Twistor Strings with Flavour

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Patra, 15 June, 2007

with J. Bedford and K. Zoubos, arXiv:0706.xxxx

Motivation

Twistor String Theory: Proposed correspondence between the open string sector of the topological B-model on $\mathbb{CP}^{3|4}$ and perturbative $\mathcal{N} = 4$ SYM. [Witten]

Simplicity of gauge theory amplitudes in fixed helicity basis. The MHV n-point gluon amplitude is proportional to

$$A = g^{n-2} \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle}$$

where $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ and $\langle \lambda_i, \lambda_j \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$.

These amplitudes are reproduced by nonperturbative effects in a holomorphic Chern-Simons theory with supertwistor space as the target space. The isometries of $\mathbb{CP}^{3|4}$ linearly encode the superconformal symmetry of $\mathcal{N} = 4$.

Unfortunately, the correspondence doesn't hold beyond tree level. Contributions from closed string B-model sector and appearance of conformal supergravity modes in the gauge theory loops. [Berkovits-Witten]

Development of MHV-formalism on gauge theory side even for less or non-susy YM. [Cachazo-Svrček-Witten]

But the twistor-inspired MHV-rules *can* be extended at loop level for $\mathcal{N} = 4$. Also for $\mathcal{N} = 1$ and pure YM.
[Brandhuber-Spence-Travaglini]
[Bedford-Brandhuber-Spence-Travaglini]

What is the quantum completion of Witten's twistor string? Some (non-topological?) B-model extension with modified target space?

Towards this end study the range of 4d gauge theories with less susy, which admit a tree-level twistor string description.

Most obvious candidates should be theories that preserve conformal invariance at loop level, order-by-order: UV-finite

- ☑ $\mathcal{N}=1$ exactly marginal deformations of $\mathcal{N}=4$ [Kulaxizi-Zoubos]

- ☑ $\mathcal{N}=1,2$ quiver gauge theories as discrete $\mathbb{CP}^{3|4}$ orbifolds [Park-Rey] [Giombi-Kulaxizi-Ricci-Robles-Llana-Trancanelli-Zoubos]

Look at $\mathcal{N}=2$ finite theories with fundamental matter:

$\mathcal{N}=2$ $\mathrm{Sp}(N_c)$ gauge theory with $N_f=4$

$\mathcal{N}=2$ $\mathrm{SU}(N_c)$ gauge theory with $N_f=2N_c$

Outline

- Review of the $\mathcal{N} = 4$ theory
- The $\mathcal{N} = 2$ $\mathrm{Sp}(N_c)$ theory with $N_f = 4$
- The $\mathcal{N} = 2$ $\mathrm{SU}(N_c)$ theory with $N_f = 2N_c$
- Conclusions

Review of the $\mathcal{N}=4$ theory

Write null momenta in terms of $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$. Twistor space is a copy of $\mathbb{CP}^3$ defined by the homogeneous co-ordinates $Z^I = (\lambda^\alpha, \mu^{\dot{\alpha}})$, where

$$\widetilde{\lambda}_{\dot{\alpha}} \rightarrow i\frac{\partial}{\partial \mu^{\dot{\alpha}}}, \quad -i\frac{\partial}{\partial \widetilde{\lambda}^{\dot{\alpha}}} \rightarrow \mu_{\dot{\alpha}}.$$

Witten showed that holomorphic λ dependence of the MHV (analytic) amplitudes means they are supported on genus zero, degree one curves in twistor space, $\mathbb{CP}^1 \subset \mathbb{CP}^3$.

Adding four fermionic co-ordinates ψ^A plus conjugates turns $\mathbb{CP}^3$ into a super-CY: $\mathbb{CP}^{3|4}$.

This is now a suitable target space for the B-model.

