BLACK HOLES
&
LARGE ORDER
QUANTUM GEOMETRY

MidEast 2007

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CERN & IPM

This talk is based on:

arxiv.org/0704.2440
[hep-th]

in collaboration with:

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MOTIVATION

IS

A

LONG STANDING

QUESTION

SETUP OF THE QUESTION

START with: M theory in 11 dimensions

$\Downarrow$

COMPACTIFY IT ON A GENERIC C.Y._3

$\mathbb{X}$

$\Downarrow$

In 5D
this generic compactification
leads to
an associated

$$\mathcal{N} = 2$$

supergravity

$$\Downarrow$$

In particular
we get bunch of

U(1) vector multiplets

$$A_i \equiv C^{\mathrm{M.T.}}_{(3)}{}_{|_{\Sigma_i}}$$

$$\Sigma_i \text{ basis for } H^2(\mathbb{X}) \quad ; \quad i = 1, .., b_2(\mathbb{X})$$

$$\Downarrow$$

Take an ensemble of

M2 branes

wrapped on the 2-cycles

$$Q = Q_i \Sigma_i \in H_2(X, \mathbb{Z}) \;\; ; \;\; i = 1, \ldots, b_2(\mathbb{X})$$

$$\Downarrow$$

Q_i is the charge under A_i

Little group of massive particles in 5D is

$$SO(4) \simeq SU(2)_L \times SU(2)_R$$

BPS condition allows either of

$$j_L \quad \text{OR} \quad j_R$$

to be turned on

$$\Downarrow$$

Can LABEL
the associated spectrum of
5D BPS particles by
charg & spin

$$(Q, m) \quad ; \quad m \leftrightarrow SU(2)_L$$

therefore
ask for

$$\Omega(Q, m)$$

Let us turn to
the
gravity side
of the
5D story

⇓

LOOK FOR
the corresponding

electrically-charged rotating
black holes in the
SUGRA+CORRECTIONS

they are in fact generalization of
the BMPV black hole

J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa 1996

⇓

COMPUTE the

BEKENSTEIN-HAWKING-WALD

ENTROPY

of these

black holes:

$$S_{\rm macro} := S_{B.H.W}(Q, m)$$

QUESTION

IS:

COMPUTE

$$\Omega(Q, m)$$

within an appropriate
microscopic
ensemble
of the M_2 branes

& SHOW THAT:

$$S_{\mathrm{micro}} := \mathrm{Log}[\Omega(\mathrm{Q}, \mathrm{m})] \equiv \mathrm{S}_{\mathrm{macro}}$$

(to all orders in the asymptotic
large "charge" expansion)

CONJECTURE :

Katz, Klemm, Vafa
1999

THE MICROSCOPIC ENSEMBLE

IS GIVEN BY THE INDEX

$$\mathcal{Z} = \mathrm{Tr}_{\mathcal{H}_Q}(-1)^m y^m = \mathrm{F}_{\mathbb{A}}$$

WHICH IS:

* THE ELLIPTIC GENUS OF THE 5D SUSY THEORY

& EQUALS

*FREE ENERGY OF: A-MODEl T.S.T ON $\mathbb{X}$

$\Downarrow$

More precisely:

the A-model TST COUNTS the DEGENERACY

$$\Omega(Q, m)$$

in the following way:

Let us:

* t^i denote the size of the 2-cycles Σ_i as measured by the kähler form of the $\mathbb{X}$

* DEFINE the GOPAKUMAR-VAFA INVARIANTS through the relation:

$$\Omega(Q, m) = \sum_r \binom{2r+2}{m+r+1} n_Q^r$$

$$\Downarrow$$

we have the **EXACT EQUALITY:**

$$F_{\mathbb{A}}(\lambda;\mathbb{X}) = \sum_{g=0}^{\infty} F_g(\vec{t})\lambda^{2g-2}$$

$$= \sum_{r=0}^{\infty} \sum_{Q\in H_2(X,\mathbb{Z})} \sum_{k=1}^{\infty} \frac{1}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2r-2} \mathrm{e}^{-kQ\cdot t}\, n_Q^r$$

MORAL of this **CONJECTURE:**

IF we **COMPUTE** $F_0, \ldots, F_{\mathrm{O}[|Q|]}$ **INDEPENDENTLY**

we can **EXTRACT** $\Omega(Q,m)$

A TECHNICAL QUESTION !

How to COMPUTE

$$F_g(\lambda, t)$$

on a COMPACT C.Y. ?

ANSWER is:

a BLESSING SYMMETRY,

MIRROR SYMMETRY !

RECIPE:

MAP A/$\mathbb{X}$ TO B/its MIRROR $\tilde{\mathbb{X}}$

Then the B-model $F_g(\lambda, t)$ on $\tilde{\mathbb{X}}$ satisfy:

$$\bar{\partial}_{\bar{l}} F_g = \frac{1}{2} C_{\bar{l}}^{ij} \left(D_i D_j F_{g-1} + \sum_{h=1}^{g-1} D_i F_h D_j F_{g-h} \right) \ ; \ \ g \geq 2$$

(HOLOMORPHIC ANOMALY EQUATIONS)

INTEGRATE these EQUATIONS with PROPER BOUNDARY CONDITIONS to OBTAIN

$$F_g(\lambda; X) = F_g(\lambda, \tilde{X})$$

NEWS !

