Geometry of all type I supersymmetric backgrounds

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SUPERSYMMETRIC SOLUTIONS

Applications

- M-theory
- String theory, duality
- Branes, Black holes
- Compactifications
- Spinorial geometry, special geometric structures
- AdS/CFT, gravity/Yang-Mills correspondences

Topics

- SUGRA Killing spinor equations
- Holonomy vs gauge symmetry
- Spinorial geometry
- All supersymmetric backgrounds of type I SUGRA
- $N = 31$
- Conclusions

KILLING SPINOR EQUATIONS (KSE)

A parallel transport equation for the supercovariant connection $\mathcal{D}$

$$\delta\psi_A| = \mathcal{D}_A\epsilon = \nabla_A\epsilon + \Sigma_A(e,F)\epsilon = 0$$

and possibly algebraic equations

$$\delta\lambda| = \mathcal{A}(e,F)\epsilon = 0$$

where ∇ is the Levi-Civita connection, $\Sigma(e,F)$ a Clifford algebra element

$$\Sigma(e,F) = \sum_k \Sigma_{[k]}(e,F)\Gamma^{[k]}$$

e frame and F fluxes, ϵ spinor, Γ gamma matrices.

- N no of linearly independent solutions ϵ.

Can the KSE be solved without any assumptions on the metric and fluxes?

REDUCED SUPERSYMMETRY

Holonomy

Hull, Duff, Liu, Tsimpis, GP

For generic D=11 and IIB backgrounds

$$\mathrm{hol}(\mathcal{D}) \subseteq SL(32, \mathbb{R})$$

because $\mathcal{R}$ takes values in $\mathfrak{sl}(32, \mathbb{R})$

For N-susy backgrounds

$$\begin{aligned} \mathrm{hol}(\mathcal{D}) &\subseteq SL(32-N, \mathbb{R}) \ltimes \oplus_N \mathbb{R}^{32-N} \\ &= \mathrm{Stab}(\epsilon) \subset SL(32, \mathbb{R}) \end{aligned}$$

The consequences are

- There may be backgrounds for any N, however see preons $(N = 31)$
- Any subbundle $\mathcal{K}$ of the Spin bundle $\mathcal{S}$ can be Killing

Gauge Symmetry G

The gauge symmetry G of the KSE are the (local) transformations such that

$$g^{-1}\mathcal{D}(e,F)g = \mathcal{D}(e^g, F^g)$$

D=11 SUGRA: $G = Spin(10,1)$
IIB SUGRA: $G = Spin(9,1) \times U(1)$

- Backgrounds related by a gauge transformation are identified
- The geometry of backgrounds is (non-uniquely) characterized by the stability subgroup stab(ϵ) of the KS in G
- $G \subset\subset \mathrm{hol}(\mathcal{D})$, e.g. 2 generic spinors in D=11 and IIB have stab$(\epsilon) = \{1\}$

- For one spinor

  D=11: stab = $SU(5)$, $Spin(7) \ltimes \mathbb{R}^9$

  Bryant, Figueroa-O'Farrill

  IIB: stab = $Spin(7) \ltimes \mathbb{R}^8$, $SU(4) \ltimes \mathbb{R}^8$, G_2

Can extended gauge symmetries help?

Spin(9,1) SPINORS

Consider $U=\mathbb{C}<e_1,\dots,e_5>$, $e_1,\dots,e_5$ orthonormal w.r.t $<,>$.

Dirac spinors: $\Delta_c=\Lambda^*(U)$

Weyl Spinors: $\Delta_c^+=\Lambda^{\rm ev}(U)$, $\Delta_c^-=\Lambda^{\rm od}(U)$.

Gamma matrices on Δ_c:

$$\begin{aligned}
&\Gamma_0\eta=-e_5\wedge\eta+e_5\lrcorner\eta\ ,\\
&\Gamma_5\eta=e_5\wedge\eta+e_5\lrcorner\eta\\
&\Gamma_i\eta=e_i\wedge\eta+e_i\lrcorner\eta\ ,\qquad i=1,\dots,4\\
&\Gamma_{5+i}\eta=ie_i\wedge\eta-ie_i\lrcorner\eta\ .
\end{aligned}$$

The Dirac inner product:

$$D(\eta,\theta)=<\Gamma_0\eta,\theta>$$

A Majorana inner product:

$$B(\eta,\theta)=<B(\eta^*),\theta>\ ,\qquad B=\Gamma_{06789}$$

The Majorana reality condition can be chosen as

$$\eta = -\Gamma_0 B(\eta^*) = \Gamma_{6789}\eta^* \ .$$

$C = \Gamma_{6789}$ is the charge conjugation matrix.