The B-model open string d.o.f. are described by the HCS action

$$S = \frac{1}{2} \int_{\mathrm{D5}} \Omega \wedge \mathrm{Tr} \left(\mathcal{A} \cdot \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right)$$

defined on a "D5"-brane sitting at the locus $\bar{\psi}^A = 0$. The superfields $\mathcal{A}$ can be expanded as

$$\mathcal{A} = A + \psi^I \lambda_I + \frac{1}{2!} \psi^I \psi^J \phi_{IJ} + \frac{1}{3!} \epsilon_{IJKL} \psi^I \psi^J \psi^K \tilde{\lambda}^L + \frac{1}{4!} \epsilon_{IJKL} \psi^I \psi^J \psi^K \psi^L G \,.$$

Via the Penrose transform the component fields get mapped to specific helicity particles in Minkowski space:

$\mathcal{N} = 4$ spectrum

How about interactions? These correspond to the *self-dual* $\mathcal{N}=4$. The full interactions arise non-perturbatively through D1-instantons wrapping the holomorphic genus zero, degree one curves with $\mathbb{C}\mathrm{P}^{3|4}$ embedding [Witten], [Nair]

$$\mu_{\dot\alpha} + x_{\alpha\dot\alpha}\lambda^\alpha = 0 \ \text{ and } \ \psi^A + \theta^A_\alpha \lambda^\alpha = 0 \ .$$

These are the same curves onto which the analytic amplitudes are localised. By integrating over the moduli space (x,θ) we get the amplitude prescription

$$\mathcal{A}_{(n)} = \int \mathrm{d}^4x \mathrm{d}^8\theta \, w_1 \cdot w_2 \cdots w_n \langle J_1 \cdot J_2 \cdots J_n \rangle$$

with the J's being D1-instanton world-volume currents and the w_i's the external particle wavefunctions.

Recover all analytic amplitudes - Extension to non-analytic

The $\mathcal{N}=2$ Sp(N) theory with $N_f=4$

Physical string realisation: N D3's living at an O7 plane with 4 D7 branes. The near horizon geometry on the D3's is $\mathrm{AdS}_5\times \mathrm{S}^5/\mathbb{Z}_2$. [Sen], [Banks-Douglas-Seiberg], [Fayyazuddin-Spalinski], [Aharony-Fayyazuddin-Maldacena]

The massless open string d.o.f can be summarised in [Gava-Narain-Sarmadi]

Component	SO(1,3)	SU(2)$_a$	SU(2)$_A$	U(1)	Sp(N)	SO(8)
A, G	$(2,2)$	1	1	0	$N(2N+1)$	1
ϕ	$(1,1)$	1	1	$+2$	$N(2N+1)$	1
$\phi^\dagger$	$(1,1)$	1	1	-2	$N(2N+1)$	1
$\lambda_{\alpha,a}$	$(2,1)$	2	1	$+1$	$N(2N+1)$	1
$\bar\lambda_{\dot\alpha,a}$	$(1,2)$	2	1	-1	$N(2N+1)$	1
z_{aA}	$(1,1)$	2	2	0	$N(2N-1)$	1
$\zeta_{\alpha,A}$	$(2,1)$	1	2	-1	$N(2N-1)$	1
$\bar\zeta_{\dot\alpha,A}$	$(1,2)$	1	2	$+1$	$N(2N-1)$	1
q_a^I	$(1,1)$	2	1	0	$2N$	8
$\eta_{\alpha I}$	$(2,1)$	1	1	-1	$2N$	8
$\bar\eta_{\dot\alpha}^I$	$(1,2)$	1	1	$+1$	$2N$	8

The gauge theory Lagrangian is in $\mathcal{N} = 1$ superfield formulation:

$$\mathcal{L} = \frac{1}{8\pi}\text{Im Tr}\left[\tau\left(\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta}\; e^{2V}\Phi^\dagger e^{-2V}\Phi\right)\right] + \int d^2\theta d^2\bar{\theta}\; Q^{\dagger I} e^{-2V} Q_I$$
$$+ \int d^2\theta d^2\bar{\theta}\; Q'^I e^{2V} Q'^\dagger_I + \text{Tr}\left(\int d^2\theta d^2\bar{\theta}\; e^{2V} Z^\dagger e^{-2V} Z + \int d^2\theta d^2\bar{\theta}\; e^{-2V} Z' e^{2V} Z'^\dagger\right)$$
$$+ \sqrt{2}\left(\int d^2\theta (Q'^I \Phi Q_I + \text{Tr}\,(Z'[\Phi, Z])) + h.c.\right)\ .$$

