

$$K = T^2 \times \prod_{i=1,\ldots,7} \frac{SU(2)_{k_i}}{U(1)}$$

where all k_i's are also taken equal to 2.

Each superconformal parafermionic block is equivalent to a $\hat{c} = 1$ system of a real fermion and a boson compactified at the self dual radius.
On the worldsheet we can define and exploit an $N = 2$ (left and right) superconformal algebra.

Since the conformal field theories are compact, the genus-zero contribution to the total string partition function vanishes.

The leading contribution is the genus-1 amplitude.
As we already discussed, the relevant genus-1 string amplitude has to be thermal.

We compute it as follows. We can start with a supersymmetric, modular invariant partition function. This implements a GSO projection, getting rid of the NSNS tachyon.

To proceed we need to identify and insert the thermal co-cycle associated to the time direction of the cosmology. For type II theories, this takes the general form

$$S\begin{bmatrix} q, & (\alpha+\bar{\alpha}) \\ p, & (\beta+\bar{\beta}) \end{bmatrix} = e^{i\pi\left(p(\alpha+\bar{\alpha}) + q(\beta+\bar{\beta}) \right)}$$

Here (p, q) are momentum and winding charges associated to the compact Euclidean time cycle.

$F_\alpha = (\alpha + \bar{\alpha})$ and $F_\beta = (\beta + \bar{\beta})$ define the spin of space-time particles: $F_\alpha = 1$ modulo 2 for fermions and $F_\alpha = 0$ modulo 2 for bosons.

When inserted in the partition function, the co-cycle destroys the cancellations between fermions and bosons and insures that the contributions of fermionic states are in accordance with the spin-statistics connection.

So it is necessary to factor out a charge lattice (p, q) associated to the Euclidean time direction.
In the model at hand, we show that this is a $\Gamma_{1,1}$ lattice at radius $R = 2\sqrt{\alpha'}$ as we argued before
$\rightarrow$ T is below Hagedorn, no winding tachyons.

Thus the genus-1 amplitude is finite. This shows that the HH wavefunction is normalizable.

There is another attractive feature of the models at hand. The characters of the parafermionic blocks exhibit a Z_4 symmetry. We can use this and its Z_2 subgroup to define orbifold models.

We show that the contributions from the twisted sectors are topological and integrable.

This introduces interesting dependence to the genus-1 amplitude (and therefore the norm of the wavefunction) on the moduli of the target space torus T^2.

There are many other models to which the construction can be generalized by changing the internal CFT K.

Finally, we have considered a two-dimensional cosmology at small level $|k|$.
This choice is mainly due to an obstruction that is difficult (but not necessarily impossible) to circumvent:

It is difficult to construct compact models with positive central charge deficit (negatively curved Euclidean backgrounds) in string theory.

We could consider

$$\frac{SL(2,R)_{-|k|}}{U(1)} \times \frac{SL(2,R)_{|k|+4}}{U(1)} \times K.$$

and take $|k|$ to be an arbitrary parameter.

But then we must provide a consistent prescription to compactify the cigar factor in order for the wavefunction to be normalizable. This is an interesting problem to persue.

We have shown how the Hartle-Hwaking proposal can be generalized to a particular class of perturbative cosmological string backgrounds.

For this class, we have shown that the norm of the wavefunction is finite. It would be interesting to use it to identify “preferred” regions in the corresponding moduli space.