

Lovelock Gravity **at the Crossroads of** **Palatini and Metric** **Formulations**

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Introduction and Motivations

- The Einstein Gravity which is described by the Einstein-Hilbert action

$$S_{EH} = \frac{1}{4\pi G_N} \int d^D x \sqrt{-g} R$$

was first formulated in search for a theory which realizes general covariance. Or invariance under general coordinate transformations.

- Although very elegant and successful in describing the real world, there are theoretical and phenomenological reasons or motivations to consider beyond Einstein Gravity theories.

- In all beyond Einstein theories, one would like to keep invariance of the theory under general coordinate transformations manifest, at least at the level when classical (geometrical space-time) makes sense. This requirement is not restrictive at all.
- **Theoretically**, one expects at *semi-classical regime*, there should be corrections to the Einstein-Hilbert action, generically of the form of a generic function of metric $g_{\mu\nu}$, the Riemann curvature $R^{\mu}_{\nu\alpha\beta}$, and its covariant derivatives...
- String theory provides a way to compute these corrections order-by-order in α' or g_s .

- **Phenomenologically**, it has been argued that theories of modified gravity may provide natural resolution to dark matter and dark energy problems and as well as a framework to address inflationary paradigm. In this context, **$f(R)$ theories of gravity**, which are the simplest modified gravity theories have been under intense study in the past five six years....

The Main Question

In a bottom-up approach and in a theoretical setting, is it possible to find criteria, besides the general covariance, like some sort of symmetry argument or alike, by means of which one can restrict form of “allowed” theories of gravity?

To start we review some observations/facts about the Einstein Gravity to see if we can find hints or clues.....

Review of some Facts about GR

- GR associates gravity to the geometric properties of space-time, the metric $g_{\mu\nu}$ and the connection $\Gamma_{\alpha\beta}^{\mu}$.
- **Metric** is a measure of **distances** on the manifold while **connection** is a measure of **parallel transport** and defines the covariant derivative.
- **Geodesics**, the paths which minimize the distance between two points, only depend on the **metric**:

$$\ddot{X}^{\mu} + \{\alpha\beta\}^{\mu} \dot{X}^{\alpha} \dot{X}^{\beta} = 0$$

$$\{\alpha\beta\}^{\mu} \equiv \frac{1}{2} g^{\mu\nu} (\partial_{\alpha} g_{\nu\beta} + \partial_{\beta} g_{\nu\alpha} - \partial_{\nu} g_{\alpha\beta}).$$

Geodesics Vs. Free Particle Paths...

- The free particle path, the path along which the parallel transport of the velocity vector remains unchanged, however, is only specified by the **connection**. If we parameterize the path by s , and denote the velocity vector field by $v^\mu(s)$, that is

$$\frac{d}{ds}v^\mu \equiv v^\nu \nabla_\nu v^\mu = v^\mu (\partial_\mu v^\nu + \Gamma_{\mu\rho}^\nu v^\rho) = 0.$$

- A geodesic becomes a free particle path if *the connection is the Levi-Civita connection*:

$$\Gamma_{\alpha\beta}^\mu = \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}$$

Palatini Vs. Metric formulations

Starting with Einstein-Hilbert action, hence, there are two interpretations, or formulations

- **The metric formulation**, in which metric is the only dynamical degree of freedom and the connection is always taken to be given by the Levi-Civita connection, and
- **The Palatini formulation**, in which both metric and the connection are treated as dynamical variables. Therefore, there are two sets of e.o.m, one for metric and the other for the connection.

