# Dual Spikes

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## ■ Introduction and Motivations ■

AdS/CFT conjecture makes a correspondence between

$\mathcal{N} = 4$ $D = 4$ $U(N)$ $SYM$ on $R \times S^3$
and
$IIB$ *string theory* on $AdS_5 \times S^5$ in the *global coordinates*.

As a consequence

deformations of $AdS_5 \times S^5 \longleftrightarrow$ gauge invariant opterators

▶ Identification of corresponding deformations on the two sides is in general difficult mostly because of the strong/weak nature of the duality; the weakly coupled world sheet theory, the $SUGRA$ limit, maps to the *strong coupling* limit of SYM with no perturbative description.

▶ There are some ways and limited cases where some of these difficulties can be bypassed:

■ There is a certain class of operators in SYM, BPS operators, which contain informations that are protected against quantum corrections i.e. one can do perturbative calculations with them with results that are "claimed to be" independent of the coupling.

■ There are cases where the correspondence can be studied beyond BPS limit when states/operators with *large quantum numbers* are involved. This is our approach in this work.

► Consider classical string configurations in $AdS_5 \times S^5$ carrying a number of classical charges such as energy and various spins and angular momenta.

• In case of large $R-charges$ $(J)$, world sheet loop expansions around certain classical string solutions are generically suppressed by inverse powers of $J$.

In SYM this translates into the distinction of a class of SYM operators, determined by the charges, whose couplings to the rest of the operators are suppressed by inverse powers of $J$.

The best known example in this context is the BMN case.

In the strict BMN limit, the world sheet expansion around a certain point-like string solution in $AdS_5 \times S^5$ terminates at one loop and the corresponding class of SYM operators are identified as the BMN operators.

The one loop action can be solved exactly and gives, as a prediction of AdS/CFT, an all loop calculation for the anomalous dimension of BMN operators in terms of an effective coupling, $\lambda' \equiv \lambda/J^2$ ($\lambda$ is the 't Hooft coupling).

• For charges only inside AdS, the world sheet loop expansions are not suppressed and the semi classical expressions are only reliable when $\lambda \gg 1$. But remarkably, for large charges, these expressions determine the *qualitative* relation between the charges even for $\lambda \ll 1$.

The best known example for this case is the folded spinning string in AdS which corresponds to twist two operators in SYM and, for large spin, have a logarithmic anomalous dimension.

▶ In this work we are interested in string solutions with large spin in AdS and which describe higher twist operators. These are generalizations of previously known "*Spiky Strings*" [Kruczenski; 2004] to new configurations which we call "*Dual Spikes*". We study these configurations first in flat space time and then in AdS.

▶ Some references

• Gubser, Klebanov, Polyakov • Berenstein, Maldacena, Nastase • Frolov, Tseytlin • Russo • Kruczenski, Tseytlin • Mikhailov • Kazakov, Marshakov, Minahan, Zarembo

• Minahan, Zarembo • Beisert, Kristjansen, Plefke, Staudacher • Bena, Polchinski, Roiban •Arutyunov, Frolov, Russo, Tseytlin • Arutyunov, Frolov, Russo, Tseytlin •Kruczenski, Ryzhov, Tseytlin • Beisert • Kruczenski

• Kruczenski • Ryang • Hofman, Maldacena • Kruczenski, Russo, Tseytlin

• Minahan • Alishahiha, A.E.M • Smedback • Khan, Larsen • Park, Tirziu, Tseytlin

## ■ Dual Spikes in Flat Background ■

Start with the following ansatz for the string

$$t = \tau, \quad \theta = \omega\ \tau + \sigma, \quad r = r(\sigma)$$

$(\tau, \sigma)$=WS coord's and in target space $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2$

NG Lagrangian $\quad \mathcal{L}_{NG} = -\sqrt{(1-\omega^2\ r^2)\ r'^2 + r^2}\ , \ (') = \partial_\sigma$

Equations of motion are satisfied with

$\frac{r'^2}{r^2} = \frac{r_c^2}{r_l^2}\ \frac{r^2 - r_l^2}{r_c^2 - r^2}\ , \quad r_c = 1/\omega\ , \quad r_l$ =const. from $\partial_\theta$ isometry