THANKS TO THE VERY RECENT PROGRESS IN SOLVING THE TST ON COMPACT C.Y.'s EXACT RESULTS FOR

$$F_g(\lambda, t)$$

ON THE QUINTIC C.Y.
UP TO GENUS 51 !

M. x. Huang, A. Klemm and S. Quackenbush 2006

A GLANCE AT THE:

MACROSCOPIC RESULTS

For the purpose of this talk

WE "JUST" CONSIDER

compact C.Y.'s with 1-kähler modulus

namely:

$$b_2(\mathbb{X}) = 1 \Rightarrow Q = d\,\Sigma_1 \;\; ; \;\; d \in \mathbb{Z}$$

Therefore we consider a

5D BPS black hole

of charge d & spin m

It has been KNOWN for a long time that
this is a BIG black hole
& its
BEKENSTEIN-HAWKING ENTROPY
is given by

$$S_{B.H.} = 2\pi\sqrt{\frac{2}{9\kappa}d^3 - m^2}$$

where:

κ is the ONLY INTERSECTION NUMBER of the C.Y.

$$\kappa = \int_{\mathbb{X}} (\eta_1)^3 \ ; \ \ \eta_1 \leftrightarrow \Sigma_1$$

BUT

THERE ARE HIGHER DERIVATIVE CORRECTIONS TO SUGRA WHOSE CONTRIBUTION TO THE ENTROPY IS CAPTURED BY THE WALD ENTROPY

In the case of 5D,
RECENTLY,
such a calculation has been done,
especially based on the (SUSY-COMPLETION of)
certain R^2 contributions to the EFFECTIVE ACTION

M. Guica, L. Huang, W. Li and A. Strominger 2005
A. Castro, J. L. Davis, P. Kraus and F. Larsen 2007
M. Alishahiha 2007

we need only to report here

the large-charge
asymptotic expansion of
the SPINLESS MACROSCOPIC ENTROPY
in the case of our
1-parameter C.Y.'s:

$$S^{\rm macro}_{0+1+\ldots}(d) = b_0 d^{3/2} + b_1 d^{1/2} + {\rm O}[d^{-1/2}]$$

$$b_0 = \frac{4\pi}{3\sqrt{2\kappa}} \quad ; \quad b_1 = c_2 \frac{\pi}{4\sqrt{2k}}$$

where: $c_2 = \int_X c_2(X) \wedge \eta_1$

This is now our purpose
to see whether

the
KKV conjecture
which proposes the

A-MODEL TST
AS THE
MICROSCOPIC INDEX THEORY OF THESE BLACK HOLES

is CLEVER enough to
CAPTURE:

THE SAME
ASYMPTOTIC DEGENERACIES

TECHNICAL WARNING !

IN ORDER TO COMPARE THE
TST RESULTS FOR THE DEGENERACIES

$$\Omega(d, m); \;\; \forall m$$

WITH THIER B.H.W COUNTERPART

$$\Downarrow$$

SHOULD COMPUTE: n_d^g

UP TO: $g \leq g_{\max} = \mathrm{O}[d]$

$$\Downarrow$$

SHOULD SOLVE: $F_g(t)$

TO VERY LARGE ORERS

$$O[d] \gg 1$$

TRICK

IS

using

proper

EXTRAPOLATION METHODS

A KNOWN EXAMPLE

IS

RICHARDSON TRANSFORMATION

THE IDEA OF RICHARDSON

Imagine that we have computed

$$\Omega(d)$$

up to

$$d_{\max} = 50$$

&

INDEPENDENTLY we have figured out that

$$S(d) = \mathrm{Log}[\Omega(d)]$$

admits the

LARGE ORDER ASYMPTOTIC EXPANSION :

$$S(d) = b_0 d^{3/2} + b_1 d^{1/2} + b_2 d^{-1/2} + \ldots$$

we want to

estimate the

coefficients of this series

{ b_0, b_1, b_2,... }

with very high accuracy

Step 0: Define a new series:

$$A(d,0) := \frac{S(d)}{d^{\frac{3}{2}}} = b_0 + b_1 d^{-1} + b_2 d^{-2} + \dots$$

This approximates b_0 up to $\mathrm{O}[\frac{1}{d}]$

Step 1: Define a new series:

$$A(d,1) = (d+1)\, A_0(d+1) - d\, A(d) = b_0 - \frac{b_1}{d^2} + \dots$$

This approximates b_0 up to $\mathrm{O}[\frac{1}{d^2}]$

$$\Downarrow$$

Step N: Define a new series:

$$A(d,N) = \sum_{k=0}^{N} \frac{A(d+k)(d+k)^N(-1)^{k+N}}{k!(N-k)!}$$

This approximates b_0 up to $\mathrm{O}[\frac{1}{d^{N+1}}]$

MORALS:

1. If we plot $A(d, N)$ in terms of d
we see that:
larger value of N,
faster it tends to the true value of b_0

2. Having found the value of b_0 ,
if we apply a similar procedure on
the series:

$$S_1(d) := S(d) - b_0 d^{3/2} = b_1 d^{\frac{1}{2}} + b_2 d^{\frac{-1}{2}} + ...$$

we'll get the coefficient b_1 & so on

.....