Palatini Vs. Metric formulations, Cont'd

- In the case of Einstein-Hilbert action, however, the two formulations lead to the same dynamics.
- To see this, start with E.H. action in the Palatini formulation:

$$S_{EH}[g_{\mu\nu}, \Gamma_{\alpha\beta}^{\mu}] = \frac{1}{4\pi G_N} \int d^D x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma)$$

and recall that

$$R^{\alpha}{}_{\mu\beta\nu} \equiv \partial_{[\beta}\Gamma_{\nu]\mu}^{\alpha} + \Gamma_{\rho[\beta}\Gamma_{\nu]\mu}^{\rho},$$

and that

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

Palatini Vs. Metric formulations, Cont'd

The equation of motion of the connection derived from the action reads as

$$\nabla_{\alpha}(\sqrt{-g}g^{\mu\nu}) = 0,$$

which for $D \neq 2$ is equivalent to

$$\nabla_{\alpha}g^{\mu\nu} = 0 \quad \Rightarrow \quad \Gamma_{\alpha\beta}^{\mu} = \frac{1}{2}g^{\mu\nu}(\partial_{\alpha}g_{\beta\nu} + \partial_{\beta}g_{\nu\alpha} - \partial_{\nu}g_{\alpha\beta})$$

That is, in the E.H. theory *equivalence of Palatini and Metric formulations and the fact that connection is found to be the Levi-Civita connection is a **dynamical** statement.*

The Main Idea...

I promote the equivalence of Palatini and metric formulations, which is a characteristic of the Einstein-Hilbert action, to a guiding principle. That is, we demand that all the generalized (modified) gravity theories should exhibit this property.

As we'll see equivalence of Palatini and Metric formulations, similarly to the general covariance is an outcome the Equivalence Principle.

The Physical Meaning of Palatini \equiv Metric

In the Palatini formulation where in principle the connection can be other than the Levi-Civita connection, a free particle does not necessarily follow a geodesic, the path which minimizes the distance.

Let us explore the physical consequences this may have.


- Consider a light ray which should follow a path of a free particle in a given background geometry. If this path is not a geodesic, then there should exist another path, a geodesic, along which an object is traveling faster than light. This is in contradiction with the basics of the Einstein general relativity.

The Equivalence Principle Requires Palatini- Metric Equivalence

- In another point of view, along a geodesic the particle will feel a force and hence gravity cannot be locally turned off, which is against the usual interpretation of the equivalence principle.
- In more technical terms, going to normal coordinates, one can (locally) trivialize *the Levi-Civita connection*. Note that this is not true for a general connection and only holds for Levi-Civita.
- *Therefore, the fact that the connection should be Levi-Civita and that free particle path should coincide with a geodesic, is the statement necessitated by the Equivalence Principle.*

The Equivalence Principle Requires Palatini- Metric Equivalence

- In the metric formulation we do not face these contradictions. Nonetheless, in a theory of modified gravity there is always the theoretical possibility of taking the Palatini or metric formulations and *a priori* there is no reason which one should be taken.
- When coupling of gravity to other fields, and in particular spinors is considered usage of Palatini formulation becomes inevitable.
- In string theory, once e.g. we compute α' -corrections, it is an implicit assumption that we are in metric formulation.



Therefore, we demand that all “physically allowed” theories of modified gravity are those which are compatible with the above statement of Equivalence Principle, *i.e.* they describe identical (classical) dynamics in both metric and Palatini formulations.

Palatini Vs. Metric formulations

- In its most general form the Lagrangian of pure gravity which is only restricted by the general covariance is a generic functional of metric $g_{\mu\nu}$, the Riemann curvature $R^\mu_{\nu\alpha\beta}$ and their covariant derivatives.
- Here I only restrict the discussion to the cases where no covariant derivative is involved explicitly, that is:

$$S_{mod.GR} = \frac{1}{4\pi G_N} \int d^D x \sqrt{-g} \mathcal{L}(g_{\mu\nu}, R^\mu_{\nu\alpha\beta}).$$

Palatini Vs. Metric formulations, Cont'd

- The above action describes two different theories, the *metric formulation* in which the Riemann curvature is expressed in terms of the Levi-Civita connection and the *Palatini formulation* which is obtained by treating metric and connection as two independent fields.
- To see the non-equivalence of these two, we explicitly work out the equations of motion for the two cases.

