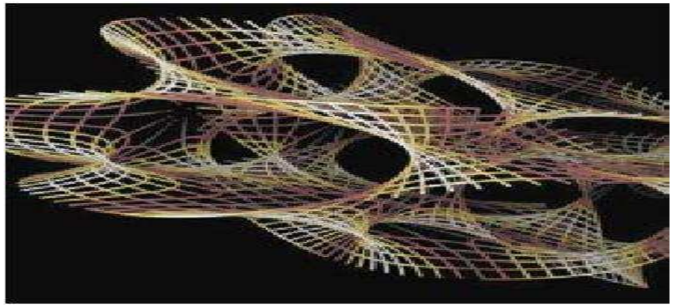
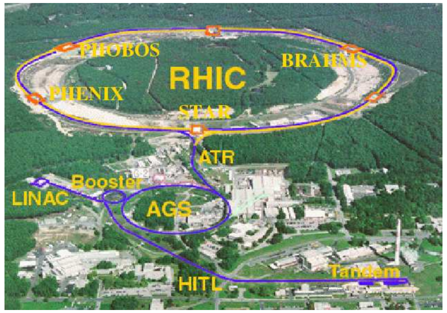
Patra - June 2007

Ringtones of black holes, hydrodynamics and phase transitions





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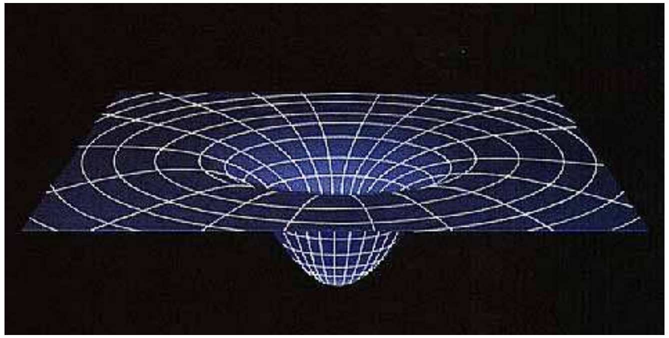


OUTLINE

- Quasi-normal modes of AdS black holes
- Heavy ion collisions and hydrodynamics
- Phase transitions in topological black holes
- Conclusion

S. Musiri, S. Ness and G. S., *Phys. Rev.* **D73** (2006) 064001.
G. Koutsoumbas, S. Musiri, E. Papantonopoulos and G. S., *JHEP* **0610** (2006) 006.
G. S., *JHEP* **0705** (2007) 042.

Quasi-normal modes of black holes

Quasi-normal modes (QNMs) describe small perturbations of a black hole.

- A black hole is a thermodynamical system whose (Hawking) temperature and entropy are given in terms of its global characteristics (total mass, charge and angular momentum).

QNMs obtained by solving a wave equation for small fluctuations subject to the conditions that the flux be

- ingoing at the horizon and
- outgoing at asymptotic infinity.

$\Rightarrow$ discrete spectrum of complex frequencies.

- imaginary part determines the decay time of the small fluctuations

$$\Im\omega = \frac{1}{\tau}$$

AdS Black Holes

AdS/CFT correspondence:

$\Rightarrow$ QNMs for AdS b.h. expected to correspond to perturbations of dual CFT.

establishment of correspondence hindered by difficulties in solving wave eq.

- In 3d: Hypergeometric equation $\therefore$ solvable

  [Cardoso, Lemos; Birmingham, Sachs, Solodukhin]

- In 5d: Heun equation $\therefore$ unsolvable.
- Numerical results in 4d, 5d and 7d

  [Horowitz, Hubeny; Starinets; Konoplya]

Asymptotic form of QNMs of AdS black holes

Approximation to the wave equation valid in the high frequency regime.

- In 3d: exact equation.
- In 5d: Heun eq. $\rightarrow$ Hypergeometric eq., as in low frequency regime.
  - analytical expression for asymptotic form of QNM frequencies
  - in agreement with numerical results.

Gravitational perturbations

AdS Schwarzschild black holes with metric in d dimensions

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \ , \quad f(r) = \frac{r^2}{R^2} + 1 - \frac{2\mu}{r^{d-3}} \ .$$

► derive analytical expressions including first-order corrections.

► results in good agreement with results of numerical analysis.

radial wave equation

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi \ ,$$

in terms of the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \ .$$

potential V from Master Equation *[Ishibashi and Kodama]*

For tensor, vector and scalar perturbations, we obtain, respectively,

[Natário and Schiappa]

$$V_{\mathsf{T}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right\}$$

$$V_{\mathsf{V}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} - \frac{rf'''(r)}{2(d-3)}\right\}$$

$$\begin{aligned}
V_{\mathsf{S}}(r) &= \frac{f(r)}{4r^2}\left[\ell(\ell+d-3)-(d-2)+\frac{(d-1)(d-2)\mu}{r^{d-3}}\right]^{-2} \\
&\times \left\{\frac{d(d-1)^2(d-2)^3\mu^2}{R^2r^{2d-8}} - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{R^2r^{d-5}}\right. \\
&+\frac{(d-4)(d-6)[\ell(\ell+d-3)-(d-2)]^2r^2}{R^2} + \frac{2(d-1)^2(d-2)^4\mu^3}{r^{3d-9}} \\
&+\frac{4(d-1)(d-2)(2d^2-11d+18)[\ell(\ell+d-3)-(d-2)]\mu^2}{r^{2d-6}} \\
&+\frac{(d-1)^2(d-2)^2(d-4)(d-6)\mu^2}{r^{2d-6}} - \frac{6(d-2)(d-6)[\ell(\ell+d-3)-(d-2)]^2\mu}{r^{d-3}} \\
&-\frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{r^{d-3}} \\
&\left.+4[\ell(\ell+d-3)-(d-2)]^3 + d(d-2)[\ell(\ell+d-3)-(d-2)]^2\right\}
\end{aligned}$$

Near the black hole singularity ($r \sim 0$),

$$V_{\mathsf{T}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{T}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots , \quad \mathcal{A}_{\mathsf{T}} = \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell+d-3)}{d-2},$$

$$V_{\mathsf{V}} = \frac{3}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{V}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots \ , \quad \mathcal{A}_{\mathsf{V}} = \frac{d^2-8d+13}{2(2d-15)} + \frac{\ell(\ell+d-3)}{d-2}$$

and

$$V_{\mathsf{S}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{S}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots ,$$

where

$$\mathcal{A}_{\mathsf{S}} = \frac{(2d^3-24d^2+94d-116)}{4(2d-5)(d-2)} + \frac{(d^2-7d+14)[\ell(\ell+d-3)-(d-2)]}{(d-1)(d-2)^2}$$

We may summarize the behavior of the potential near the origin by

$$V = \frac{j^2-1}{4r_*^2} + \mathcal{A}\, r_*^{-\frac{d-1}{d-2}} + \dots$$

where $j = 0$ (2) for scalar and tensor (vector) perturbations.

for large r,

$$V = \frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} + \dots \quad , \qquad \bar{r}_* = \int_0^\infty \frac{dr}{f(r)}$$

where $j_\infty = d-1$, $d-3$ and $d-5$ for tensor, vector and scalar perturbations, respectively.

After rescaling the tortoise coordinate $(z = \omega r_*)$, wave equation

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}}\mathcal{H}_1\right)\Psi = 0,$$

where

$$\mathcal{H}_0 = \frac{d^2}{dz^2} - \left[\frac{j^2-1}{4z^2} - 1\right] \;\; , \;\; \mathcal{H}_1 = -\mathcal{A}\, z^{-\frac{d-1}{d-2}}.$$

By treating $\mathcal{H}_1$ as a perturbation, we may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-\frac{d-3}{d-2}}\Psi_1(z) + \dots$$

and solve wave eq. perturbatively.

The zeroth-order wave equation,

$$\mathcal{H}_0 \Psi_0(z) = 0,$$

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1 \sqrt{z}\, J_{\frac{j}{2}}(z) + A_2 \sqrt{z}\, N_{\frac{j}{2}}(z).$$

For large z, it behaves as

$$\begin{aligned} \Psi_0(z) &\sim \sqrt{\frac{2}{\pi}} \left[A_1 \cos(z - \alpha_+) + A_2 \sin(z - \alpha_+) \right], \\ &= \frac{1}{\sqrt{2\pi}} (A_1 - iA_2) e^{-i\alpha_+} e^{iz} + \frac{1}{\sqrt{2\pi}} (A_1 + iA_2) e^{+i\alpha_+} e^{-iz} \end{aligned}$$

where $\alpha_\pm = \frac{\pi}{4}(1 \pm j)$.

large z ($r \longrightarrow \infty$)

wavefunction ought to vanish $\therefore$ acceptable solution

$$\Psi(r_*) = B\sqrt{\omega(r_* - \bar{r}_*)}\ J_{\frac{j_\infty}{2}}(\omega(r_* - \bar{r}_*))$$

NB: $\Psi \longrightarrow 0$ as $r_* \longrightarrow \bar{r}_*$, as desired.

Asymptotically, it behaves as

$$\Psi(r_*) \sim \sqrt{\frac{2}{\pi}}B\cos\left[\omega(r_* - \bar{r}_*) + \beta\right] \ , \quad \beta = \frac{\pi}{4}(1 + j_\infty)$$

match this to asymptotic behavior in the vicinity of the black-hole singularity along the Stokes line $\Im z = \Im(\omega r_*) = 0$

$\Rightarrow$ constraint on the coefficients $A_1,\ A_2$,

$$A_1 \tan(\omega\bar{r}_* - \beta - \alpha_+) - A_2 = 0.$$

impose boundary condition at the horizon

$$\Psi(z) \sim e^{iz} \ , \quad z \rightarrow -\infty \ ,$$

$\Rightarrow$ second constraint

analytically continue wavefunction near the origin to negative values of z.