THE 13 1-PARAMETER C.I.CYS WHICH WE STUDY

DEFINITION AND RELEVANT DATA:

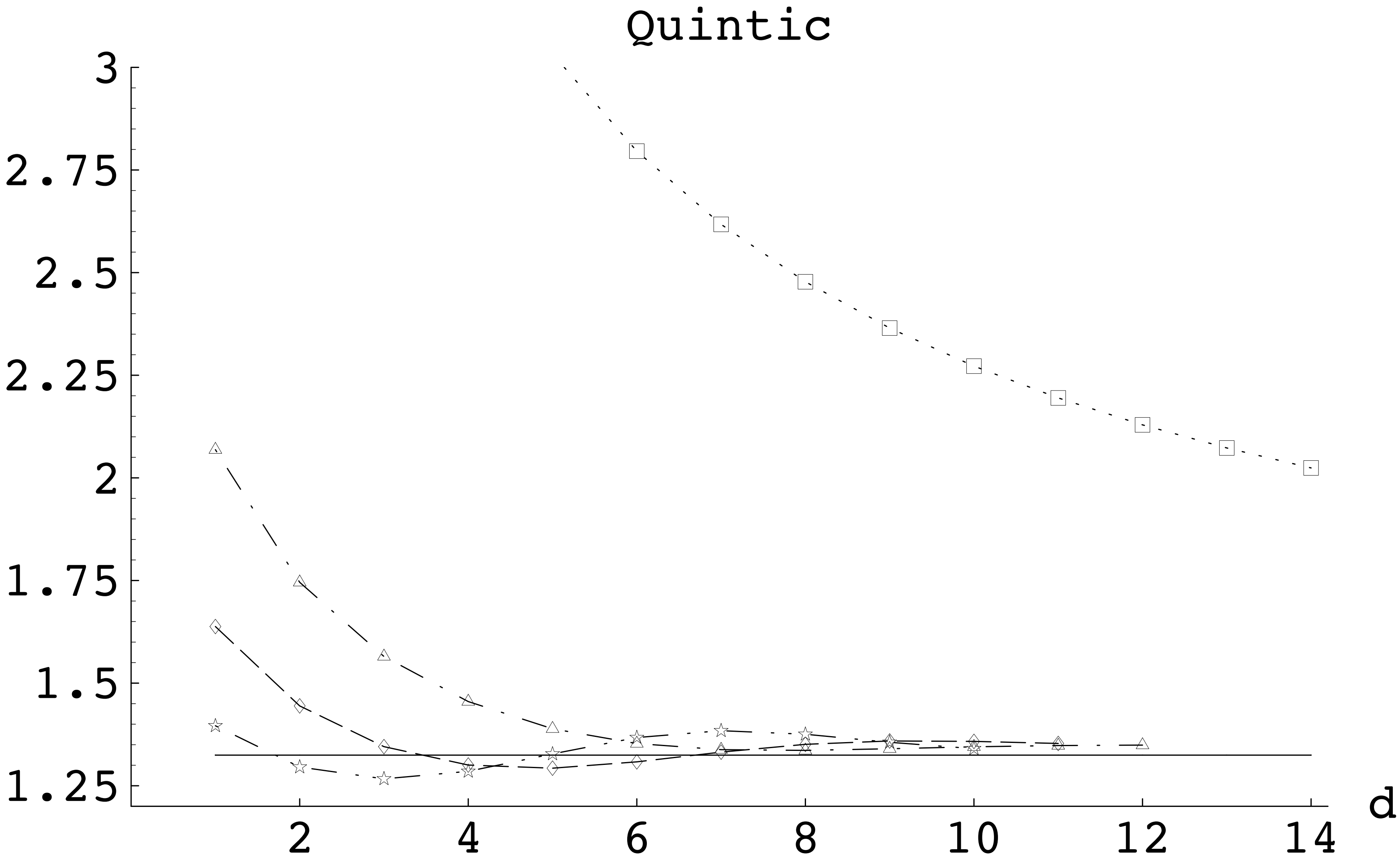
CY	χ	$c_2 \cdot \eta$	κ	CY	χ	$c_2 \cdot \eta$	κ
$X_5(1^5)$	-200	50	5	$X_6(1^4,2)$	-204	42	3
$X_8(1^4,4)$	-296	44	2	$X_{10}(1^3,2,5)$	-288	34	1
$X_{3,3}(1^6)$	-144	54	9	$X_{4,2}(1^6)$	-176	56	8
$X_{3,2,2}(1^7)$	-144	60	12	$X_{2,2,2,2}(1^8)$	-128	64	16
$X_{4,3}(1^5,2)$	-156	48	6	$X_{4,4}(1^4,2^2)$	-144	40	4
$X_{6,2}(1^5,3)$	-256	52	4	$X_{6,4}(1^3,2^2,3)$	-156	32	2
$X_{6,6}(1^2,2^2,3^2)$	-120	32	1				

A C.I.CY of degree $\{d_1, \ldots, d_k\}$ in a weighted projective space $\mathbb{P}^{l-1}(\omega_1, \ldots, \omega_l)$ is denoted : $\mathbb{X}_{d1,\ldots,d_k}(\omega_1, \ldots, \omega_l)$

$$c_2.\eta = \int_{\mathbb{X}} c_2(\mathbb{X}) \wedge \eta_1 \quad ; \quad \kappa = \int_{\mathbb{X}} (\eta_1)^3 \quad ; \quad \eta_1 \leftrightarrow \Sigma_1$$