Palatini Vs. Metric formulations, Cont'd

- E.o.M in Metric formulation:

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \frac{1}{2} \mathcal{L} g^{\mu\nu} + 2 \nabla_{\alpha} \nabla_{\beta} \frac{\partial \mathcal{L}}{\partial R_{\mu\alpha\beta\nu}} = 0,$$
$$\nabla_{\alpha} g^{\mu\nu} = 0$$

- E.o.M in Palatini formulation:

$$\text{e.o.m for connection : } \nabla_{\alpha} \left(\sqrt{-g} \frac{\partial \mathcal{L}}{\partial R^{\mu}_{\nu\alpha\beta}} \right) = 0,$$

$$\text{e.o.m for metric : } \frac{\partial}{\partial g_{\mu\nu}} (\sqrt{-g} \mathcal{L}) = 0.$$

- In general the above two sets of equations are not identical.

Requiring Palatini-Metric Equivalence

- Requiring the Palatini-metric equivalence amounts to demanding that the Levi-Civita connection should be a solution to the Palatini e.o.m for the connection.

- This leads to

$$\frac{\partial^2 \mathcal{L}}{\partial R_{\mu\nu\beta\alpha} \partial R_{\rho\sigma\lambda\gamma}} \nabla_{\alpha} R_{\rho\sigma\lambda\gamma} = 0.$$

- The Palatini or metric descriptions of the same action are hence equivalent if and only if the above equation is satisfied for all curvature tensors, regardless of the equation of motion for metric.

Requiring Palatini-Metric Equivalence, Cont'd

- Recalling the Bianchi identity

$$\nabla_{\alpha} R_{\rho\sigma\lambda\gamma} + \nabla_{\rho} R_{\sigma\alpha\lambda\gamma} + \nabla_{\sigma} R_{\alpha\rho\lambda\gamma} = 0,$$

The Palatini-Metric Equivalence condition becomes

$$\frac{\partial^2 \mathcal{L}}{\partial R_{\mu\nu\beta\alpha} \partial R_{\rho\sigma\lambda\gamma}} = \frac{\partial^2 \mathcal{L}}{\partial R_{\mu\nu\beta\sigma} \partial R_{\alpha\rho\lambda\gamma}} = \frac{\partial^2 \mathcal{L}}{\partial R_{\mu\nu\beta\rho} \partial R_{\sigma\alpha\lambda\gamma}} \quad (\star).$$

- Note also that the other (Bianchi) identity, $R^{\mu}_{[\alpha\beta\gamma]} = 0$, yields

$$\frac{\partial \mathcal{L}}{\partial R^{\mu}_{[\alpha\beta\gamma]}} = 0 \quad (\circledast).$$

Specifying the Solutions

The Lovelock Gravity Theories are those which fulfill the Palatini-metric equivalence requirement.

To see this let us have a de tour to Lovelock gravity theory.

A de tour to Lovelock Gravity

- About 35 years ago, David Lovelock used the following assumptions to restrict the form of higher derivative corrections to the *Einstein tensor*:
 1. The generalization of the Einstein tensor, hereafter denoted by $A_{\mu\nu}$, should be a symmetric tensor of rank two; $A_{\mu\nu} = A_{\nu\mu}$.
 2. $A_{\mu\nu}$ is concomitant of the metric and its first two derivatives, $A_{\mu\nu} = A_{\mu\nu}(g, \partial g, \partial^2 g)$.
 3. $A_{\mu\nu}$ is divergence free, $\nabla^\mu A_{\mu\nu} = 0$.

A de tour to Lovelock Gravity, Cont'd

- In a series of theorems Lovelock proved that *the above three conditions uniquely fixes the Lagrangian density in a D dimensional space-time, to the Lovelock Gravity*

$$\mathcal{L}_{Lovelock} = \sum_{p=0}^{\lfloor \frac{D+1}{2} \rfloor} a_p \delta_{\nu_1 \dots \nu_{2p}}^{\mu_1 \dots \mu_{2p}} R_{\mu_1 \mu_2}^{\nu_1 \nu_2} \dots R_{\mu_{2p-1} \mu_{2p}}^{\nu_{2p-1} \nu_{2p}} \quad (\ast)$$

where $\lfloor \frac{D+1}{2} \rfloor$ represents the integer part of $\frac{D+1}{2}$ and

$$\delta_{\nu_1 \dots \nu_N}^{\mu_1 \dots \mu_N} = \det \begin{vmatrix} \delta_{\nu_1}^{\mu_1} & \dots & \delta_{\nu_N}^{\mu_1} \\ \vdots & & \vdots \\ \delta_{\nu_1}^{\mu_N} & \dots & \delta_{\nu_N}^{\mu_N} \end{vmatrix} .$$