- rotation of z by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-2}$ near the origin in the complex r-plane.

using

$$J_\nu(e^{-i\pi}z) = e^{-i\pi\nu}J_\nu(z)\ , \quad N_\nu(e^{-i\pi}z) = e^{i\pi\nu}N_\nu - 2i\cos\pi\nu\, J_\nu(z)$$

for $z < 0$, the wavefunction changes to

$$\Psi_0(z) = e^{-i\pi(j+1)/2}\sqrt{-z}\,\left\{\left[A_1 - i(1+e^{i\pi j})A_2\right]\,J_{\frac{j}{2}}(-z) + A_2 e^{i\pi j}\,N_{\frac{j}{2}}(-z)\right\}$$

whose asymptotic behavior is given by

$$\Psi \sim \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}}\left[A_1 - i(1+2e^{j\pi i})A_2\right]\,e^{-iz} + \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}}\left[A_1 - iA_2\right]\,e^{iz}$$

$\Rightarrow$ second constraint

$$A_1 - i(1+2e^{j\pi i})A_2 = 0$$

constraints compatible provided

$$\left|\begin{array}{cc} 1 & -i(1+2e^{j\pi i}) \\ \tan(\omega\bar{r}_* - \beta - \alpha_+) & -1 \end{array}\right| = 0$$

∴ quasi-normal frequencies

$$\omega \bar{r}_* = \frac{\pi}{4}(2 + j + j_\infty) - \tan^{-1} \frac{i}{1 + 2e^{j\pi i}} + n\pi$$

[Natário and Schiappa]

First-order corrections

[Musiri, Ness and Siopsis]

To first order, the wave equation becomes

$$\mathcal{H}_0\Psi_1 + \mathcal{H}_1\Psi_0 = 0$$

The solution is

$$\Psi_1(z) = \sqrt{z}\, N_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'}\, J_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}} - \sqrt{z}\, J_{\frac{j}{2}}(z) \int_0^z dz' \frac{\sqrt{z'}\, N_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}}$$

$\mathcal{W} = 2/\pi$ is the Wronskian.

$\therefore$ wavefunction up to first order

$$\Psi(z) = \{A_1[1 - b(z)] - A_2 a_2(z)\}\sqrt{z} J_{\frac{j}{2}}(z) + \{A_2[1 + b(z)] + A_1 a_1(z)\}\sqrt{z} N_{\frac{j}{2}}(z)$$

where

$$\begin{aligned} a_1(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\; z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') J_{\frac{j}{2}}(z') \\ a_2(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\; z'^{-\frac{1}{d-2}} N_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z') \\ b(z) &= \frac{\pi\mathcal{A}}{2}\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\; z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z') \end{aligned}$$

$\mathcal{A}$ depends on the type of perturbation.

asymptotically

$$\Psi(z) \sim \sqrt{\frac{2}{\pi}}\,[A_1' \cos(z-\alpha_+) + A_2' \sin(z-\alpha_+)]\ ,$$

where

$$A_1' = [1-\bar{b}]A_1 - \bar{a}_2 A_2 \ \ , \quad A_2' = [1+\bar{b}]A_2 + \bar{a}_1 A_1$$

and we introduced the notation

$$\bar{a}_1 = a_1(\infty) \ \ , \quad \bar{a}_2 = a_2(\infty) \ \ , \quad \bar{b} = b(\infty)\ .$$

First constraint modified to

$$A_1' \tan(\omega \bar{r}_* - \beta - \alpha_+) - A_2' = 0$$

$\therefore$

$$[(1-\bar{b})\tan(\omega\bar{r}_*-\beta-\alpha_+)-\bar{a}_1]A_1-[1+\bar{b}+\bar{a}_2\tan(\omega\bar{r}_*-\beta-\alpha_+)]A_2 = 0$$

For second constraint,

$\hookrightarrow$ approach the horizon

$\hookrightarrow$ rotate by $-\pi$ in the z-plane

$$a_1(e^{-i\pi}z) = e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}a_1(z) ,$$
$$a_2(e^{-i\pi}z) = e^{-i\pi\frac{d-3}{d-2}}\left[e^{i\pi j}a_2(z) - 4\cos^2\frac{\pi j}{2}a_1(z) - 2i(1+e^{i\pi j})b(z)\right] ,$$
$$b(e^{-i\pi}z) = e^{-i\pi\frac{d-3}{d-2}}\left[b(z) - i(1+e^{-i\pi j})a_1(z)\right]$$

$\therefore$ in the limit $z \to -\infty$,

$$\Psi(z) \sim -ie^{-ij\pi/2}B_1\cos(-z-\alpha_+) - ie^{ij\pi/2}B_2\sin(-z-\alpha_+)$$

where

$$\begin{aligned} B_1 &= A_1 - A_1 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] \\ &\quad - A_2 e^{-i\pi\frac{d-3}{d-2}}\left[e^{+i\pi j}\bar{a}_2 - 4\cos^2\frac{\pi j}{2}\bar{a}_1 - 2i(1+e^{+i\pi j})\bar{b}\right] \\ &\quad - i(1+e^{i\pi j})\left[A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}\bar{a}_1\right] \\ B_2 &= A_2 + A_2 e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1 e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}\bar{a}_1 \end{aligned}$$

$\therefore$ second constraint

$$[1 - e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1 + \bar{b})]A_1 - [i(1+2e^{i\pi j}) + e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1 + e^{i\pi j}\bar{a}_2 - i\bar{b})]A_2 = 0$$

compatibility of the two first-order constraints,

$$\begin{vmatrix} 1+\bar{b}+\bar{a}_2\tan(\omega\bar{r}_*-\beta-\alpha_+) & i(1+2e^{i\pi j})+e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1+e^{i\pi j}\bar{a}_2-i\bar{b}) \\ (1-\bar{b})\tan(\omega\bar{r}_*-\beta-\alpha_+)-\bar{a}_1 & 1-e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1+\bar{b}) \end{vmatrix} = 0$$

$\Rightarrow$ first-order expression for quasi-normal frequencies,

$$\begin{aligned} \omega\bar{r}_* &= \frac{\pi}{4}(2+j+j_\infty)+\frac{1}{2i}\ln 2+n\pi \\ &\quad -\frac{1}{8}\left\{6i\bar{b}-2ie^{-i\pi\frac{d-3}{d-2}}\bar{b}-9\bar{a}_1+e^{-i\pi\frac{d-3}{d-2}}\bar{a}_1+\bar{a}_2-e^{-i\pi\frac{d-3}{d-2}}\bar{a}_2\right\} \end{aligned}$$

where

$$\begin{aligned} \bar{a}_1 &= \frac{\pi\mathcal{A}}{4}\left(\frac{n\pi}{2\bar{r}_*}\right)^{-\frac{d-3}{d-2}} \frac{\Gamma(\frac{1}{d-2})\Gamma(\frac{j}{2}+\frac{d-3}{2(d-2)})}{\Gamma^2(\frac{d-1}{2(d-2)})\Gamma(\frac{j}{2}+\frac{d-1}{2(d-2)})} \\ \bar{a}_2 &= \left[1+2\cot\frac{\pi(d-3)}{2(d-2)}\cot\frac{\pi}{2}\left(-j+\frac{d-3}{d-2}\right)\right]\bar{a}_1 \\ \bar{b} &= -\cot\frac{\pi(d-3)}{2(d-2)}\,\bar{a}_1 \end{aligned}$$

▶ first-order correction is $\sim O(n^{-\frac{d-3}{d-2}})$.

4d

compare with numerical results *[Cardoso, Konoplya and Lemos]*

set the AdS radius $R = 1$: radius of horizon r_H related to black hole mass μ by

$$2\mu = r_H^3 + r_H$$

$f(r)$ has two more (complex) roots, r_- and its complex conjugate, where

$$r_- = e^{i\pi/3}\left(\sqrt{\mu^2 + \frac{1}{27}} - \mu\right)^{1/3} - e^{-i\pi/3}\left(\sqrt{\mu^2 + \frac{1}{27}} + \mu\right)^{1/3}$$

The integration constant in the tortoise coordinate is

$$\bar{r}_* = \int_0^\infty \frac{dr}{f(r)} = -\frac{r_-}{3r_-^2 + 1}\ln\frac{r_-}{r_H} - \frac{r_-^*}{3r_-^{*2} + 1}\ln\frac{r_-^*}{r_H}$$

Scalar perturbations

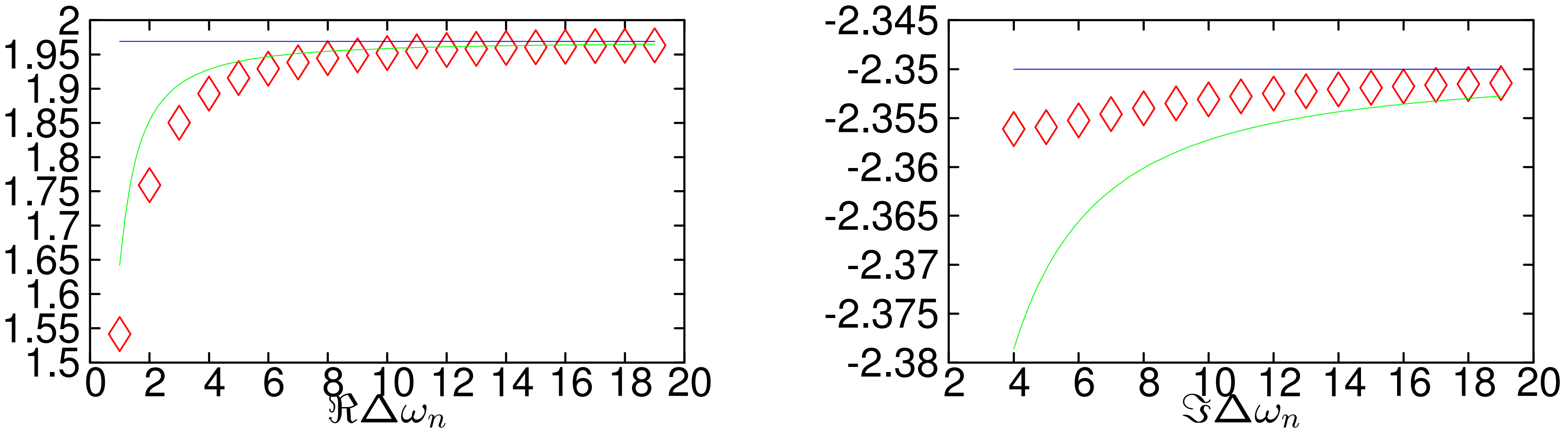


Fig. 1: $r_H = 1$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_{\mathsf{S}}\Gamma^4(\frac{1}{4})}{16\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\,, \quad \mathcal{A}_{\mathsf{S}} = \frac{\ell(\ell+1)-1}{6}$$

only the first-order correction is ℓ-dependent.

In the limit of large horizon radius ($r_H \approx (2\mu)^{1/3} \gg 1$),

$$\bar{r}_* \approx \frac{\pi(1+i\sqrt{3})}{3\sqrt{3}r_H}$$

Numerically for $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.508 + 0.293i}{r_H^2\sqrt{n}}$$

which compares well with the result of numerical analysis,

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.581 - 0.41i$$

including both leading order and offset.