The "horizontal line" in the plot denotes: b_0^{MACRO}

$b_0^{\mathrm{TST}}(d \mid \{0,1,2,3\})$



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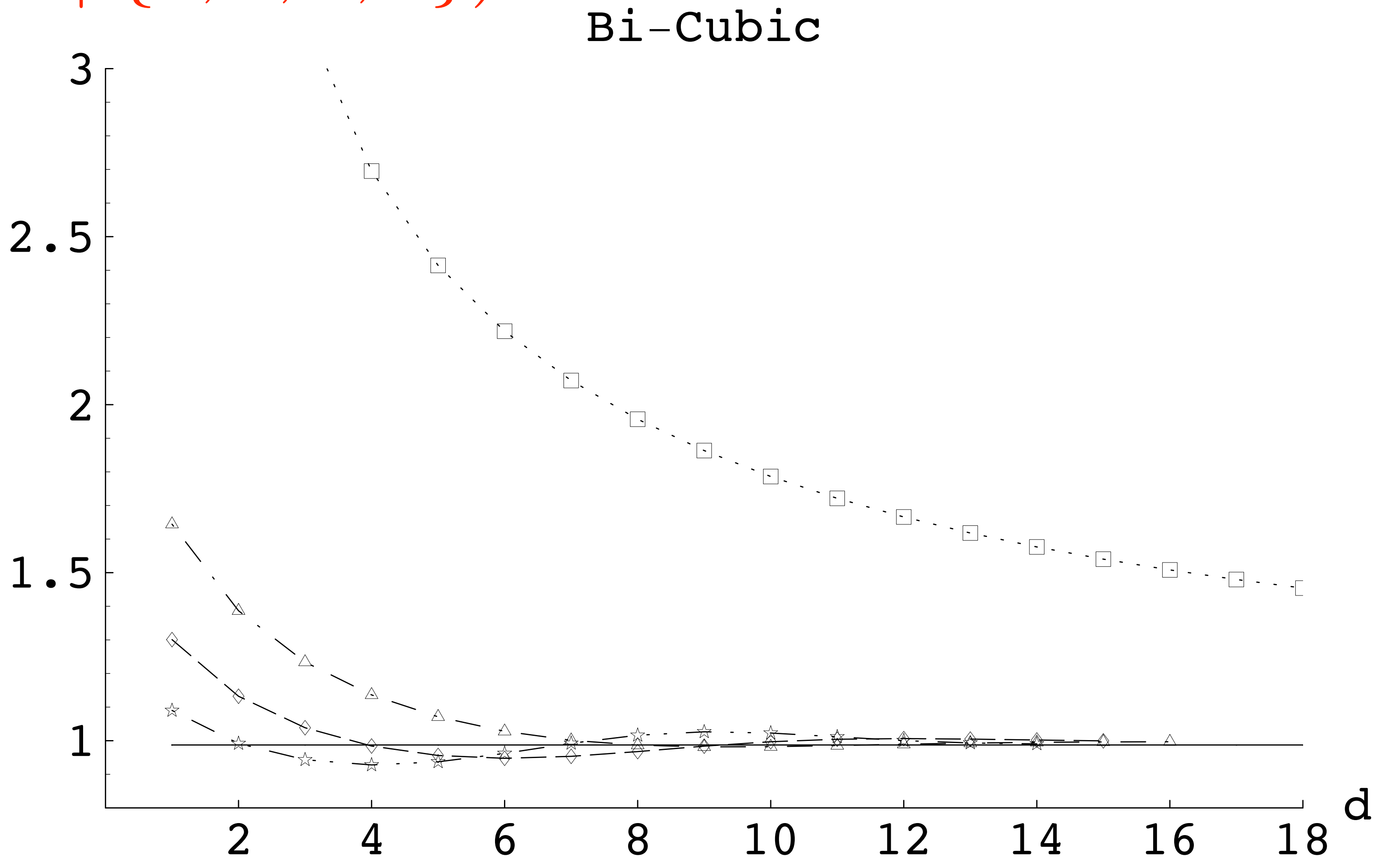
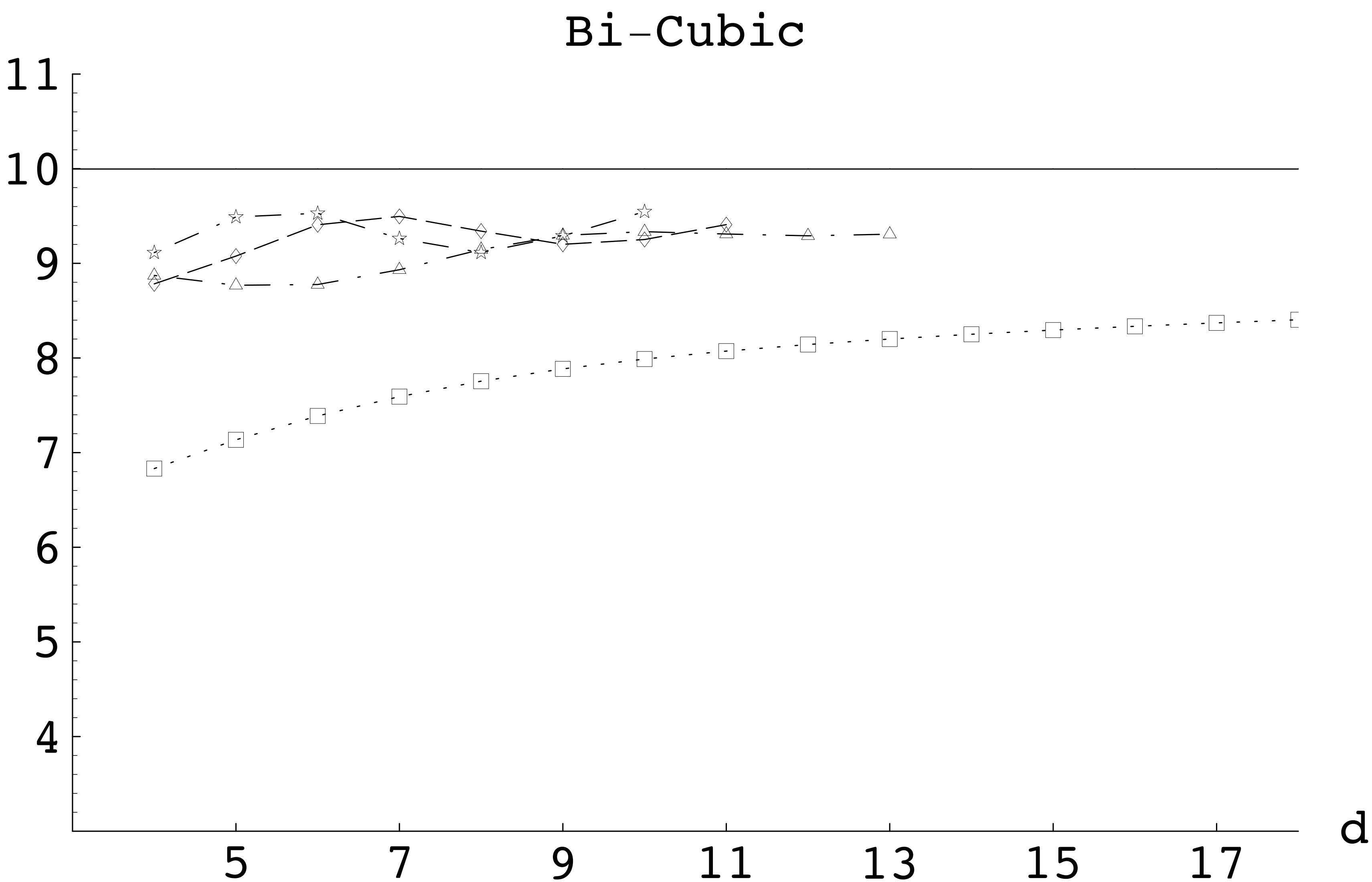