This describes a segment of string stretching between a cusp at $r_c$ and a lobe at $r_l$

with $r_1 = min(r_l, r_c)$ , $r_2 = max(r_l, r_c)$ and $a = \frac{r_c}{r_l}$

$$E_{seg} = \frac{1}{2\pi}\frac{1}{r_c}\int_{r_1}^{r_2} dr r \frac{|r_c^2 - r_l^2|}{\sqrt{(r^2 - r_l^2)(r_c^2 - r^2)}} = \frac{1}{4}\frac{r_c}{a^2}|a^2 - 1|$$

$$J_{seg} = \frac{1}{2\pi}\int_{r_1}^{r_2} dr r \sqrt{\frac{r^2 - r_l^2}{r_c^2 - r^2}} = \frac{1}{8}\frac{r_c^2}{a^2}|a^2 - 1|$$

$$\Delta\theta = \frac{r_l}{r_c}\int_{r_1}^{r_2}\frac{dr}{r}\sqrt{\frac{r_c^2 - r^2}{r_2 - r_l^2}} = \frac{\pi}{2}\frac{|a - 1|}{a}$$

We now demand that $2n$ number of segments make up a closed string and hence we have $n$ number of spikes on the string

$$\frac{2a}{|a-1|} = n$$

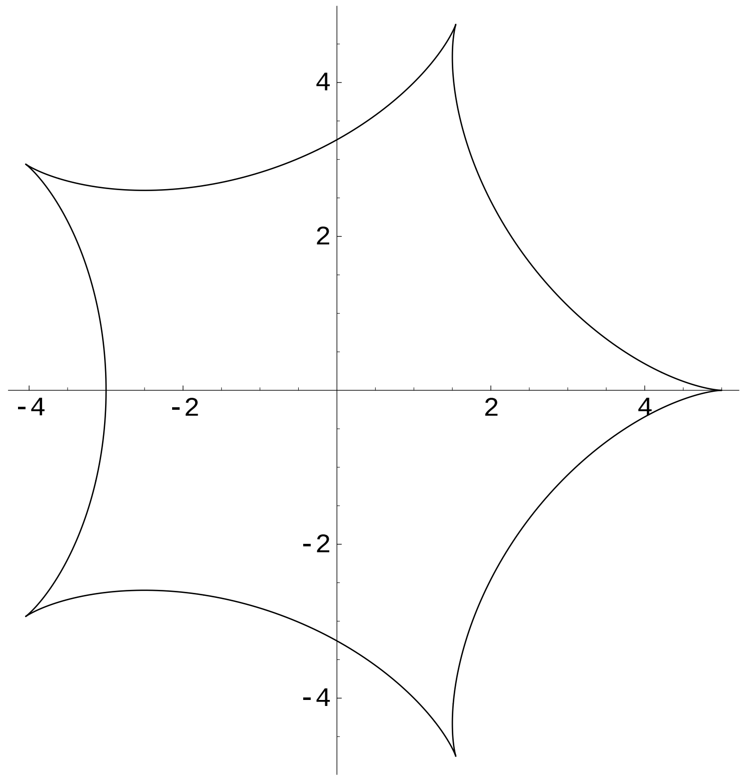
The total $E$ and $J$ of the closed string is

$$E = \frac{r_c}{a}(a+1)\ , \qquad J = \frac{r_c^2}{2a}(a+1)\ , \qquad E = 2\frac{J}{r_c}$$

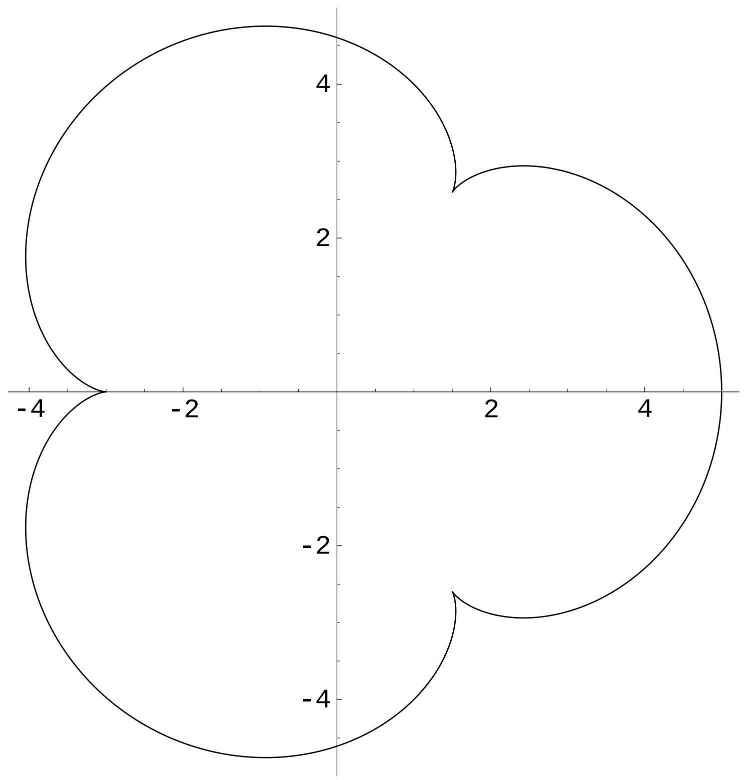
Therefore to each pair, $(a, r_c)$, corresponds a unique string configuration provided that the periodicity condition is satisfied for some integer $n$.