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.654 + 0.458i}{\sqrt{n}}$$

In Fig. 1 we compare with data from numerical analysis. We plot the gap

$$\Delta\omega_n = \omega_n - \omega_{n-1}$$

because the offset does not always agree with numerical results.

▶ numerical estimates of the offset ought to be improved.

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.093 + 0.561i}{\sqrt{n}}$$

to be compared with the result of numerical analysis,

$$(\omega_n)_{\text{numerical}} \approx (1.61 - 0.6i)n + 2.7 - 0.37i$$

The two estimates of the offset disagree with each other.

Tensor perturbations

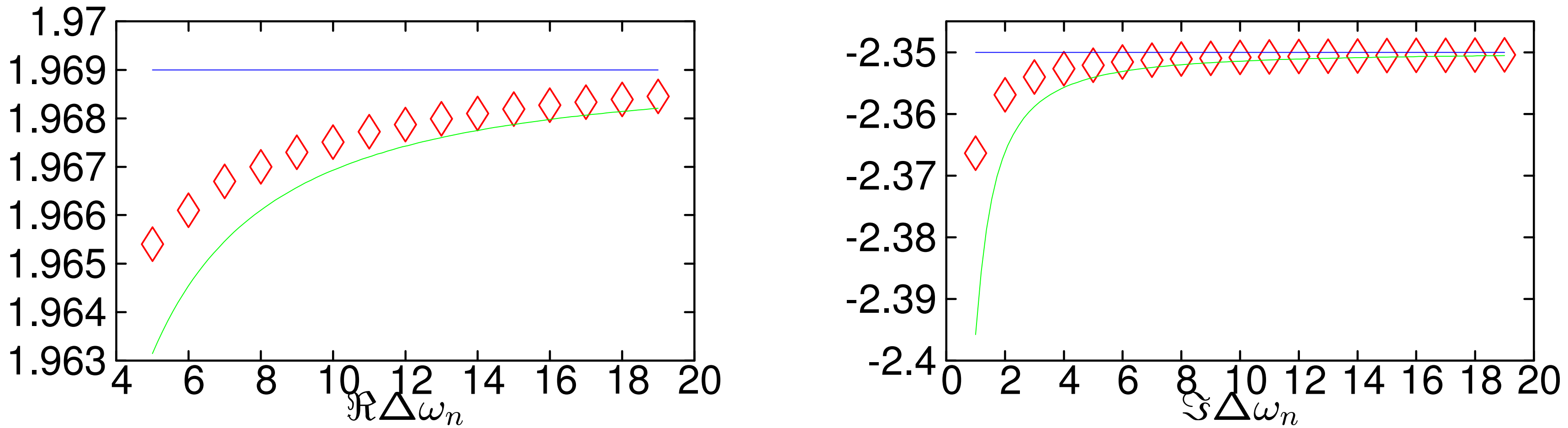


Fig. 2: $r_H = 1$ and $\ell = 0$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_\mathsf{T}\Gamma^4(\frac{1}{4})}{16\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}\,, \quad \mathcal{A}_\mathsf{T} = \frac{3\ell(\ell+1)+1}{6}$$

Numerically for large r_H and $\ell = 0$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{0.102 + 0.0586i}{r_H^2\sqrt{n}}$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.131 + 0.0916i}{\sqrt{n}}$$

in good agreement with the result of numerical analysis (Fig. 2), including the offset.

For a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 2.182 - 0.615i + \frac{0.489 + 0.251i}{\sqrt{n}}$$

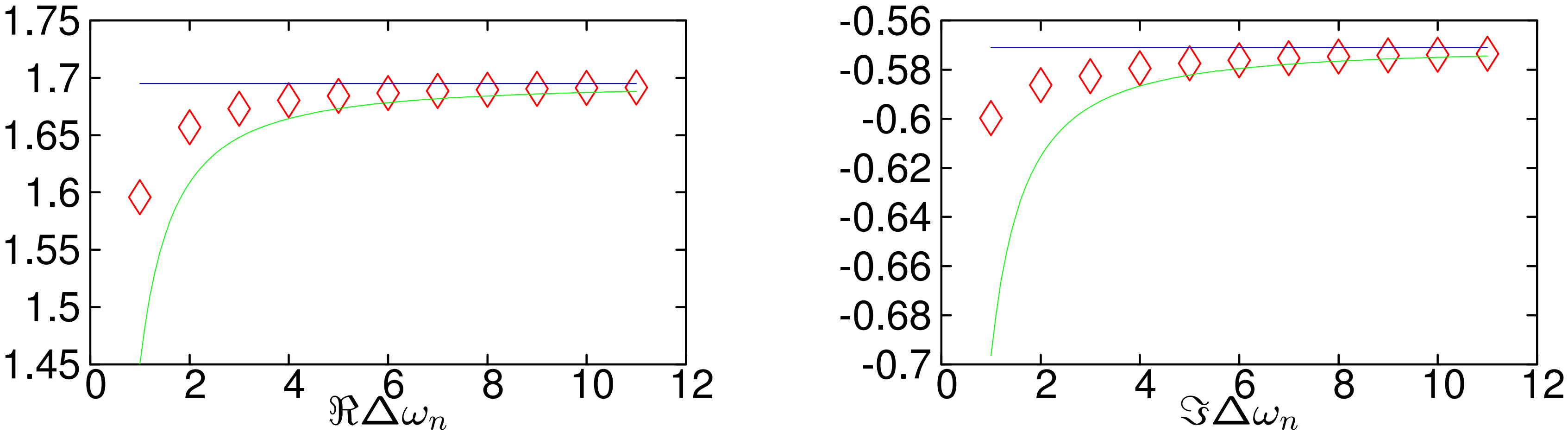


Fig. 3: $r_H = 0.2$ and $\ell = 0$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

Vector perturbations

$$\omega_n \bar{r}_* = \left(n + \frac{1}{4}\right)\pi + \frac{i}{2}\ln 2 + e^{i\pi/4}\frac{\mathcal{A}_V \Gamma^4(\frac{1}{4})}{48\pi^2}\sqrt{\frac{\bar{r}_*}{2\mu n}}, \quad \mathcal{A}_V = \frac{\ell(\ell+1)}{2} + \frac{3}{14}$$

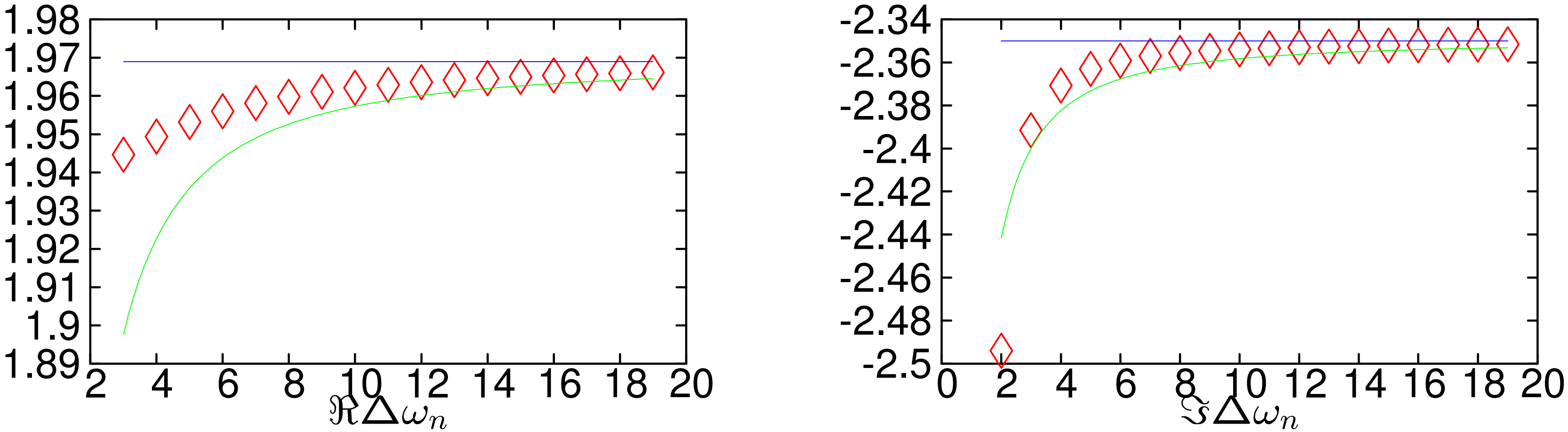


Fig. 4: $r_H = 1$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical (eq. ()) compared with numerical data (diamonds).

Numerically for large r_H and $\ell = 2$,

$$\frac{\omega_n}{r_H} = (1.299 - 2.250i)n + 0.573 - 0.419i + \frac{8.19 + 6.29i}{r_H^2\sqrt{n}}$$

to be compared with the result of numerical analysis,

$$\left(\frac{\omega_n}{r_H}\right)_{\text{numerical}} \approx (1.299 - 2.25i)n + 0.58 - 0.42i$$

For an intermediate black hole, $r_H = 1$, we obtain

$$\omega_n = (1.969 - 2.350i)n + 0.752 - 0.370i + \frac{0.741 + 0.519i}{\sqrt{n}}$$

and for a small black hole, $r_H = 0.2$, we obtain

$$\omega_n = (1.695 - 0.571i)n + 0.487 - 0.0441i + \frac{1.239 + 0.6357i}{\sqrt{n}}$$

estimates of the offset agree for large r_H but diverge as $r_H \to 0$.