The COMPLETE TABLE OF THE MICRO-MACRO COMPARISON AT THE LEADING ORDER

Calabi-Yau	d_{max}	$A(d_{max}-3,3)$	$b_0=\frac{4\pi}{3\sqrt{2\kappa}}$	error
$X_5(1^5)$	14	1.35306	1.32461	2.15 %
$X_6(1^4,2)$	10	1.75559	1.71007	2.66 %
$X_8(1^4,4)$	7	2.11454	2.0944	0.96 %
$X_{10}(1^3,2,5)$	5	2.99211	2.96192	1.02 %
$X_{3,3}(1^6)$	17	1.00204	0.987307	1.49 %
$X_{4,2}(1^6)$	15	1.07031	1.0472	2.21 %
$X_{3,2,2}(1^7)$	10	0.821169	0.855033	-3.96 %
$X_{2,2,2,2}(1^8)$	13	0.722466	0.74048	-2.43 %
$X_{4,3}(1^5,2)$	11	1.21626	1.2092	0.58 %
$X_{6,2}(1^5,3)$	11	1.52785	1.48096	3.17 %
$X_{4,4}(1^4,2^2)$	7	1.42401	1.48096	-3.85 %
$X_{6,4}(1^3,2^2,3)$	5	2.06899	2.0944	-1.21 %
$X_{6,6}(1^2,2^2,3^2)$	4	2.95082	2.96192	-0.37 %