The flavour symmetry group in this notation is given under the maximal embedding U(1) × SU(4) ⊂ SO(8). An additional series of helicity-dependent rescalings leads to the self-dual truncation:

$$\mathcal{L} = \text{Tr}\left[-\frac{1}{2}GF + D\phi^\dagger D\phi + i\bar{\lambda}^a \not{D}\lambda_a - \lambda^a\lambda_a\phi^\dagger\right]$$
$$- \text{Tr}\left[\frac{1}{2}Dz^{aA}Dz_{Aa} + i\bar{\zeta}^A \not{D}\zeta_A + z^{aA}[\lambda_a, \zeta_A] + \zeta^A\zeta_A\phi\right]$$
$$- \frac{1}{2}Dq^a{}_M Dq^M{}_a - i\bar{\eta}_M \not{D}\eta^M + q^a{}_M\lambda_a\eta^M - \frac{1}{2}\eta_M\phi\eta^M\ .$$

On the twistor side we perform a super-orientifold

$$(a) \quad \psi^a \to \psi^a \quad , \quad \psi^A \to -\psi^A$$

$$(b) \quad \mathcal{A}^i{}_j \to \mathcal{A}^j{}_i \; ; a = 1, 2 \; , \; A = 3, 4$$

The superfield $\mathcal{A}$ decomposes into an adjoint and an anti-symmetric Sp(N) piece yielding part of the spectrum

$$\begin{aligned} \mathcal{A} &= \mathcal{V} + \mathcal{Z} \\ &= (A + \psi^a \lambda_a + \psi^1 \psi^2 \phi + \psi^3 \psi^4 \phi^\dagger + \epsilon_{cd} \psi^3 \psi^4 \psi^c \tilde{\lambda}^d + \psi^1 \psi^2 \psi^3 \psi^4 G) \\ &+ (\psi^A \zeta_A + \psi^a \psi^B z_{aB} + \epsilon_{CD} \psi^1 \psi^2 \psi^C \tilde{\zeta}^D) \end{aligned}$$

We still need to recover the fundamental d.o.f.

This is done by introducing a new "flavour" brane, sitting at the locus $\psi^A = 0$, $\bar{\psi}^{a,A} = 0$

$$\mathcal{Q}_X = \psi^A Q_{AX} = \psi^A(\eta_{AX} + \psi^a q_{aAX} + \psi^1\psi^2\tilde{\eta}_{AX}).$$

X is chosen to be an index of Sp(2), associated with the maximal embedding Sp(2) × SU(2) ⊂ SO(8).

The full HCS action is

$$S = \frac{1}{2}\int_{\mathrm{D5}} \Omega \wedge \left(\mathrm{Tr}[\mathcal{A}\cdot\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}] + \mathcal{Q}^X\cdot\bar{\partial}\mathcal{Q}_X + \mathcal{Q}^X\wedge\mathcal{A}\wedge\mathcal{Q}_X\right)\,,$$

while in component form

$$\begin{aligned} S_{HCS} = \int_{\mathbb{CP}^3} \Omega' \wedge \mathrm{Tr}[&G\wedge F + \phi^\dagger\wedge\bar{D}\phi - \tilde{\lambda}^a\wedge\bar{D}\lambda_a + \lambda^a\wedge\lambda_a\wedge\phi^\dagger + \frac{1}{2}z^{Aa}\wedge\bar{D}z_{aA} \\ &- \tilde{\zeta}^A\wedge\bar{D}\zeta_A + z^{aA}\wedge\lambda_a\wedge\zeta_A + \zeta^A\wedge\zeta_A\wedge\phi] + \tilde{\eta}^{XA}\wedge\bar{D}\eta_{AX} \\ &- \frac{1}{2}q^{aAX}\wedge\bar{D}q_{aAX} + q^{aAX}\wedge\lambda_a\wedge\eta_{AX} - \frac{1}{2}\eta^{AX}\wedge\phi\wedge\eta_{AX}\,. \end{aligned}$$

We have recovered the spectrum. To compare interactions derive Feynman rules for gauge theory and calculate amplitude ratios with the analytic ones evaluated from Witten's prescription on the twistor side.