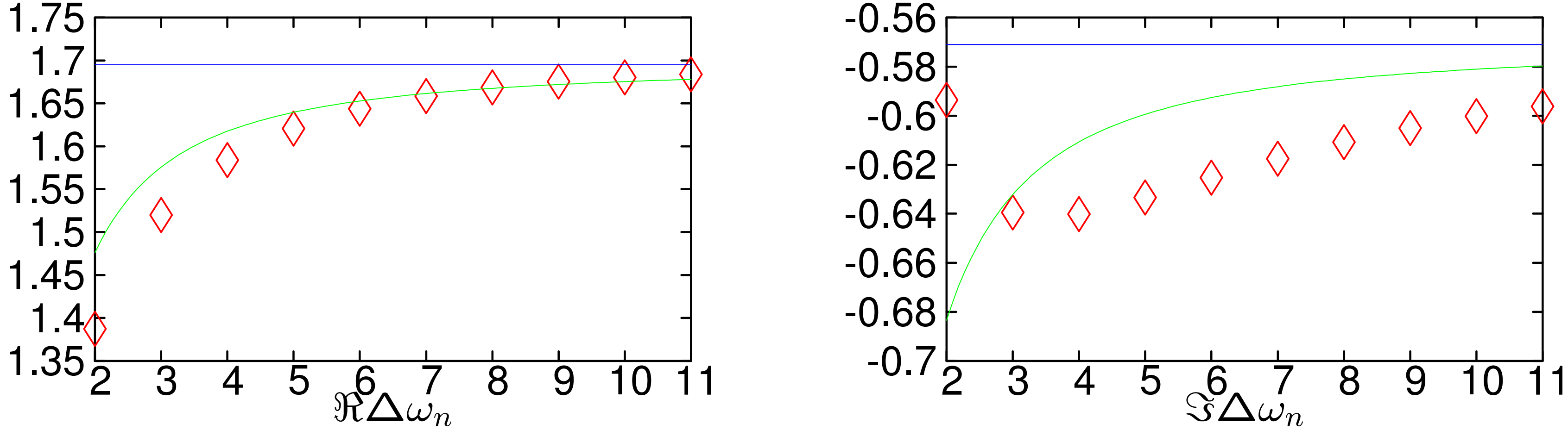


Fig. 5: $r_H = 0.2$ and $\ell = 2$: zeroth (horizontal line) and first order (curved line) analytical (eq. ()) compared with numerical data (diamonds).

Electromagnetic perturbations

electromagnetic potential

$$V_{\text{EM}} = \frac{\ell(\ell+1)}{r^2} f(r).$$

Near the origin,

$$V_{\text{EM}} = \frac{j^2 - 1}{4r_*^2} + \frac{\ell(\ell+1) r_*^{-3/2}}{2\sqrt{-4\mu}} + \dots ,$$

where $j = 1$ - vanishing potential to zeroth order!

▶ need to include first-order corrections for QNMs.

QNMs

$$\omega \bar{r}_* = n\pi - \frac{i}{4} \ln n + \frac{1}{2i} \ln \left(2(1+i)\mathcal{A}\sqrt{\bar{r}_*} \right) , \quad \mathcal{A} = \frac{\ell(\ell+1)}{2\sqrt{-4\mu}}$$

▶ correction behaves as $\ln n$.

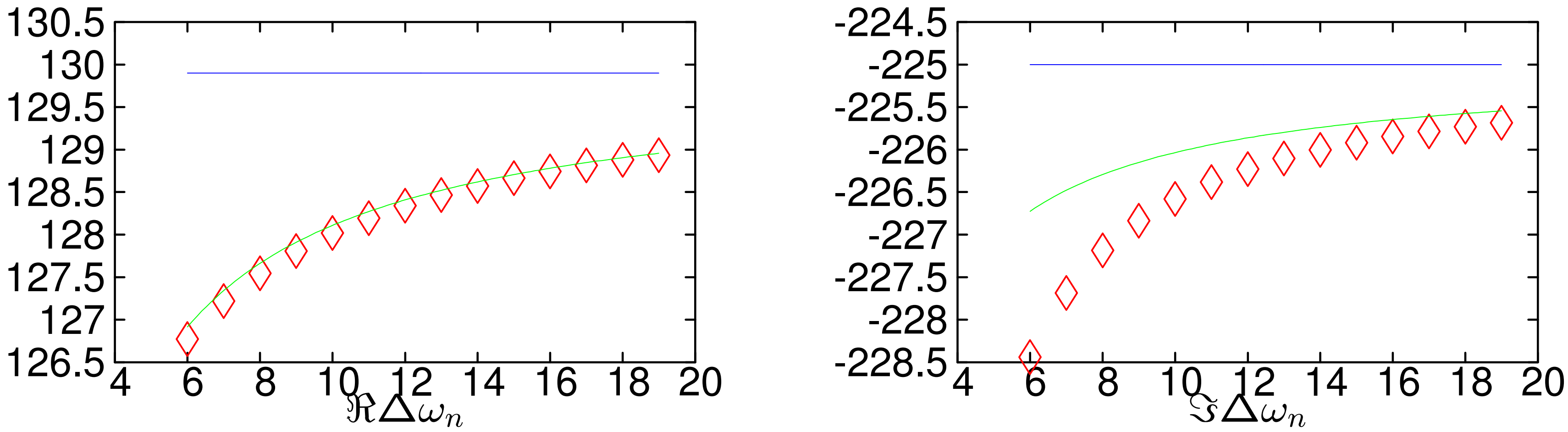


Fig. 6: $r_H = 100$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

For a large black hole, we obtain the spectrum

$$\frac{\Delta\omega_n}{r_H} \approx \frac{3\sqrt{3}(1-i\sqrt{3})}{4}\left(1 - \frac{i}{4\pi n} + \dots\right) = 1.299 - 2.25i - \frac{0.179 + 0.103i}{n} + \dots$$

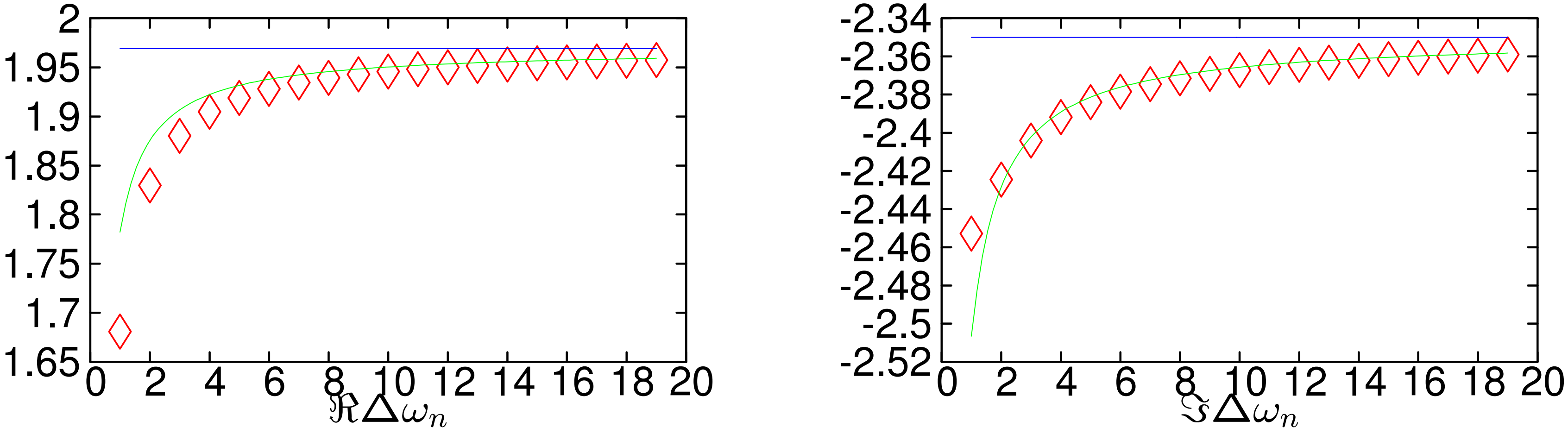


Fig. 7: $r_H = 1$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

For an intermediate black hole, $r_H = 1$,

$$\omega_n = (1.969 - 2.350i)n - (0.187 + 0.1567i)\ln n + \dots$$

and for a small black hole, $r_H = 0.2$,

$$\omega_n = (1.695 - 0.571i)n - (0.045 + 0.135i)\ln n + \dots$$

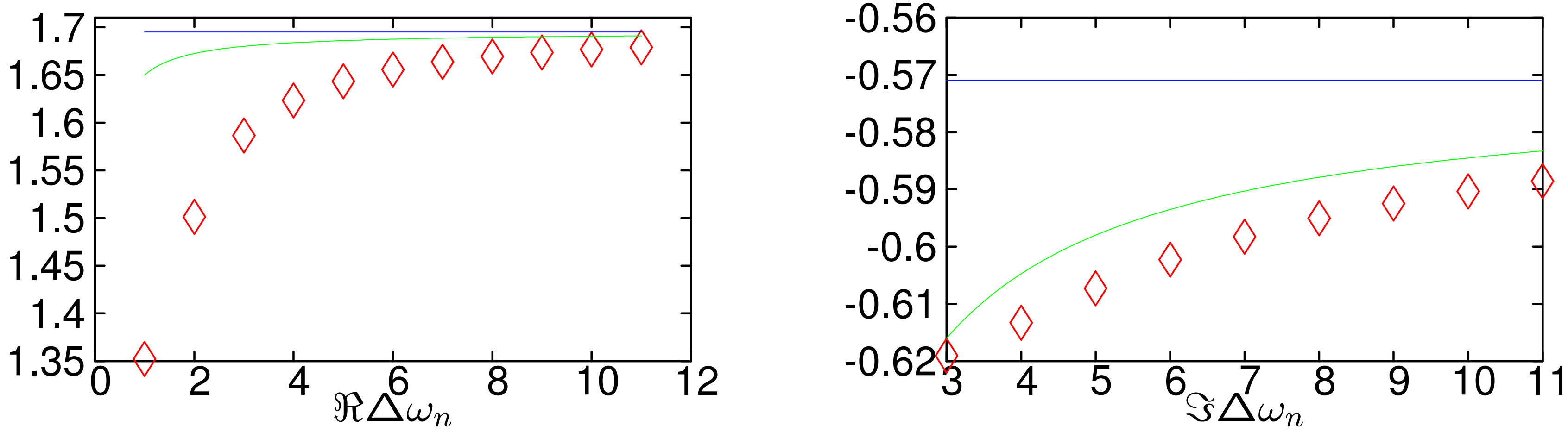
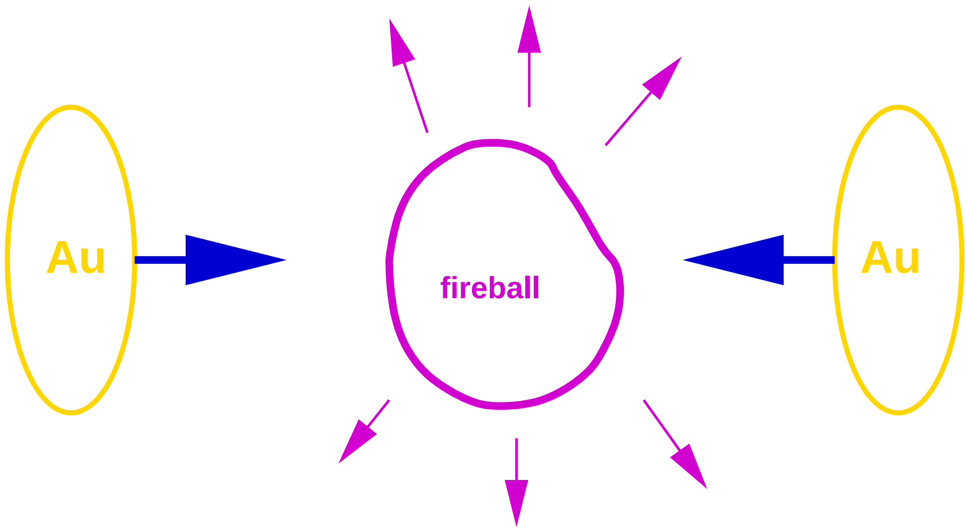


Fig. 8: $r_H = 0.2$ and $\ell = 1$: zeroth (horizontal line) and first order (curved line) analytical compared with numerical data (diamonds).