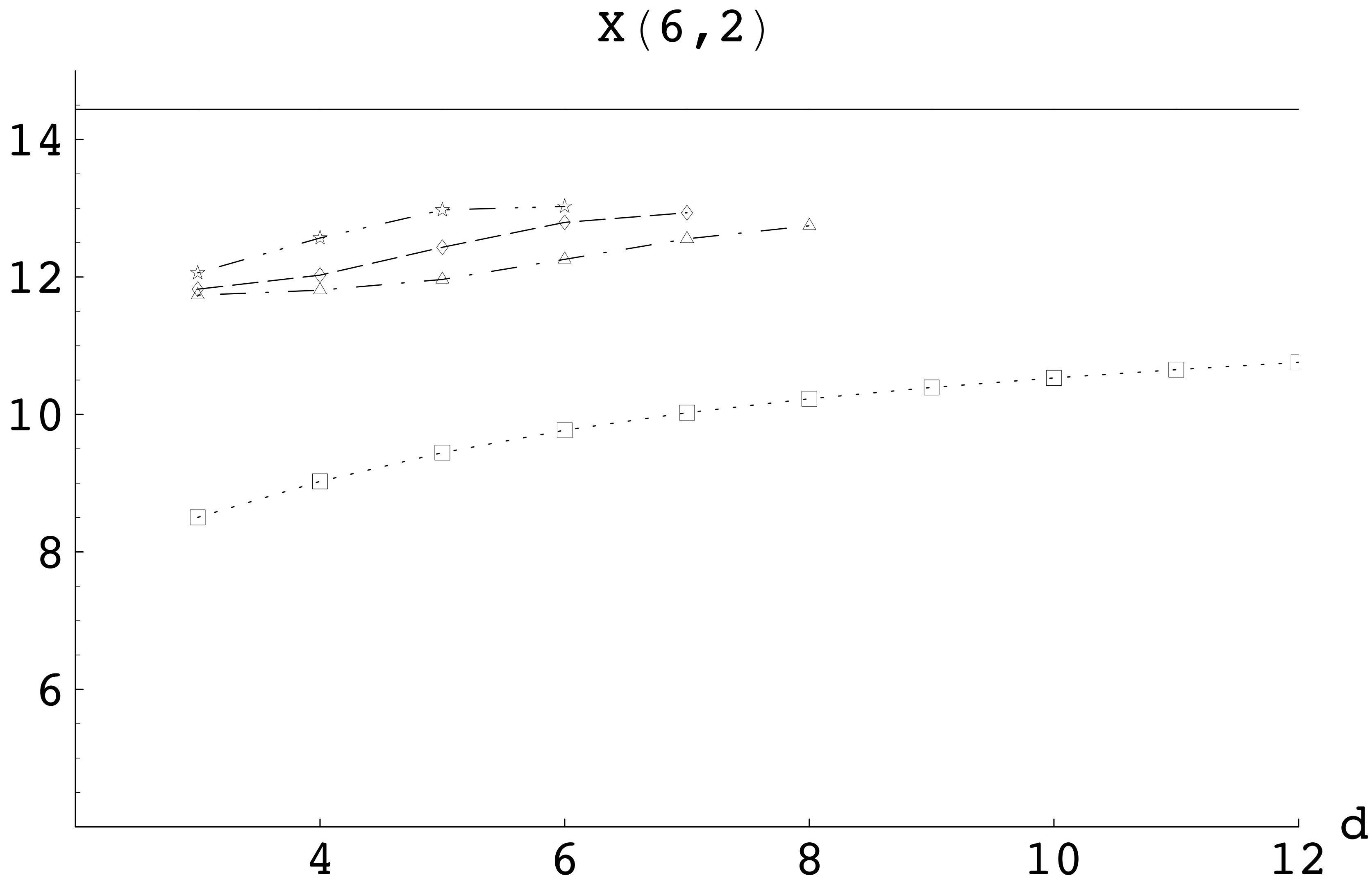
The "horizontal line" in the plot denotes: b_1^{MACRO}

$b_1^{\mathrm{TST}}(d \mid \{0,1,2,3\})$



The "horizontal line" in the plot denotes: b_1^{MACRO}

$b_1^{\mathrm{TST}}(d \mid \{0,1,2,3\})$



The COMPLETE TABLE OF THE MICRO-MACRO COMPARISON AT THE SUBLEADING ORDER

Calabi-Yau	d_{max}	$A_1(d_{max}-3,3)$	$b_1=\frac{\pi c_2}{4\sqrt{2\kappa}}$	error	estimated b_2
$X_5(1^5)$	14	11.2668	12.4182	-9.27 %	-11.9503
$X_6(1^4,2)$	10	11.9237	13.4668	-11.5 %	-12.1848
$X_8(1^4,4)$	7	14.0537	17.2788	-18.7 %	-14.9973
$X_{10}(1^3,2,5)$	5	15.2509	18.8823	-19.2 %	-14.9817
$X_{3,3}(1^6)$	17	9.29062	9.99649	-7.06 %	-9.63958
$X_{4,2}(1^6)$	15	10.0226	10.9956	-8.85 %	-10.7834
$X_{3,2,2}(1^7)$	10	8.45163	9.61912	-12.1 %	-9.3828
$X_{2,2,2,2}(1^8)$	13	7.84595	8.88577	-11.7 %	-8.88773
$X_{4,3}(1^5,2)$	11	9.5981	10.8828	-11.8 %	-9.96404
$X_{6,2}(1^5,3)$	11	12.5614	14.4394	-13.0 %	-14.2582
$X_{4,4}(1^4,2^2)$	7	9.70091	11.1072	-12.7 %	-9.41295
$X_{6,4}(1^3,2^2,3)$	5	11.1008	12.5664	-11.7 %	-10.0821
$X_{6,6}(1^1,2^2,3^3)$	4	11.1378	12.2179	-8.84 %	-8.15739

A SIMILAR ANALYSIS
OF THE
SPINNING CASE

$$\Omega(d, m \neq 0)$$

SHOWS
A QUALITATIVELY
AND
QUANTITATIVELY
SIMILAR
MATCHING BETWEEN
THE TST PREDICTIONS
AND
THE MACROSCOPIC RESULTS

WE SKIP TO THE ANALYTIC SECTOR OF OUR STUDY

ANALYTIC
ASYMPTOTIC RESULTS:

TST PREDICTIONS
FOR
DEGENERACIES OF:
SMALL BHs
IN
K3-FIBRATIONS

HERE WE CONSIDER:

$$\begin{array}{ccc} K3 & \rightarrow & \mathbb{X} \\ & & \downarrow \\ & & S \end{array}$$

& RESTRICT THE M_2-BRANE CHARGE Q

TO SIT IN THE $K3$ FIBER:

$$Q \in \mathrm{Pic}(K3)$$

THIS WAY

WE ENGINEER:

5D BPS

SMALL

BLACK HOLES

WHICH ARE NECESSARILY

SPINLESS

$$\Downarrow$$

$$\mathrm{Log}[\Omega^{\mathrm{TST}}(Q)] \longleftarrow \text{compare} \longrightarrow S^{B.H.W}(Q)$$