Two distinct cases are identified: $a > 1$ and $a < 1$

• $a > 1$ results in the "*Spiky Strings*" found in [Kruczenski; 2004]; closed strings with $n$ spikes $n = 2, 3, ....$ For example for $(a, r_c) = (5/3, 5)$



• $a < 1$ results in *"Dual Spikes"*; closed strings with $n$ spikes $n = 1, 2.....$ For example for $(a, r_c) = (3/5, 3)$



One can easily check that the energy determined by the pair $(a, r_c)$ remains invariant if we switch to the pair $(1/a, r_c/a)$. This transformation takes one from a "*Spiky String*" with $n$ spikes to a "*Dual Spiky String*" with $n-2$ spikes or vice versa.

$$a \to \frac{1}{a}\ , \quad r_c \to \frac{r_c}{a}\ , \quad n \to n + 2\frac{|1-a|}{1-a}\ , \quad J \to \frac{J}{a}\ , \quad E \to E$$

The $a = 1$ case describes a circular string $(n \to \infty)$ and is a *self dual* configuration.

▶ One can find these two sets of solutions from the Polyakov action for strings in the target space $ds^2 = -dt^2 + dx^2 + dy^2$

$$x = r_c \frac{|a-1|}{2a} \cos(\frac{a+1}{a-1}\sigma_+) + r_c \frac{a+1}{2a} \cos(\sigma_-)$$

$$y = r_c \frac{|a-1|}{2a} \sin(\frac{a+1}{a-1}\sigma_+) + r_c \frac{a+1}{2a} \sin(\sigma_-)$$

$$t = r_c \frac{a+1}{a} \tau$$

The transformation $(a, r_c) \to (1/a, r_c/a)$ amounts to $y_L \to -y_L$ which is a $T - Duality$ in the $y$ direction.

## ■ Dual Spikes in AdS ■

We find spiky strings in $AdS_3 \subset AdS_5$ with the metric

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2$$

The string ansatz is

$$t = \tau \ , \qquad \theta = \omega\tau + \sigma \ , \qquad \rho = \rho(\sigma)$$

The radius of $AdS$ is chosen to be one and the dimensionless worldsheet coupling constant is denoted by $1/\sqrt{\lambda}$ where it is understood that $\lambda$ is the 't Hooft coupling in the dual field theory, $N = 4$ SYM.

The NG Lagrangian is

$$\mathcal{L}_{NG} = -\frac{\sqrt{\lambda}}{2\pi}\sqrt{(\cosh^2 \rho - \omega^2 \sinh^2 \rho)\rho'^2 + \sinh^2 \rho \cosh^2 \rho}$$

The equations of motion are solved with

$$\frac{\rho'}{\sinh 2\rho} = \frac{1}{2}\frac{\sqrt{\cosh 2\rho_c - 1}}{\sinh 2\rho_l}\sqrt{\frac{\cosh^2 2\rho - \cosh^2 2\rho_l}{\cosh 2\rho_c - \cosh 2\rho}}$$

where

$$\sinh^2 \rho_c \equiv \frac{1}{\omega^2 - 1} \quad (\omega > 1) \ , \qquad \rho_l = Const. \ (\partial_\theta \ isometry)$$

This describes a segment of string stretching between a cusp at $\rho_c$ and a lobe at $\rho_l$. The assumption $\rho_l < \rho_c$ gives the spikes of [Kruczenski;2004]. In the following we assume $\rho_l > \rho_c$ which results in dual spikes.