All first-order analytical results are in good agreement with numerical results.

"A second unexpected connection comes from studies carried out using the Relativistic Heavy Ion Collider, a particle accelerator at Brookhaven National Laboratory. This machine smashes together nuclei at high energy to produce a hot, strongly interacting plasma. Physicists have found that some of the properties of this plasma are better modeled (via duality) as a tiny black hole in a space with extra dimensions than as the expected clump of elementary particles in the usual four dimensions of spacetime. The prediction here is again not a sharp one, as the string model works much better than expected. String-theory skeptics could take the point of view that it is just a mathematical spinoff. However, one of the repeated lessons of physics is unity - nature uses a small number of principles in diverse ways. And so the quantum gravity that is manifesting itself in dual form at Brookhaven is likely to be the same one that operates everywhere else in the universe."

– Joe Polchinski



AdS/CFT correspondence and hydrodynamics

[Policastro, Son and Starinets]

correspondence between $\mathcal{N} = 4$ SYM in the large N limit and type-IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

- in strong coupling limit of field theory, string theory is reduced to classical supergravity, which allows one to calculate all field-theory correlation functions.

  $\hookrightarrow$ nontrivial prediction of gauge theory/gravity correspondence

entropy of $\mathcal{N} = 4$ SYM theory in the limit of large 't Hooft coupling is precisely 3/4 the value in zero coupling limit.

long-distance, low-frequency behavior of any interacting theory at finite temperature must be described by fluid mechanics (hydrodynamics).

universality: hydrodynamics implies very precise constraints on correlation functions of conserved currents and stress-energy tensor:

- correlators fixed once a few transport coefficients are known.

Hydrodynamics

conserved current: j^μ

chemical potential $\mu = 0$, so in thermal equilibrium

$$\langle j^0 \rangle = 0$$

retarded thermal Green function

$$G^R_{\mu\nu}(\omega, \boldsymbol{q}) = -i \int d^4x \, e^{-iq\cdot x} \, \theta(t) \langle [j_\mu(x), \, j_\nu(0)] \rangle \, ,$$

where $q = (\omega, \boldsymbol{q})$, $x = (t, \boldsymbol{x})$

▶ determines response to a small external source coupled to the current.

ω and $\boldsymbol{q}$ small:

- external perturbation varies slowly in space and time
- macroscopic hydrodynamic description for its evolution is possible.

diffusion equation

$$\partial_0 j^0 = D \nabla^2 j^0 \, ,$$

where D is a diffusion constant with dimension of length.

$\Rightarrow$ overdamped mode, dispersion relation

$$\omega = -iDq^2\,,$$

pole at $\omega = -iDq^2$ in the complex ω-plane, in the retarded correlation functions of j^0

stress-energy tensor $T^{\mu\nu}$

$$\partial_0\tilde{T}^{00} + \partial_i T^{0i} = 0\,,$$
$$\partial_0 T^{0i} + \partial_j \tilde{T}^{ij} = 0\,,$$

where

$$\tilde{T}^{00} = T^{00} - \rho, \qquad \rho = \langle T^{00}\rangle\,,$$
$$\tilde{T}^{ij} = T^{ij} - p\delta^{ij} = -\frac{1}{\rho+p}\Big[\eta\Big(\partial_i T^{0j} + \partial_j T^{0i} - \frac{2}{3}\delta^{ij}\partial_k T^{0k}\Big) + \zeta\delta^{ij}\partial_k T^{0k}\Big],$$

ρ (p): energy density (pressure)

η (ζ): shear (bulk) viscosity.

two types of eigenmodes:

- the shear modes - transverse fluctuations of momentum density T^{0i}, with a purely imaginary eigenvalue

$$\omega = -iDq^2 \quad , \quad D = \frac{\eta}{\rho + p} \, ,$$

- sound wave - simultaneous fluctuation of energy density T^{00} and longitudinal component of momentum density T^{0i}, with dispersion relation

$$\omega = u_s q - \frac{i}{2} \frac{1}{\rho + p} \left(\zeta + \frac{4}{3} \eta \right) q^2 \, , \qquad u_s^2 = \frac{\partial p}{\partial \rho} \, .$$

conformal theory $\Rightarrow$ stress-energy tensor is traceless, so

$$\rho = 3p \quad , \quad \zeta = 0 \quad , \quad u_s = \frac{1}{\sqrt{3}}$$

Gravity

The non-extremal 3-brane background is a solution of type-IIB low energy equations of motion.

In the near-horizon limit $r \ll R$, the metric becomes

$$ds_{10}^2 = \frac{(\pi T R)^2}{u}\left(-f(u)dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2 \, ,$$

where $T = \frac{r_0}{\pi R^2}$ is Hawking temperature, $u = \frac{r_0^2}{r^2}$, $f(u) = 1 - u^2$.

The horizon corresponds to $u = 1$, spatial infinity to $u = 0$.

gauge theory/gravity correspondence:

- background metric with non-extremality parameter r_0 is dual to $\mathcal{N} = 4$ $SU(N)$ SYM at finite temperature T in the limit of $N \to \infty$, $g_{YM}^2 N \to \infty$.

Shear Mode

compute two-point function of stress-energy tensor in the boundary theory

- consider small perturbation of metric

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$$

where $g_{\mu\nu}^{(0)}$ is given by

$$ds_5^2 = \frac{\pi^2 T^2 R^2}{u} \left(-f(u) dt^2 + d\boldsymbol{x}^2 \right) + \frac{R^2}{4 f(u) u^2} du^2 \, .$$

Einstein equations:

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\nu}^{(0)} + \mathcal{R}_{\mu\nu}^{(1)} + \cdots = \frac{2\Lambda}{3} g_{\mu\nu} \, , \Lambda = \frac{-6}{R^2}$$

To first order in $h_{\mu\nu}$,

$$\mathcal{R}_{\mu\nu}^{(1)} = -\frac{4}{R^2} h_{\mu\nu} \, .$$

assume perturbation $\sim e^{-i\omega t + iqz}$

fix gauge

$$h_{u\mu} = 0$$

3 classes of perturbations:

- $h_{xy} \neq 0$, or $h_{xx} = -h_{yy} \neq 0$;
- h_{xt} and $h_{xz} \neq 0$, or h_{yt} and $h_{yz} \neq 0$;
- h_{tz} and all diagonal elements of $h_{\mu\nu}$ are nonzero, and $h_{xx} = h_{yy}$ (sound wave).

Perturbation with $h_{xy} \neq 0$

$$h''_{xy} + \frac{1-3u^2}{uf} h'_{xy} + \frac{1}{(2\pi T)^2 f^2 u} \left(f \frac{\partial^2 h_{xy}}{\partial z^2} - \frac{\partial^2 h_{xy}}{\partial t^2} \right) - \frac{1+u^2}{fu^2} h_{xy} = 0$$

Introduce $\phi = \frac{u h_{xy}}{(\pi T R)^2} = h^x_y$

Fourier component $\phi_k(u)$ satisfies eq. for minimally coupled scalar

$$\phi''_k - \frac{1+u^2}{uf}\phi'_k + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2}\phi_k = 0\,.$$

For incoming wave at horizon,

$$\phi_k(u) = (1-u)^{-i\mathfrak{w}/2} F_k(u)\,,$$

where $F_k(u)$ is regular at $u = 1$ and can be written as a series

$$F_k(u) = 1 - \frac{i\mathfrak{w}}{2} \ln \frac{1+u}{2} + \frac{\mathfrak{w}^2}{8} \left[\left(\ln \frac{1+u}{2} + 8(1 - \frac{\mathfrak{q}^2}{\mathfrak{w}^2}) \right) \ln \frac{1+u}{2} - 4\, \mathrm{Li}_2 \frac{1-u}{2} \right] + O(\mathfrak{w}^3)\,,$$

retarded Green function

$$G_{\mu\nu,\lambda\rho}(\omega, \boldsymbol{q}) = -i \int d^4x\, e^{-iq\cdot x}\, \theta(t) \langle [T_{\mu\nu}(x),\, T_{\lambda\rho}(0)] \rangle\,.$$

Deduce

$$G_{xy,xy}(\omega, \boldsymbol{q}) = -\frac{N^2T^2}{16}\left(i\,2\pi T\omega + q^2\right) .$$

shear viscosity of strongly coupled $\mathcal{N} = 4$ SYM plasma (Kubo formula)

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega}\int dt\, d\boldsymbol{x}\, e^{i\omega t}\, \langle [T_{xy}(x),\, T_{xy}(0)]\rangle = \frac{\pi}{8}N^2T^3 .$$

NEXT: Calculation involving different components leads to same result!

Perturbation with $h_{tx} \neq 0$, $h_{xz} \neq 0$

In this channel correlation functions have diffusion pole.