Example

For Weyl spinor $a1 + be_{1234}$, $a, b \in \mathbb{C}$, the reality condition gives

$$\eta = a1 + a^* e_{1234} \ .$$

Two Majorana spinors: $1 + e_{1234}$ and $i1 - ie_{1234}$.

- $\mathrm{stab}(1 + e_{1234}) = Spin(7) \ltimes \mathbb{R}^8$
- $\mathrm{stab}(1+e_{1234}, i(1-e_{1234})) = SU(4) \ltimes \mathbb{R}^8$
- Δ_c has an oscillator basis, $\mu = 0, 1, \dots, 4$

  $1, \quad e_\mu = e_\mu \wedge 1, \quad e_{\mu\nu} = e_\mu \wedge e_\nu \wedge 1, \quad \dots$

SPINORIAL GEOMETRY

Gillard, Gran, GP

The ingredients of the spinorial method to classify supergravity backgrounds are

- Gauge symmetry of KSE

  Effective for backgrounds with small and large number of susies

- Spinors in terms of forms

  Convenient notation

- An oscillator basis in the space of spinors

  Allows to extract the geometric information from the KSE

TYPE I SUPERGRAVITY

Gran, Lohrmann, GP

Gran, Roest, Sloane, GP

Gravitino

$$\mathcal{D}\epsilon = \hat{\nabla}\epsilon = \nabla\epsilon + \frac{1}{2}H\epsilon = 0$$

H torsion, and

$$\mathrm{hol}(\hat{\nabla}) = G = Spin(9,1)$$

In addition

$$\hat{\nabla}\epsilon = 0 \Rightarrow \hat{R}\epsilon = 0$$

So either

$$\hat{R} = 0$$

and M is a group Manifold ($dH = 0$), or

$$\mathrm{stab}(\epsilon) \neq \{1\}$$

- All the parallel spinors can be determined
- The parallel spinors are singlets of subgroups of $Spin(9,1)$.
- There are similarities with the parallel spinors on Riemannian manifolds, e.g. CY.

L	$Stab(\epsilon_1, \dots, \epsilon_L)$	ϵ
1	$Spin(7) \ltimes \mathbb{R}^8$	$1 + e_{1234}$
2	$SU(4) \ltimes \mathbb{R}^8$	1
3	$Sp(2) \ltimes \mathbb{R}^8$	$1,\ i(e_{12} + e_{34})$
4	$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$1,\ e_{12}$
5	$SU(2) \ltimes \mathbb{R}^8$	$1,\ e_{12},\ e_{13} + e_{24}$
6	$U(1) \ltimes \mathbb{R}^8$	$1,\ e_{12},\ e_{13}$
8	$\mathbb{R}^8$	$1,\ e_{12},\ e_{13},\ e_{14}$
2	G_2	$1 + e_{1234},\ e_{15} + e_{2345}$
4	$SU(3)$	$1,\ e_{15}$
8	$SU(2)$	$1,\ e_{12},\ e_{15},\ e_{25}$
16	$\{1\}$	Δ^+

Table 1: In the columns are the number, isotropy groups and representatives of parallel spinors, respectively.

- There are compact K and non-compact $K \ltimes \mathbb{R}^8$ isotropy groups stab(ϵ).
- Some stab(ϵ) are different from those that appear in the Berger list for Riemannian manifolds.

Dilatino

$$d\Phi\epsilon - \frac{1}{2} H\epsilon = 0$$

Some of the parallel spinors may not solve the dilatino KSE. Having solved the gravitino KSE to solve the dilatino KSE, the gauge group that can be used is

$$\Sigma(\mathcal{P}) = \{\ell \in Spin(9,1) | \ \ell\mathcal{P} \subseteq \mathcal{P}\}$$

$\mathcal{P}$ is the space of parallel spinors.

- Killing spinors or their normals are represented by orbits of subgroups of $\Sigma(\mathcal{P})$ in $\mathcal{P}$. All Killing spinors are determined.
- Killing spinors may have trivial isotropy group in $\Sigma(\mathcal{P})$.