COMPACTIFICATIONS OF M-THEORY

ON K3-FIBRATIONS

GENERICALLY ADMIT A

CELEBRATED HETEROTIC-DUAL DESCRIPTION

IN WHICH

THE HETEROTIC COUPLING

IS IDENTIFIED WITH

THE SIZE OF THE BASE

THEREFORE
IN THE LIMIT:

$$\mathrm{VOL}[\mathcal{S}] \longrightarrow \infty$$

CAN EXTRACT

EXACT NON-PERTURBATIVE INFORMATION

IN THE M-THEORY SIDE

BASED ON 1-LOOP CALCULATIONS

IN THE HETEROTIC SIDE

IN PARTICULAR :

THE EXACT GENERATING FUNCTION

OF THE GOPAKUMAR-VAFA INVARIANTS

$$n^r_Q$$

ASSOCIATED TO THE

FIBER-MEMBRANES

CAN BE EXTRACTED IN CLOSED ANALYTIC FORMS

THIS WAY

THE PARTITION FUNCTION OF

THE FIBER-MEMBRANES

IS COMPUTED AS THE FOLLOWING:

$$\sum_{Q\in \mathrm{Pic}(\mathrm{K3})} \sum_{m=-\infty}^{\infty} \Omega(Q,m)\mathrm{e}^{2\pi i(m\nu+Q^2/2\,\sigma)}$$

$$= f_{\mathbb{X}}(\mathrm{e}^{2\pi i\sigma})\xi^2(\nu,\sigma)$$

WHERE:

$f_{\mathbb{X}}(\mathrm{e}^{2\pi i\sigma})$: A MODEL-DEPENDENT MODULAR FORM KNOWN FOR VARIOUS CHOICES OF $\mathbb{X}$

&

$$\xi(\nu,\sigma) = 2\sin(\pi\nu)\frac{\eta^3(\sigma)}{\vartheta_1(\nu|\sigma)}$$

$$f_{\mathbb{X}}(\mathrm{e}^{2\pi i\sigma})$$

FOR 3 CASES OF INTEREST:

1. $\mathbb{X} = K3 \times T^2 \quad \longrightarrow \quad f_{\mathbb{X}} = \eta(\sigma)^{-24}$

(THE CLASSIC BMPV BLACK HOLE)

2. STU model $\longrightarrow \quad f_{\mathbb{X}} = -2\dfrac{E_4(\mathrm{e}^{2\pi\mathrm{i}\sigma})\,\mathrm{E}_6(\mathrm{e}^{2\pi\mathrm{i}\sigma})}{\eta(\sigma)^{12}}$

3. $\mathbb{X} = \text{Enriques C.Y.} \quad \longrightarrow \quad f_{\mathbb{X}} = -2\,\eta(2\sigma)^{-12}$

ANALYTIC EVALUATION OF THE MICROSCOPIC DEGENERACIES:

IN ORDER TO COMPARE THE TST PREDICTIONS
WITH
THEIR MACROSCOPIC COUNTERPARTS

SHOULD EVALUATE THE LARGE CHARGE ASYMPTOTIC EXPANSION
OF THE FOURIER INTEGRAL:

$$\Omega(N) = \int_{-\frac{1}{2}+i0^+}^{\frac{1}{2}+i0^+} \mathrm{d}\sigma \int_0^1 \mathrm{d}\nu \, \mathrm{e}^{-2\pi i N \sigma} \Phi(\nu, \sigma)$$

$$N = Q^2/2 \in \mathbb{Z} \quad ; \quad \Phi(\nu, \sigma) = f_X(p) \xi^2(\nu, \sigma)$$

THIS ASYMPTOTIC EVALUATION CAN BE PERFORMED EXACTLY MODULO THE EXPONENTIALLY SUPPRESSED CORRECTIONS

$$S^{\mathrm{TST}}_{\mathrm{K3\times T^2}} \sim 4\pi\sqrt{N} - \frac{29}{4}\mathrm{Log}(N) + \mathrm{Log}\left(\frac{\sqrt{2}}{\pi}\right) - \frac{591}{32\pi}\frac{1}{\sqrt{N}} + \mathrm{O}[N^{-1}]$$

$$S^{\mathrm{TST}}_{\mathrm{STU}} \sim 4\pi\sqrt{N} - \frac{9}{4}\mathrm{Log}(N) + \mathrm{Log}\left(\frac{\sqrt{8}}{\pi}\right) + \frac{129}{32\pi}\frac{1}{\sqrt{N}} + \mathrm{O}[N^{-1}]$$

$$S^{\mathrm{TST}}_{E}(Q) \sim \pi\sqrt{8N} - \frac{17}{4}\mathrm{Log}(N) + \mathrm{O}[N^{0}]$$

THE LEADING MICROSCOPIC ENTROPY IN A NUTSHELL

$$S^{TST}(Q) \sim 2\pi\sqrt{\frac{c_{\mathcal{S}}}{12}\, Q^2}$$

$$c_{\mathcal{S}} = \int c_2(\mathbb{X}) \wedge \eta_{\mathcal{S}}$$

$C_{\mathcal{S}} = 12$: Enriques

$C_{\mathcal{S}} = 24$: Other Cases including the ST model

THE MACROSCOPIC COUNTERPARTS

CAN USE THE 5D-4D CONNECTION

RECIPE:

1. MAP:

$$\text{D6 charge} \;:\; p0 = \text{T.N.charge}$$

$$\text{D2 charge}: \; q_i = \frac{Q_i}{p^0}$$

$$\text{D0 charge}: q_0 = \frac{2j_L}{(p^0)^2}$$

2. COMPUTE THE B.H.W ENTROPY FROM THE PREPOTENTIAL:

$$\mathcal{F} = -\frac{1}{2}C_{ab}\frac{X^{\mathcal{S}}X^aX^b}{X^0} - \frac{1}{6}C_{abc}\frac{X^aX^bX^c}{X^0}$$

$$C_{abc} = \int_{\mathbb{X}} \eta_a \wedge \eta_b \wedge \eta_c \quad ; \quad \eta_a \in H_2(K3;\mathbb{Z})$$

$$C_{ab} = \int_{\mathbb{X}} \eta_{\mathcal{S}} \wedge \eta_a \wedge \eta_b$$

RESULT:

DOING SO

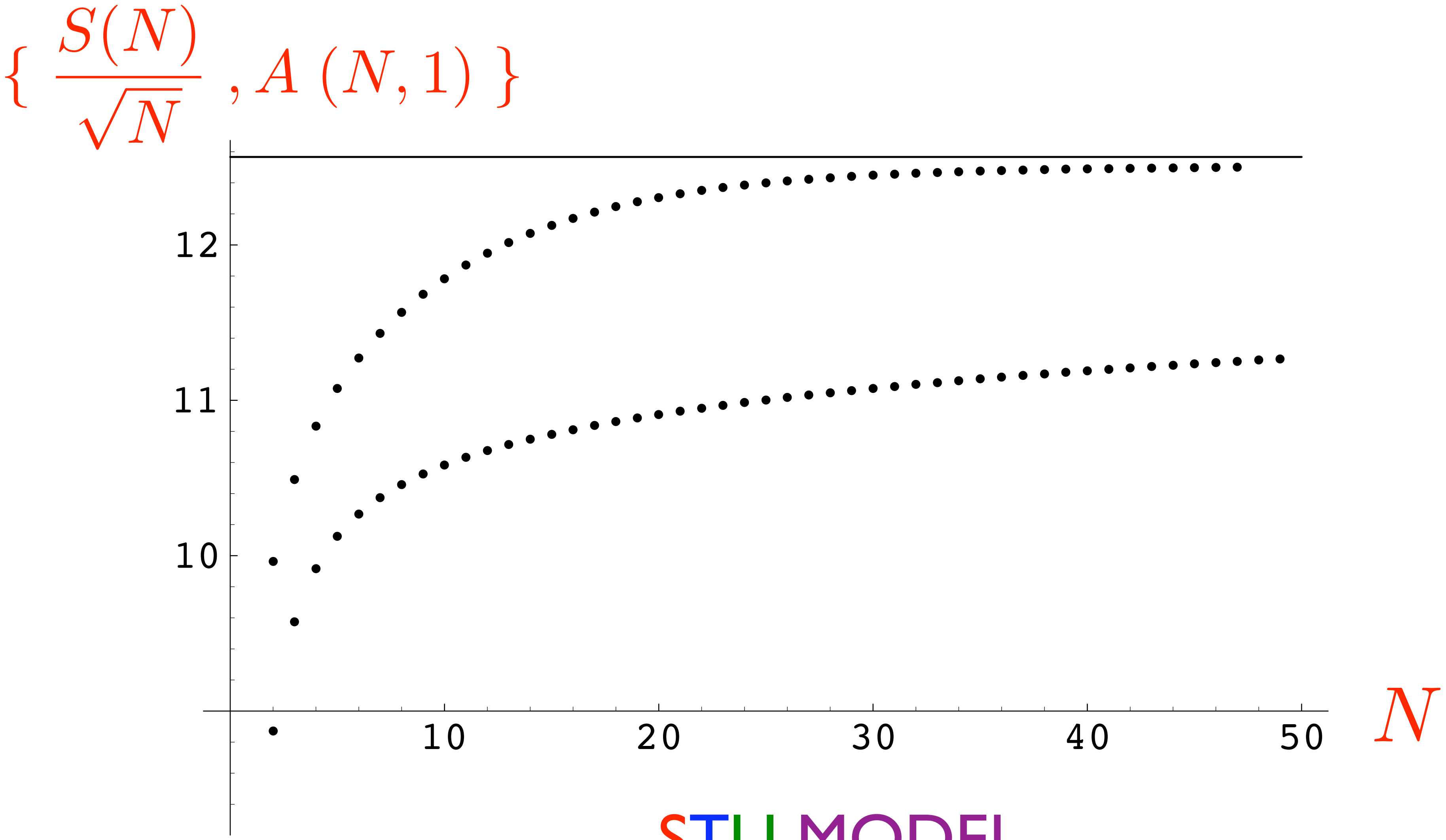
WE GET PERFECT MATCHING

WITH THE MACROSCOPIC ENTROPIES

AT THE LEADING ORDER

FOR ALL THE CASES.

The "horizontal line" in the plot denotes the expected value from the macroscopic analysis: 4π



STU MODEL

A QUESTION:

IN VIEW OF THE INTERPRETATION OF

THE A-MODEL TST
AS THE
MICROSCOPIC THEORY
OF THE GENERALIZED BMPV BLACK HOLES,

CAN WE LEARN ABOUT
THE
NONPERTURBATIVE COMPLETION
OF
THE TOPOLOGICAL STRING

THE OTHER WAY AROUND ?