Einstein equations for Fourier components of $H_t = \frac{uh_{tx}}{(\pi TR)^2}$, $H_z = \frac{uh_{zx}}{(\pi TR)^2}$:

$$H_t' + \frac{\mathfrak{q}f}{\mathfrak{w}} H_z' = 0\,,$$
$$H_t'' - \frac{1}{u} H_t' - \frac{\mathfrak{w}\mathfrak{q}}{uf} H_z - \frac{\mathfrak{q}^2}{uf} H_t = 0\,,$$
$$H_z'' - \frac{1+u^2}{uf} H_z' + \frac{\mathfrak{w}^2}{uf^2} H_z + \frac{\mathfrak{w}\mathfrak{q}}{uf^2} H_t = 0\,.$$

Deduce

$$H_z = \frac{uf}{\mathfrak{w}\mathfrak{q}} H_t'' - \frac{f}{\mathfrak{w}\mathfrak{q}} H_t' - \frac{\mathfrak{q}}{\mathfrak{w}} H_t\,.$$

$$H_t''' - \frac{2u}{f} H_t'' + \frac{2uf - \mathfrak{q}^2 f + \mathfrak{w}^2}{uf^2} H_t' = 0\,.$$

In the low-frequency, long-wavelength limit, solve as before...

$$H_t'(u) = (1-u)^{-i\mathfrak{w}/2} G(u)$$

$$G'' - \left(\frac{2u}{f} - \frac{i\mathfrak{w}}{1-u} \right) G' + \frac{1}{f} \left(2 + \frac{i\mathfrak{w}}{2} - \frac{\mathfrak{q}^2}{u} + \frac{\mathfrak{w}^2 [4 - u(1+u)^2]}{4uf} \right) G = 0 \,.$$

$$G(u) = C \left[u - i\mathfrak{w} \left(1 - u - \frac{u}{2} \ln \frac{1+u}{2} \right) + \frac{\mathfrak{q}^2 (1-u)}{2} \right] + O \left(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2, \mathfrak{q}^4 \right) \,.$$

$$C = \frac{\mathfrak{q}^2 H_t^0 + \mathfrak{q}\mathfrak{w} H_z^0}{i\mathfrak{w} - \frac{\mathfrak{q}^2}{2}} \,.$$

Deduce correlators

$$\begin{aligned} G_{tx,tx}(\omega, \boldsymbol{q}) &= \frac{N^2 \pi T^3 q^2}{8(i\omega - \mathcal{D} q^2)} + \mathcal{O}(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2, \mathfrak{q}^4) \,, \\ G_{tx,xz}(\omega, \boldsymbol{q}) &= -\frac{N^2 \pi T^3 \omega q}{8(i\omega - \mathcal{D} q^2)} + \mathcal{O}(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2, \mathfrak{q}^4) \,, \\ G_{xz,xz}(\omega, \boldsymbol{q}) &= \frac{N^2 \pi T^3 \omega^2}{8(i\omega - \mathcal{D} q^2)} + \mathcal{O}(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2, \mathfrak{q}^4) \,, \end{aligned}$$

where $\mathcal{D} = \frac{1}{4\pi T}$

Deduce η:

▶ recall from hydrodynamics $\mathcal{D} = \frac{\eta}{\rho+p}$.

Entropy:

$$s = \frac{3}{4} s_0 = \frac{\pi^2}{2} N^2 T^3 \,,$$

where s_0 is entropy at zero coupling.

From $s = \frac{\partial P}{\partial T}$, $\rho = 3p$, deduce $\rho + p = \frac{\pi^2}{2} N^2 T^4$, $\quad \therefore$

$$\eta = \frac{\pi}{8} N^2 T^3 \;, \quad \frac{\eta}{s} = \frac{1}{4\pi}$$

▶ agrees with Kubo formula.

▶ no agreement unless $s = \frac{3}{4} s_0$.

behavior of η as a function of the 't Hooft coupling

$$\eta = f_\eta(g^2_{\text{YM}} N) N^2 T^3$$

where $f_\eta(x) \sim \frac{1}{-x^2 \ln x}$ for $x \ll 1$ and $f_\eta(x) = \frac{\pi}{8}$ for $x \gg 1$.

- At weak coupling,

$$\frac{\eta}{s} \gg \frac{1}{4\pi}$$

Conformal soliton flow

the holographic image on Minkowski space of the global AdS_5-Schwarzschild black hole is a spherical shell of plasma first contracting and then expanding.

▶ conformal map from $S^{d-2} \times \mathbb{R}$ to $(d-1)$-dim Minkowski space

[Friess, Gubser, Michalogiorgakis, Pufu]

QNMs $\Rightarrow$ properties of plasma

- $$\frac{v_2}{\delta} = \frac{1}{6\pi} \mathrm{Re} \frac{\omega^4 - 40\omega^2 + 72}{\omega^3 - 4\omega} \sin \frac{\pi\omega}{2}$$
  - $v_2 = \langle \cos 2\phi \rangle$ at $\theta = \frac{\pi}{2}$ (mid-rapidity), average with respect to energy density at late times
  - $\delta = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}$ (eccentricity at time $t = 0$).

  Numerically, $\frac{v_2}{\delta} = 0.37$, *cf.* with result from RHIC data, $\frac{v_2}{\delta} \approx 0.323$

  [PHENIX Collaboration, arXiv:nucl-ex/0608033]

- thermalization time

$$\tau = \frac{1}{2|\mathrm{Im}\,\omega|} \approx \frac{1}{8.6T_{\mathsf{peak}}} \approx 0.08 \text{ fm/c} \ , \quad T_{\mathsf{peak}} = 300 \text{ MeV}$$

cf. with RHIC result $\tau \sim 0.6$ fm/c

[Arnold, Lenaghan, Moore, Yaffe, Phys. Rev. Lett. **94** *(2005) 072302]*

Not in agreement, but encouragingly small

▶ perturbative QCD yields $\tau \gtrsim 2.5$ fm/c.

[Baier, Mueller, Schiff, Son; Molnar, Gyulassy]

Analytical calculation of low-lying QNMs

[G. S., hep-th/0702079]

Vector perturbations

introduce the coordinate

$$u = \left(\frac{r_H}{r}\right)^{d-3}$$

wave equation

$$-(d-3)^2 u^{\frac{d-4}{d-3}} \hat{f}(u) \left(u^{\frac{d-4}{d-3}} \hat{f}(u) \Psi'\right)' + \hat{V}_{\mathsf{V}}(u)\Psi = \hat{\omega}^2 \Psi \quad , \quad \hat{\omega} = \frac{\omega}{r_H}$$

where prime denotes differentiation with respect to u and

$$\hat{f}(u) \equiv \frac{f(r)}{r^2} = 1 - u^{\frac{2}{d-3}} \left(u - \frac{1-u}{r_H^2}\right)$$

$$\hat{V}_{\mathsf{V}}(u) \equiv \frac{V_{\mathsf{V}}}{r_H^2} = \hat{f}(u) \left\{ \hat{L}^2 + \frac{(d-2)(d-4)}{4} u^{-\frac{2}{d-3}} \hat{f}(u) - \frac{(d-1)(d-2)\left(1 + \frac{1}{r_H^2}\right)}{2} u \right\}$$

where $\hat{L}^2 = \frac{\ell(\ell+d-3)}{r_H^2}$

First consider large black hole limit $r_H \to \infty$ keeping $\hat{\omega}$ and $\hat{L}$ fixed (small). Factoring out the behavior at the horizon ($u = 1$)

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} F(u)$$

the wave equation simplifies to

$$\mathcal{A} F'' + \mathcal{B}_{\hat{\omega}} F' + \mathcal{C}_{\hat{\omega},\hat{L}} F = 0$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}} (1 - u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} &= -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega},\hat{L}} &= \hat{L}^2 + \frac{(d-2)[d-4-3(d-2)u^{\frac{d-1}{d-3}}]}{4} u^{-\frac{2}{d-3}} \\
&\quad - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2 \frac{\hat{\omega}^2}{(d-1)^2} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad -(d-3)\frac{i\hat{\omega}}{d-1} \frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

solve perturbatively:

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0$$

where

$$\begin{aligned} \mathcal{H}_0 F &\equiv \mathcal{A}F'' + \mathcal{B}_0 F' + \mathcal{C}_{0,0} F \\ \mathcal{H}_1 F &\equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F' + (\mathcal{C}_{\hat{\omega},\hat{L}} - \mathcal{C}_{0,0})F \end{aligned}$$

Expanding the wavefunction perturbatively,

$$F = F_0 + F_1 + \dots$$

at zeroth order we have

$$\mathcal{H}_0 F_0 = 0$$

whose acceptable solution is

$$F_0 = u^{\frac{d-2}{2(d-3)}}$$

regular at horizon ($u = 1$) and boundary ($u = 0$, or $\Psi \sim r^{-\frac{d-2}{2}} \to 0$ as $r \to \infty$).

Wronskian

$$\mathcal{W} = \frac{1}{u^{\frac{d-4}{d-3}}(1 - u^{\frac{d-1}{d-3}})}$$

Another linearly independent solution

$$\check{F}_0 = F_0 \int \frac{\mathcal{W}}{F_0^2}$$

unacceptable $\because$ diverges at both horizon ($\check{F}_0 \sim \ln(1-u)$ for $u \approx 1$) and boundary ($\check{F}_0 \sim u^{-\frac{d-4}{2(d-3)}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-4}{2}} \to \infty$ as $r \to \infty$).

At first order we have

$$\mathcal{H}_0 F_1 = -\mathcal{H}_1 F_0$$

whose solution may be written as

$$F_1 = F_0 \int \frac{\mathcal{W}}{F_0^2} \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}}$$

The limits of the inner integral may be adjusted at will

$\because$ this amounts to adding an arbitrary amount of the unacceptable solution.

To ensure regularity at the horizon, choose one of the limits at $u = 1$

- integrand is regular at the horizon, by design.

at the boundary ($u = 0$),

$$F_1 = \breve{F}_0 \int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} + \text{regular terms}$$

The coefficient of the singularity ought to vanish,

$$\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} = 0$$

$\Rightarrow$ constraint on the parameters (dispersion relation)

$$\mathbf{a}_0 \hat{L}^2 - i\mathbf{a}_1 \hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0$$

After some algebra, we arrive at

$$\mathbf{a}_0 = \frac{d-3}{d-1} \; , \quad \mathbf{a}_1 = d - 3$$

The coefficient $\mathbf{a}_2$

- may also be found explicitly for each dimension d,

- it cannot be written as a function of d in closed form.
- it does not contribute to the dispersion relation at lowest order.
- E.g., for $d = 4, 5$, we obtain, respectively

$$\mathbf{a}_2 = \frac{65}{108} - \frac{1}{3}\ln 3 \ , \quad \frac{5}{6} - \frac{1}{2}\ln 2$$

quadratic in $\hat{\omega}$ eq. has two solutions,

$$\hat{\omega}_0 \approx -i\frac{\hat{L}^2}{d-1} \ , \quad \hat{\omega}_1 \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\hat{L}^2}{d-1}$$

In terms of frequency ω and quantum number ℓ,

$$\omega_0 \approx -i\frac{\ell(\ell+d-3)}{(d-1)r_H} \ , \quad \frac{\omega_1}{r_H} \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\ell(\ell+d-3)}{(d-1)r_H^2}$$

The smaller of the two, ω_0,

- is inversely proportional to the radius of the horizon,
- is not included in the asymptotic spectrum.