L	$Stab(\epsilon_1, \ldots, \epsilon_L)$	$\Sigma(\mathcal{P})$
1	$Spin(7) \ltimes \mathbb{R}^8$	$Spin(1,1)$
2	$SU(4) \ltimes \mathbb{R}^8$	$Spin(1,1) \times U(1)$
3	$Sp(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(2)$
4	$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(1) \times Sp(1)$
5	$SU(2) \ltimes \mathbb{R}^8$	$Spin(1,1) \times Sp(2)$
6	$U(1) \ltimes \mathbb{R}^8$	$Spin(1,1) \times SU(4)$
8	$\mathbb{R}^8$	$Spin(1,1) \times Spin(8)$
2	G_2	$Spin(2,1)$
4	$SU(3)$	$Spin(3,1) \times U(1)$
8	$SU(2)$	$Spin(5,1) \times SU(2)$
16	$\{1\}$	$Spin(9,1)$

Table 2: In the columns are the numbers of parallel spinors, their isotropy groups and the $\Sigma(\mathcal{P})$ groups, respectively.

- The $\Sigma(\mathcal{P})$ groups are a product of a $Spin$ group and a R-symmetry group, reminiscent of lower-dimensional supergravities.

There are backgrounds for any N

L	$\Sigma(\mathcal{P})$	N
1	$Spin(1,1)$	1(1)
2	$Spin(1,1) \times U(1)$	1(1), 2(1)
3	$Spin(1,1) \times SU(2)$	1(1), 2(1), 3(1)
4	$Spin(1,1) \times Sp(1) \times Sp(1)$	1(1), 2(1), 3(1), 4(1)
5	$Spin(1,1) \times Sp(2)$	1(1), 2(1), 3(1), 4(1), 5(1)
6	$Spin(1,1) \times SU(4)$	1(1), 2(1), 3(1), 4(1), 5(1), 6(1)
8	$Spin(1,1) \times Spin(8)$	1(1), 2(1), 3(1), 4(1), 5(1), 6(1), 7(1), 8(1)
2	$Spin(2,1)$	1(1), 2(1)
4	$Spin(3,1) \times U(1)$	1(1), 2(2), 3(2), 4(1)
8	$Spin(5,1) \times SU(2)$	1(1), 2(2), 3(3), 4(6), 5(3), 6(2), 7(1), 8(1)
16	$Spin(9,1)$	1(1), 2(2), 3(1), 4(2), 5(1), 6(1), 8(2), 10(1), 12(1), 14(1), 16(1)

Table 3: In the columns are the $\Sigma(\mathcal{P})$ groups that arise from the solution of the gravitino and dilatino Killing spinor equations and the number N of supersymmetries, respectively. The number in parenthesis indicates the different cases that arise in the dilatino Killing spinor equation for a given N.

- If $N = 16$, then the spacetime is locally isometric to $\mathbb{R}^{9,1}$

Geometry of N=L Backgrounds

Gran, Lohrmann, GP

(i). stab(ϵ) compact

- The spacetime admits 1 timelike, and 2 (G_2), 3 ($SU(3)$) and 5 ($SU(2)$) spacelike $\hat{\nabla}$-parallel one-forms.
- The commutator $[X, Y]$ of any two X, Y, $\hat{\nabla}$-parallel vector fields, and so Killing, is also $\hat{\nabla}$-parallel.
- The commutator is determined by H

Two assumptions

- The parallel spinors are Killing
- The $\hat{\nabla}$-parallel vectors constructed from Killing spinor bilinears span a Lie algebra $\mathfrak{h}$ of a group $\mathcal{H}$.

The spacetime is a principal bundle $M = P(\mathcal{H}, B, \pi)$ equipped with a instanton-like connection λ with curvature $\mathcal{F}$.

The metric and H of the background can be written as

$$ds^2 = \eta_{ab}\lambda^a\lambda^b + \pi^* d\tilde{s}^2$$
$$H = \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda \wedge \mathcal{F}^b + \pi^* \tilde{H}$$

The base space B admits an integrable, conformally balanced K-structure, compatible with a connection, $\hat{\tilde{\nabla}}$, with skew-symmetric torsion associated with the pair $(d\tilde{s}^2, \tilde{H})$.

In addition

$$dH = \eta_{ab}\mathcal{F}^a \wedge \mathcal{F}^b + \pi^* d\tilde{H}$$

i.e. part of dH is specified by the first Pontrjagin form of P

G_2

$\mathfrak{h}= \mathfrak{sl}(2,\mathbb{R})$ or $\mathbb{R} \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$

$$\tilde{H} = -\frac{r}{6}(d\varphi, \star\varphi)\varphi + \star d\varphi + \star(\tilde{\theta}_\varphi \wedge \varphi)$$

Ivanov, et al

$$\tilde{\theta}_\varphi = 2d\Phi \ ,$$
$$d \star \varphi = -\tilde{\theta}_\varphi \wedge \star\varphi$$

$r = 0$ if $\mathfrak{h}$ abelian, and $r = 1$ if $\mathfrak{h}$ non-abelian, where

$$\tilde{\theta}_\varphi = \star(\star d\varphi \wedge \varphi)$$

is the Lee form of the G_2-invariant form φ.