We find agreement for a large set of amplitudes up to the same constant normalisation factor:

$\langle \lambda^a, \phi^\dagger, \bar{\lambda}^b, \phi \rangle$ $\quad \langle \phi^\dagger, {z^a}_A, {z^b}_B, \phi \rangle$ $\quad \langle \lambda^a, \lambda^b, \zeta_A, \zeta_B \rangle$ $\quad \langle \lambda^a, {z^b}_B, {z^c}_C, \lambda^d, \phi^\dagger \rangle$

$\langle \lambda^a, \zeta_A, \bar{\zeta}_B, \bar{\lambda}^b \rangle$ $\quad \langle \phi^\dagger, {q^a}_A, {q^b}_B, \phi \rangle$ $\quad \langle \phi_1, \phi_2, \phi_3^\dagger, \phi_4^\dagger \rangle$ $\quad \langle \phi, {q^a}_A, {q^b}_B, \eta_C, \eta_D \rangle$

$\langle \eta_A, \lambda^a, \eta_B, \lambda^b \rangle$ $\quad \langle {z^a}_A, \zeta_C, \bar{\zeta}_D, {z^b}_B \rangle$ $\quad \langle \eta_A, \lambda^a, \bar{\lambda}^b, \bar{\eta}_B \rangle$

$\langle {z^a}_A, {z^b}_B, {z^c}_C, {z^d}_D \rangle$ $\quad \langle {q^a}_A, {q^b}_B, {q^c}_C, {q^d}_D \rangle$

These include 4-point amplitudes with fundamental, antisymmetric and adjoint external particles and two 5-point.

NB1: The integration over the D1-instanton moduli space has only included $\mathbb{CP}^1$ topologies. $\mathbb{RP}^2$'s $\sim \mathcal{O}(g^3)$ will not contribute to tree-level analytic amplitudes.

NB2: We are encoding part of the flavour symmetry geometrically. Above agreement forces us to identify $\mathrm{SU}(2) \subset \mathrm{SO}(8)$ with $\mathrm{SU}(2)_A$. Twistor strings describe theory with global $\mathrm{SU}(2)_A \times \mathrm{Sp}(2)$.

NB3: The pre-analytic amplitudes corresponding to interactions arising only from the self-dual vertices vanish. This could be an indication for classical integrability of the self-dual theory.

NB4: We have used $\mathrm{Sp}(N_c)$ properties while constructing the actions but the amplitudes are colour-stripped. Expect similar behaviour for $\mathrm{SU}(N_c)$.

The $\mathcal{N}=2$ SU(N_c) theory with $N_f = 2N_c$

Physical string description as N_c fractional D3's probing N_f fractional D7's on an $\mathbb{R}^{1,5}\times\mathbb{R}^4/\mathbb{Z}_2$ orbifold. For $N_f = 2N_c$ the theory becomes superconformal.
[Bertolini-Di Vecchia-Frau-Lerda-Marotta]

The D7's now lie along the orbifolded co-ordinates. The D3's are constrained from moving in these directions. The spectrum is

Component	SO(1,3)	SU(2)$_a$	SU(2)$_A$	U(1)	SU(N)	SU($2N$)
A, G	$(2,2)$	1	1	0	N^2-1	1
ϕ	$(1,1)$	1	1	$+2$	N^2-1	1
$\phi^\dagger$	$(1,1)$	1	1	-2	N^2-1	1
$\lambda_{\alpha,a}$	$(2,1)$	2	1	$+1$	N^2-1	1
$\bar\lambda_{\dot\alpha,a}$	$(1,2)$	2	1	-1	N^2-1	1
$q^I_a, q^\dagger_{aI}$	$(1,1)$	2	1	0	$N, \overline{N}$	$\overline{2N}, 2N$
$\eta^I_\alpha, \bar\eta'^I_\alpha$	$(2,1)$	1	1	-1	N	$\overline{2N}$
$\bar\eta_{\dot\alpha I}, \eta'_{\dot\alpha I}$	$(1,2)$	1	1	$+1$	$\overline{N}$	$2N$