The other solution, ω_1,

- is a crude estimate of the first overtone in the asymptotic spectrum.
- shares important features with asymptotic spectrum:
  - it is proportional to r_H
  - dependence on ℓ is $O(1/r_H^2)$.

The approximation may be improved by including higher-order terms

- ▶ Inclusion of higher orders also increases the degree of the polynomial in the dispersion relation whose roots then yield approximate values of more QNMs.
- ▶ this method reproduces the asymptotic spectrum albeit not in an efficient way.

Include finite size effects:

$\hookrightarrow$ use perturbation (assuming $1/r_H$ is small) and replace $\mathcal{H}_1$ by

$$\mathcal{H}_1' = \mathcal{H}_1 + \frac{1}{r_H^2}\mathcal{H}_H$$

where

$$\mathcal{H}_H F \equiv \mathcal{A}_H F'' + \mathcal{B}_H F' + \mathcal{C}_H F$$

$$\begin{aligned}
\mathcal{A}_H &= -2(d-3)^2 u^2 (1-u) \\
\mathcal{B}_H &= -(d-3)u \left[(d-3)(2-3u) - (d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}} u^{\frac{d-1}{d-3}} \right] \\
\mathcal{C}_H &= \frac{d-2}{2} \left[d-4-(2d-5)u-(d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}} u^{\frac{d-1}{d-3}} \right]
\end{aligned}$$

Interestingly, zeroth order wavefunction F_0 is eigenfunction of $\mathcal{H}_H$,

$$\mathcal{H}_H F_0 = -(d-2) F_0$$

$\therefore$ first-order finite-size effect is simple shift of angular momentum

$$\hat{L}^2 \to \hat{L}^2 - \frac{d-2}{r_H^2}$$

$\therefore$ QNMs of lowest frequency are modified to

$$\omega_0 = -i \frac{\ell(\ell+d-3)-(d-2)}{(d-1)r_H} + O(1/r_H^2)$$

For $d = 4, 5$, we have respectively,

$$\omega_0 = -i\frac{(\ell-1)(\ell+2)}{3r_H} \ , \quad -i\frac{(\ell+1)^2-4}{4r_H}$$

in agreement with numerical results

[Cardoso, Konoplya and Lemos; Friess, Gubser, Michalogiorgakis and Pufu]

AdS/CFT correspondence

dual to AdS Schwarzschild bh: gauge theory fluid on boundary of AdS ($S^{d-2} \times \mathbb{R}$).

consider the fluid dynamics ansatz

$$u_i = \mathcal{K} e^{-i\Omega\tau} \mathbb{V}_i$$

u_i: (small) velocity of a point in the fluid, $\mathbb{V}_i$: vector harmonic on S^{d-2}.

Demanding that this ansatz satisfy standard eqs of linearized hydrodynamics,

$\Rightarrow$ constraint on the frequency of the perturbation Ω which yields

$$\Omega = -i\frac{\ell(\ell+d-3)-(d-2)}{(d-1)r_H} + O(1/r_H^2)$$

[Michalogiorgakis and Pufu]

in perfect agreement with its dual counterpart.

Scalar perturbations

$\widehat{V}_{\mathsf{V}}$ replaced by

$$\begin{aligned}
\widehat{V}_{\mathsf{S}}(u) &= \frac{\widehat{f}(u)}{4}\left[\widehat{m} + \left(1+\frac{1}{r_H^2}\right)u\right]^{-2} \\
&\times \Bigg\{ d(d-2)\left(1+\frac{1}{r_H^2}\right)^2 u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\widehat{m}\left(1+\frac{1}{r_H^2}\right)u^{\frac{d-5}{d-3}} \\
&+(d-4)(d-6)\widehat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2\left(1+\frac{1}{r_H^2}\right)^3 u^3 \\
&+2(2d^2-11d+18)\widehat{m}\left(1+\frac{1}{r_H^2}\right)^2 u^2 \\
&+\frac{(d-4)(d-6)\left(1+\frac{1}{r_H^2}\right)^2}{r_H^2}u^2 - 3(d-2)(d-6)\widehat{m}^2\left(1+\frac{1}{r_H^2}\right)u \\
&-\frac{6(d-2)(d-4)\widehat{m}\left(1+\frac{1}{r_H^2}\right)}{r_H^2}u + 2(d-1)(d-2)\widehat{m}^3 + d(d-2)\frac{\widehat{m}^2}{r_H^2}\Bigg\}
\end{aligned}$$

where $\widehat{m} = 2\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H^2} = \frac{2(\ell+d-2)(\ell-1)}{(d-1)(d-2)r_H^2}$

In the large black hole limit $r_H \to \infty$ with $\hat{m}$ fixed, potential simplifies

$$\begin{aligned}\hat{V}_{\mathsf{S}}^{(0)}(u) &= \frac{1-u^{\frac{d-1}{d-3}}}{4(\hat{m}+u)^2}\Big\{d(d-2)u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}u^{\frac{d-5}{d-3}} \\ &+(d-4)(d-6)\hat{m}^2u^{-\frac{2}{d-3}} + (d-2)^2u^3 \\ &+2(2d^2-11d+18)\hat{m}u^2 - 3(d-2)(d-6)\hat{m}^2u + 2(d-1)(d-2)\hat{m}^3\Big\}\end{aligned}$$

- additional singularity due to double pole of scalar potential at $u = -\hat{m}$.
- desirable to factor out the behavior not only at the horizon, but also at the boundary and the pole of the scalar potential,

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} \frac{u^{\frac{d-4}{2(d-3)}}}{\hat{m}+u} F(u)$$

$\therefore$ wave equation

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega}}F = 0$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} &= -(d-3)u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\left[\frac{d-4}{u} - \frac{2(d-3)}{\hat{m}+u}\right] \\
&\quad -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega}} &= -u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})\left[-\frac{(d-2)(d-4)}{4u^2} - \frac{(d-3)(d-4)}{u(\hat{m}+u)} + \frac{2(d-3)^2}{(\hat{m}+u)^2}\right] \\
&\quad -\left[\left\{d-4-(2d-5)u^{\frac{d-1}{d-3}}\right\}u^{\frac{d-5}{d-3}} + 2(d-3)\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u}\right]\left[\frac{d-4}{2u} - \frac{d-3}{\hat{m}+u}\right] \\
&\quad -(d-3)\frac{i\hat{\omega}}{d-1}\frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad +\frac{\hat{V}_{\mathsf{S}}^{(0)}(u)-\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2\frac{\hat{\omega}^2}{(d-1)^2}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

Define zeroth-order wave equation $\mathcal{H}_0 F_0 = 0$, where

$$\mathcal{H}_0 F \equiv \mathcal{A}F'' + \mathcal{B}_0 F'$$

Acceptable zeroth-order solution

$$F_0(u) = 1$$

- plainly regular at all singular points ($u = 0, 1, -\hat{m}$).
- corresponds to a wavefunction vanishing at the boundary ($\Psi \sim r^{-\frac{d-4}{2}}$ as $r \to \infty$).

Wronskian

$$\mathcal{W} = \frac{(\hat{m} + u)^2}{u^{\frac{2d-8}{d-3}}(1 - u^{\frac{d-1}{d-3}})}$$

Unacceptable solution: $\check{F}_0 = \int \mathcal{W}$

- can be written in terms of hypergeometric functions.
- for $d \geq 6$, has a singularity at the boundary, $\check{F}_0 \sim u^{-\frac{d-5}{d-3}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-6}{2}} \to \infty$ as $r \to \infty$.
- for $d = 5$, acceptable wavefunction $\sim r^{-1/2}$; unacceptable $\sim r^{-1/2} \ln r$
- for $d = 4$, roles of F_0 and $\check{F}_0$ reversed; results still valid.
- $\check{F}_0$ is also singular (logarithmically) at the horizon ($u = 1$).

Working as in the case of vector modes, we arrive at the first-order constraint

$$\int_0^1 \frac{\mathcal{C}_{\hat{\omega}}}{\mathcal{A}\mathcal{W}} = 0$$

$\because \quad \mathcal{H}_1 F_0 \equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0) F_0' + \mathcal{C}_{\hat{\omega}} F_0 = \mathcal{C}_{\hat{\omega}}$

$\therefore$ dispersion relation

$$\mathbf{a}_0 - \mathbf{a}_1 i\hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0$$

After some algebra, we obtain

$$\mathbf{a}_0 = \frac{d-1}{2}\,\frac{1+(d-2)\hat{m}}{(1+\hat{m})^2} \;, \quad \mathbf{a}_1 = \frac{d-3}{(1+\hat{m})^2} \;, \quad \mathbf{a}_2 = \frac{1}{\hat{m}}\{1 + O(\hat{m})\}$$

For small $\hat{m}$, the quadratic equation has solutions

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\,\hat{m}}$$

related to each other by $\hat{\omega}_0^{+} = -\hat{\omega}_0^{-*}$

▶ general symmetry of the spectrum.

Finite size effects at first order amount to a shift of the coefficient $\mathbf{a}_0$ in the dispersion relation

$$\mathbf{a}_0 \to \mathbf{a}_0 + \frac{1}{r_H^2}\mathbf{a}_H$$

after some tedious but straightforward algebra, we obtain

$$\mathbf{a}_H = \frac{1}{\hat{m}}\{1 + O(\hat{m})\}$$

The modified dispersion relation yields the modes

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\,\hat{m} + 1}$$

in terms of the quantum number ℓ,

$$\omega_0^{\pm} \approx -i(d-3)\,\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H} \pm \sqrt{\frac{\ell(\ell+d-3)}{d-2}}$$

in agreement with numerical results

[Friess, Gubser, Michalogiorgakis and Pufu]

- imaginary part inversely proportional to r_H, as in vector case
- finite real part independent of r_H
  $\hookrightarrow$ speed of sound $v_s = \frac{1}{\sqrt{d-2}}$ (due to conformal invariance)

AdS/CFT correspondence

perturb gauge theory fluid on the boundary of AdS ($S^{d-2} \times \mathbb{R}$) using the ansatz

$$u_i = \mathcal{K} e^{-i\Omega\tau} \nabla_i \mathbb{S} \quad , \qquad \delta p = \mathcal{K}' e^{-i\Omega\tau} \mathbb{S}$$

u_i: (small) velocity of a point in the fluid,

δp: pressure perturbation,

$\mathbb{S}$: scalar harmonic on S^{d-2}.