In addition, λ , is a $\mathfrak{h}$-valued, $\mathfrak{g}_2 \subset \Lambda^2(\mathbb{R}^7)$ instanton

$\mathrm{hol}(\hat{\tilde{\nabla}}) \subseteq G_2$

$SU(3)$

$\mathfrak{h} = \mathbb{R} \oplus^3 \mathfrak{u}(1)$, $\mathbb{R} \oplus \mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$, $\mathfrak{cw}_4$

If $\mathfrak{h}$ abelian, $\mathrm{hol}(\hat{\hat{\tilde{\nabla}}}) \subseteq SU(3)$ and λ an abelian $\mathfrak{s}u(3) \subset \Lambda^2(\mathbb{R}^6)$ Donaldson connection (B Hermitian).

if $\mathfrak{h}$ non-abelian, $\mathrm{hol}(\hat{\hat{\tilde{\nabla}}}) \subseteq U(3)$ and λ is a $\mathfrak{h}$-valued $\mathfrak{u}(3) \subset \Lambda^2(\mathbb{R}^6)$ Donaldson connection

$SU(2)$

$\mathfrak{h} = \mathbb{R} \oplus^5 \mathfrak{u}(1)$, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$, $\mathfrak{cw}_6$

$\mathrm{hol}(\hat{\hat{\tilde{\nabla}}}) \subseteq SU(2)$ and λ a $\mathfrak{h}$-valued, instanton on B

(ii) $\mathrm{stab}(\epsilon) = K \ltimes \mathbb{R}^8$ non-compact

The metric and torsion are

$$ds^2 = 2e^+e^- + \delta_{ij}e^ie^j$$
$$H = e^+ \wedge de^- + e^- \wedge (\rho + \sigma) + \frac{1}{3!}H_{ijk}e^i \wedge e^j \wedge e^k$$

where $\rho \in \mathfrak{k}$ and $\sigma \in \mathfrak{k}^\perp$.

- All H is determined in terms of geometry apart from ρ.
- M admits a single $\hat{\nabla}$-parallel null vector field, and so Killing, with non-vanishing rotation.
- If the rotation vanishes, the spacetime is a pp-wave propagating in a manifold B with skew-symmetric torsion and a K-structure.

$SU(4) \ltimes \mathbb{R}^8$: Dilatino KSE

Gran, Roest, Sloane, GP

$\hat{\nabla}$-Parallel forms

$$e^- \ , \quad e^- \wedge \omega_I \ , \quad e^- \wedge \chi$$

ω_I hermitian form, χ $(4,0)$-form. This is equivalent to $\mathrm{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$.

$SU(4) \ltimes \mathbb{R}^8$	de^-	$\mathcal{N}$	Stab_Σ	ϵ
$N=1$	$\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) \neq 0$	$\{1\}$	$1 + e_{1234}$
$N=2$	$\mathfrak{su}(4) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = 0$	$\{1\}$	1

Table 4: The differences in the geometry of $N=1$ and $N=2$ backgrounds are in the non-vanishing components of de^- and $\mathcal{N}(I)$.

The remaining conditions of the dilatino Killing spinor equation are

$$(d\Phi)_i + \frac{1}{8}(\mathcal{N} \cdot (\mathrm{Re}\,\chi))_i - \frac{1}{2}(\theta_{\omega_I})_i - \frac{1}{2} H_{-+i} = 0 \ , \qquad \partial_+ \Phi = 0 \ .$$