The gauge theory Lagrangian is in $\mathcal{N}=1$ notation

$$\mathcal{L} = \frac{1}{8\pi}\text{Im Tr}\left[\tau\left(\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar\theta\; e^{2V}\Phi^\dagger e^{-2V}\Phi\right)\right] + \int d^2\theta d^2\bar\theta\; Q^{\dagger I}e^{-2V}Q_I$$
$$+ \int d^2\theta d^2\bar\theta\; Q'^I e^{2V} Q'^\dagger_I + \sqrt{2}\int d^2\theta\left(Q'^I\Phi Q_I + h.c.\right)\;,$$

with a series of helicity-asymmetric rescalings leading to the following self-dual truncation

$$\mathcal{L} = \text{Tr}\left[-\frac{1}{2}GF + D\phi^\dagger D\phi + i\bar\lambda^a \not{D}\lambda_a - \lambda^a\lambda_a\phi^\dagger\right] - Dq^{\dagger aI}Dq_{aI}$$
$$- i\bar\eta^I \not{D}\eta_I - i\eta'^I \not{D}\bar\eta'_I - \eta'^I\phi\eta_I + q^{\dagger aI}\lambda_a\eta_I - \eta'^I\lambda^a q_{aI}\;.$$

On the twistor side we only orbifold the fermionic directions

$$\psi^a \to \psi^a \quad , \quad \psi^A \to -\psi^A \;;\; \psi^a = \psi^{1,2} \;,\; \psi^A = \psi^{3,4}$$

This projects out all but the following component fields

$$\mathcal{A} \;=\; (A + \psi^a \lambda_a + \psi^1\psi^2\phi + \psi^3\psi^4\phi^\dagger + \epsilon_{cd}\psi^3\psi^4\psi^c\tilde{\lambda}^d + \psi^1\psi^2\psi^3\psi^4 G)$$

The "flavour" branes correspond to the introduction of the superfields

$$\begin{aligned}\mathcal{Q}_K &= \psi^A Q_{AK} = \psi^A \left(\eta_{AK} + \psi^a q_{aAK} + \psi^1\psi^2\tilde{\eta}'_{AK}\right)\\ \mathcal{Q}^{\dagger K} &= \psi^A Q^{\dagger K}_A = \psi^A \left(\eta'^K_A + \psi^a q^{\dagger K}_{aA} + \psi^1\psi^2\tilde{\eta}^K_A\right) .\end{aligned}$$

K is an index of SU(N), coming from the following maximal embedding SU(N) $\times$ SU(2) $\subset$ SU($2N$).

The HCS action is

$$S = \int_{\mathrm{D5}} \Omega \wedge \left(\frac{1}{2} \mathrm{Tr}[\mathcal{A} \cdot \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}] + \mathcal{Q}^{\dagger K} \cdot \bar{\partial} \mathcal{Q}_K + \mathcal{Q}^{\dagger K} \wedge \mathcal{A} \wedge \mathcal{Q}_K \right)$$

The SU(N) spectrum has been recovered. The colour-stripped amplitude ratios work out in the same manner as before with identical relative normalisation.

$$\langle q^{\dagger a}{}_A, q^b{}_B, q^{\dagger c}{}_C, q^d{}_D \rangle \qquad \langle \phi, q^a{}_A, q^{\dagger b}{}_B, \eta_C, \eta'_D \rangle$$

NB: In this case the twistor string encodes the flavour symmetry on the spacetime side exactly.

Conclusions

- [x] We established a correspondence between two classes of $\mathcal{N}=2$ UV-finite gauge theories with fundamental matter and twistor string theory
- [x] This required the introduction of some new objects in the B-model on $\mathbb{CP}^{3|4}$, which we called "flavour" branes
- [ ] Description of these as proper B-model states?
- [x] An SU(2) part of the flavour group realised geometrically
- [ ] Exact mechanism?
- [ ] Is the flavour group for $\mathrm{Sp}(N_c)$ broken to $\mathrm{SO}(8) \to \mathrm{Sp}(2) \times \mathrm{SU}(2)$?
- [x] This reinforces the belief that finite 4d gauge theories should have a tree level twistor string description.
- [ ] Use this correspondence to learn something new about branes in topological string theories.