Demanding that this ansatz satisfy eqs of linearized hydrodynamics,

$\Rightarrow$ frequency of perturbation Ω in perfect agreement with our analytic result.

Tensor perturbations

Unlike the other two cases, asymptotic spectrum is entire spectrum.
In large bh limit, wave equation

$$-(d-3)^2(u^{\frac{2d-8}{d-3}} - u^3)\Psi'' - (d-3)[(d-4)u^{\frac{d-5}{d-3}} - (2d-5)u^2]\Psi'$$
$$+\left\{\hat{L}^2 + \frac{d(d-2)}{4}u^{-\frac{2}{d-3}} + \frac{(d-2)^2}{4}u - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}}\right\}\Psi = 0$$

For zeroth-order eq., set $\hat{L} = 0 = \hat{\omega}$
$\hookrightarrow$ two solutions are ($\Psi = F_0$ at zeroth order)

$$F_0(u) = u^{\frac{d-2}{2(d-3)}} \quad , \quad \check{F}_0(u) = u^{-\frac{d-2}{2(d-3)}} \ln\left(1 - u^{\frac{d-1}{d-3}}\right)$$

Neither behaves nicely at both ends ($u = 0, 1$)

$\therefore$ both are unacceptable.

$\therefore$ impossible to build a perturbation theory to calculate small frequencies.

in agreement with numerical results and in accordance with the

AdS/CFT correspondence

- there is no ansatz that can be built from tensor spherical harmonics $\mathbb{T}_{ij}$ satisfying the linearized hydrodynamic eqs because of the conservation and tracelessness properties of $\mathbb{T}_{ij}$.

PHASE TRANSITIONS IN TOPOLOGICAL BLACK HOLES

[Koutsoumbas, Musiri, Papantonopoulos and Siopsis]

MTZ black hole *[Martinez, Troncoso, Zanelli]*

4d gravity with negative cosmological constant ($\Lambda = -3$) and a scalar field

$$I = \int d^4x\sqrt{-g}\left[\frac{R+6}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] ,$$

potential is given by

$$V(\phi) = -\frac{3}{4\pi G}\sinh^2\sqrt{\frac{4\pi G}{3}}\phi ,$$

static black hole solution with topology $\mathbb{R}^2 \times \Sigma$, where Σ is a 2d manifold of negative constant curvature

$$ds^2 = \frac{r(r+2G\mu)}{(r+G\mu)^2}\left[-\left(r^2 - \left(1+\frac{G\mu}{r}\right)^2\right)dt^2 + \left(r^2 - \left(1+\frac{G\mu}{r}\right)^2\right)^{-1}dr^2 + r^2 d\sigma^2\right],$$

and the scalar field is

$$\phi = \sqrt{\frac{3}{4\pi G}}\,\text{Arctanh}\frac{G\mu}{r + G\mu} .$$

For $\phi = 0$, Topological Black Hole (TBH)

$$ds_0^2 = -\left[r^2 - 1 - \frac{2G\mu}{r}\right] dt^2 + \left[r^2 - 1 - \frac{2G\mu}{r}\right]^{-1} dr^2 + r^2 d\sigma^2 .$$

difference between the TBH and MTZ free energies

$$\Delta F = F_{TBH} - F_{MTZ} = -\frac{\sigma}{8\pi G}(T - T_0)^3 \pi^3 + \dots ,$$

$\Rightarrow$ phase transition between MTZ and TBH at the critical temperature $T_0 = \frac{1}{2\pi}$.

QNMs of electromagnetic perturbations

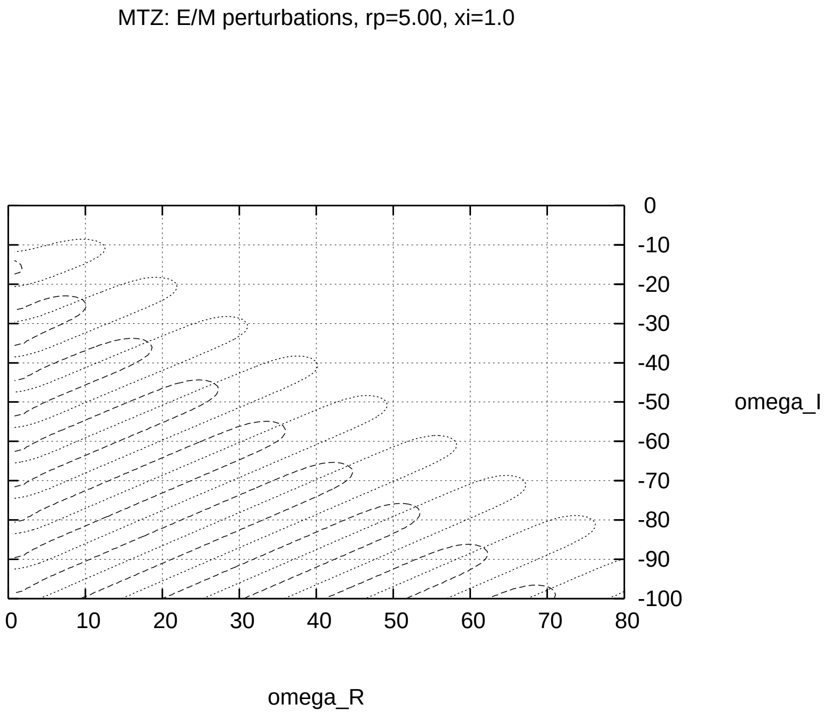
wave equation

$$f(r)\frac{d}{dr}\left(f(r)\frac{d\Psi_\omega}{dr}\right)+\left[\omega^2-\left(\xi^2+\frac{1}{4}\right)\frac{f(r)}{r^2}\right]\Psi_\omega=0\,,$$

where

$$f_{MTZ}(r)=r^2-\left(1+\frac{G\mu}{r}\right)^2\quad,\quad f_{TBH}(r)=r^2-1-\frac{2G\mu}{r}\,.$$

calculate QNMs both analytically and numerically - excellent agreement!



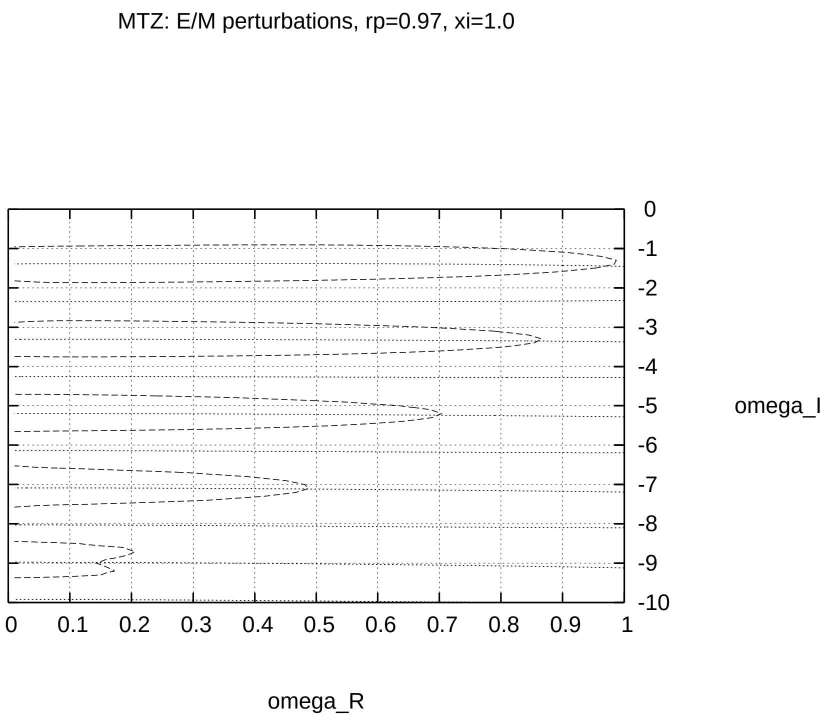
Contours displaying the lines $\text{Re}\,[\psi_0(\omega)] = 0$ (dashed lines) and $\text{Im}\,[\psi_0(\omega)] = 0$ (dotted lines) on the complex $\omega-$plane for MTZ Black Holes and $r_+ = 5.0$.

QNMs lie on a straight line with *negative* slope; spacing is constant.

Below critical T

striking feature of QNMs: slope is positive. Intersections no longer lie along a straight line; spacing changes.

▶ finite number of QNMs.



Contours displaying the lines Re $[\psi_0(\omega)] = 0$ (dashed lines) and Im $[\psi_0(\omega)] = 0$ (dotted lines) on the complex $\omega-$plane for MTZ Black Holes and $r_+ = 0.97$.

r_+	$G\mu$	T	$\Delta\omega_R^{Ana}$	$\Delta\omega_I^{Ana}$	$\Delta\omega_R^{Num}$	$\Delta\omega_I^{Num}$
10.000	90.00	3.023	18.96	−20.26	19.00	−20.15
5.000	20.00	1.434	8.91	−10.25	8.77	−10.45
2.000	2.000	0.477	2.75	−4.21	2.70	−4.14
1.050	0.0525	0.175	0.32	−2.15	0.27	−2.16
1.000	0.000	0.160	0.00	−2.00	0.00	−2.00

analytical and numerical values of QNMs of EM perturbations of MTZ Black Holes.

ω_R^{Num}	ω_I^{Num}	$\Delta\omega_R^{Num}$	$\Delta\omega_I^{Num}$
0.973	−1.496	-	-
0.864	−3.351	0.109	1.905
0.701	−5.239	0.163	1.888
0.486	−7.114	0.215	1.875
0.143	−8.980	0.343	1.866

Numerical results for QNMs of EM perturbations for MTZ Black Holes with $r_+ = 0.97$.

Two time scales:

- oscillation time scale $\tau_R \equiv 1/\omega_R$
- damping time scale $\tau_I \equiv 1/\omega_I$

Above critical T $(r_+ > 1)$, the scalar field is absorbed by the black hole and τ_I is small

$\therefore$ perturbations fall off rather rapidly with time.

Below critical T $(r_+ < 1)$, the black hole is dressed up with the scalar field and τ_I is large

$\therefore$ perturbations last longer.

At critical T $(r_+ = 1)$, we have a change of slope

▶ transient configuration - second order phase transition.

CONCLUSIONS

- Quasi-normal modes are a powerful tool in understanding hydrodynamic behavior of gauge theory fluid at strong coupling
- RHIC's fireball can be described by a dual black hole
- RHIC and LHC may probe black holes and provide information on string theory as well as non-perturbative QCD effects.