A de tour to Lovelock Gravity, Cont'd

- Note that in the Lovelock setting connection has been taken to be the Levi-Civita connection.
- The p^{th} term is called the p^{th} order Lovelock gravity.
- At **zeroth** Lovelock gravity is the **cosmological constant**,
- The **first** order is the **Einstein-Hilbert** action.
- In **second** order it is the **Gauss-Bonnet** term. In $D = 4$ we have only up to second order terms.
- The compact form of its higher orders becomes more involved, e.g. [F. Mueller-Hoissen, PLB163, '85, J. T. Wheeler, NPB268, '86].

Palatini-Metric Equivalence Implies Lovelock

- To show the claim, it is enough to show that the Einstein tensor computed requiring the Palatini-metric equivalence satisfies the three Lovelock conditions.
- To see this we note that

Metric Formulation : $A_{metric}^{\mu\nu} \equiv \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{-g}\mathcal{L})$

Palatini Formulation : $A_{Palatini}^{\mu\nu} \equiv \frac{\partial}{\partial g_{\mu\nu}} (\sqrt{-g}\mathcal{L})$

Proof of the claim

- The Palatini-metric equivalence is then stated as:

$$A_{metric}^{\mu\nu} \equiv A_{Palatini}^{\mu\nu}$$

- The first and third Lovelock conditions are immediate in the Lagrangian language.
- The second condition is immediate once we use the Palatini definition of the Einstein tensor, *i.e.*

$$A_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \mathcal{L}$$

and recall that in the metric formulation the Riemann tensor only contains up to second order derivatives of the metric. $\mathcal{QED} \blacksquare$.

Another proof

- As an alternative proof, one may try to solve the equations (\star) , (\circledast) .
- In order that it is enough to recall that the determinant δ is invariant under the even permutations of its rows.
- To show the uniqueness, one may start with an ansatz of the form (\circledast) but replace the tensor δ with a generic tensor M . It is straightforward to show that only δ leads to the Lagrangians with the properties we would like to have.

Conclusions and Remarks

⊛ We showed that within the class of

$$\mathcal{L} = \mathcal{L}(g_{\mu\nu}, R^{\mu}_{\nu\alpha\beta})$$

theories of gravity the Palatini-metric equivalence **uniquely** fixes \mathcal{L} to the **Lovelock** gravity theories.

⊛ We discussed that Palatini-Metric equivalence is implied by the **Equivalence Principle** (in the Palatini formulation).

⊛ Being a statement of the Equivalence Principle, we propose that the Palatini-Metric equivalence should be viewed as an outcome of any theory of quantum gravity in the semiclassical regime, *i.e.*, it should be viewed as a guiding principle in a bottom-up approach.


Remarks and open problems

⊛ Within this class of Lagrangians we showed that this is only **Lovelock** gravity theories which fulfill the “*Palatini-Metric Equivalence Principle*” (PMEP).

⊛ Here we set torsion equal to zero and considered Lagrangians without derivatives of curvature. It is desirable to see how **PMEP** restricts Lagrangians without these simplifying assumptions.

Remarks and open problems

- ⊛ Besides the pure gravity cases, the **PMEP** can also be used to restrict the form of matter-gravity coupling, in particular form of the non-minimal couplings.
- ⊛ It is very interesting to check whether string theory corrections to (super)gravity also satisfy the **PMEP**. One should note that string theory corrections always come with a field redefinition ambiguity. It is conceivable that the **PMEP** can be used to fix this field redefinition ambiguity.



Viva PMEP

THE END