$\mathbb{R}^8$: Dilatino KSE

$\hat{\nabla}$-parallel forms

$$e^- \ , \quad e^- \wedge \psi$$

$SU(2) \ltimes \mathbb{R}^8$	de^-	$\mathcal{N}$	θ
$N=1$	$\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I), \mathcal{N}(J), \mathcal{N}(L)$ $\mathcal{N}(Q), \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$-$
$N=2$	$\mathfrak{su}(4) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=0, \mathcal{N}(J), \mathcal{N}(J),$ $\mathcal{N}(Q), \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$-$
$N=3$	$\mathfrak{sp}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=0, \mathcal{N}(L),$ $\mathcal{N}(Q), \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}$
$N=4$	$(\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=0$ $\mathcal{N}(Q), \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}=\theta_{\omega_L}$
$N=5$	$\mathfrak{su}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=$ $\mathcal{N}(Q)=0, \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}=$ $\theta_{\omega_L}=\theta_{\omega_Q}$
$N=6$	$\mathfrak{u}(1) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=$ $\mathcal{N}(Q)=\mathcal{N}(T)=0, \mathcal{N}(U) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}=$ $\theta_{\omega_L}=\theta_{\omega_Q}=\theta_{\omega_T}$
$N=7$	$\mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=$ $\mathcal{N}(Q)=\mathcal{N}(T)=\mathcal{N}(U)=0$	$\theta_{\omega_I}=\theta_{\omega_J}=\theta_{\omega_L}=$ $\theta_{\omega_Q}=\theta_{\omega_T}=\theta_{\omega_U}$
$N=8$	$\mathbb{R}^8$	$H_{ijk}=0$	

Table 5: As in previous cases, the differences in the geometry of descendants, $N<L$, are in the non-vanishing components of de^-, and $\mathcal{N}(I)$, $\mathcal{N}(J)$, $\mathcal{N}(L)$, $\mathcal{N}(Q)$, $\mathcal{N}(T)$ and $\mathcal{N}(U)$ and the relation between the Lee forms. $-$ indicates that there is no relation between the Lee forms. It is assumed that the remaining conditions of the dilatino Killing spinor equation of $N=1$ supersymmetric backgrounds are valid.

Holonomy Reduction

Consider $SU(4)\ltimes\mathbb{R}^8$. Field equations,

$$dH = 0 \ , \quad \mathrm{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$$

imply that

$$\tau_1 = H_{+ij}\omega_I^{ij} e^+ \ , \quad \tau_2 = \partial_+ \Phi e^+ \ ,$$
$$\tau_3 = \mathcal{N} \ , \quad \tau_4 = 2d\Phi - \theta_{\omega_I}$$

are $\hat{\nabla}$-parallel. The consequences for $K \ltimes \mathbb{R}^8$ cases are

- The existence of descendants requires that $\mathrm{hol}(\hat{\nabla}) \subset \mathrm{stab}(\epsilon)$.
- If $\mathrm{hol}(\hat{\nabla}) = \mathrm{stab}(\epsilon)$, then the gravitino KSE imply the dilatino ones and all parallel are Killing $L = N$.

For compact stability subgroups there are descendants with $\mathrm{hol}(\hat{\nabla}) = \mathrm{stab}(\epsilon)$.

N=31 is not IIB

Gran, Gutowski, Roest, GP

Preons are solutions that preserve 31 supersymmetries in type II.

31 spinors span a hyperplane and have a unique normal ν w.r.t. a suitable inner product in the space of IIB spinors.

The gauge symmetry can be used to choose the normal ν as

stab(ν)	spinor ν
$Spin(7) \ltimes \mathbb{R}^8$	$(a+ib)(e_5+e_{12345})$
$SU(4) \ltimes \mathbb{R}^8$	$(a+ib)e_5+(c+id)e_{12345}$
G_2	$a(e_5+e_{12345})+b(e_1+e_{234})$

Choose the Killing spinors orthogonal to ν. Then

$$\mathcal{A}\epsilon_r = 0 \ , \quad r = 1, \dots 31$$

implies that

$$P = G = 0$$

The remaining KSE are linear over the complex numbers and so the number of Killing spinors preserved is even. So there are no IIB preons.

- There are no IIA preons

Bandos, Azcarraga, Varela

N=31 D=11

Gran, Gutowski, Roest, GP

In D=11 SUGRA there are two types on normals to the hyperplanes of 31 Killing spinors

The gauge symmetry can be used to choose the normal ν as

stab(ν)	spinor ν
$(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$	$1 + e_{1234}$
$SU(5)$	$1 + e_{12345}$

Choose the Killing spinors ϵ orthogonal to ν. In this case

$$\mathrm{hol}(\mathcal{D}) = \mathbb{R}^{31}$$

But the integrability condition

$$\mathcal{R}\epsilon = 0$$

the Bianchi and Field equations imply that

$$\mathcal{R} = 0$$

So there are no M-preons.

SUMMARY

- The KSE of type I supergravity backgrounds has been solved in ALL cases, and the geometry has been understood.
- There are no type II backgrounds with N=31 supersymmetries. There is a classification for N=32.
- In $D = 11$, the $N = 32$ backgrounds have been classified. There are no N=31 backgrounds.