Defining $u = \cosh 2\rho$, the angle covered by each segment is

$$\Delta\theta = \sqrt{\frac{u_l^2 - 1}{u_c - 1}} \int_{u_c}^{u_l} \frac{du}{u^2 - 1} \frac{\sqrt{u - u_c}}{\sqrt{u_l^2 - u^2}}$$

To make a closed string we require that $(\rho_c, \rho_l)$ are chosen such that $\Delta\theta = \pi/n$. This gives a closed string with $n$ spikes pointing towards the origin (dual spikes) by gluing $2n$ number of these segments. One can show that the gluing process gives regular functions for $\rho$ and $\theta$ as functions of $\tilde{\sigma}(\sigma)$ with $\frac{d\sigma}{d\tilde{\sigma}} = 0$ at the position of cusps.

For the closed string we find

$$\Delta\theta = \frac{\sqrt{u_l^2 - 1}}{\sqrt{2u_l(u_c - 1)}} \ \{\frac{u_c + 1}{u_l + 1}\Pi(n_1, p) - \frac{u_c - 1}{u_l - 1}\Pi(n_2, p)\}$$

$$E - \omega J = \frac{2n}{2\pi}\sqrt{\lambda}\frac{2\sqrt{u_l}}{\sqrt{u_c - 1}} \ \{E(p) - \frac{u_c + u_l}{2u_l}K(p)\}$$

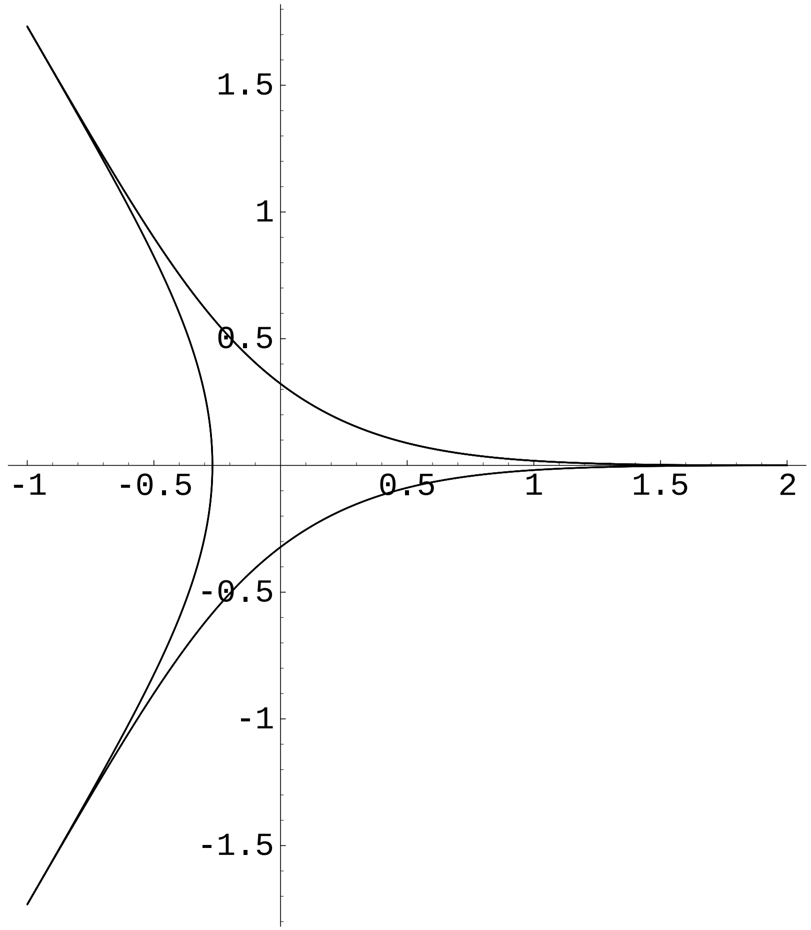
$$J = \frac{2n}{2\pi}\sqrt{\lambda}\frac{\sqrt{u_c + 1}}{2\sqrt{2u_l}} \ \{(1 + u_l)K(p) - 2u_l E(p) + (u_l - 1)\Pi(n_1, p)\}$$

where

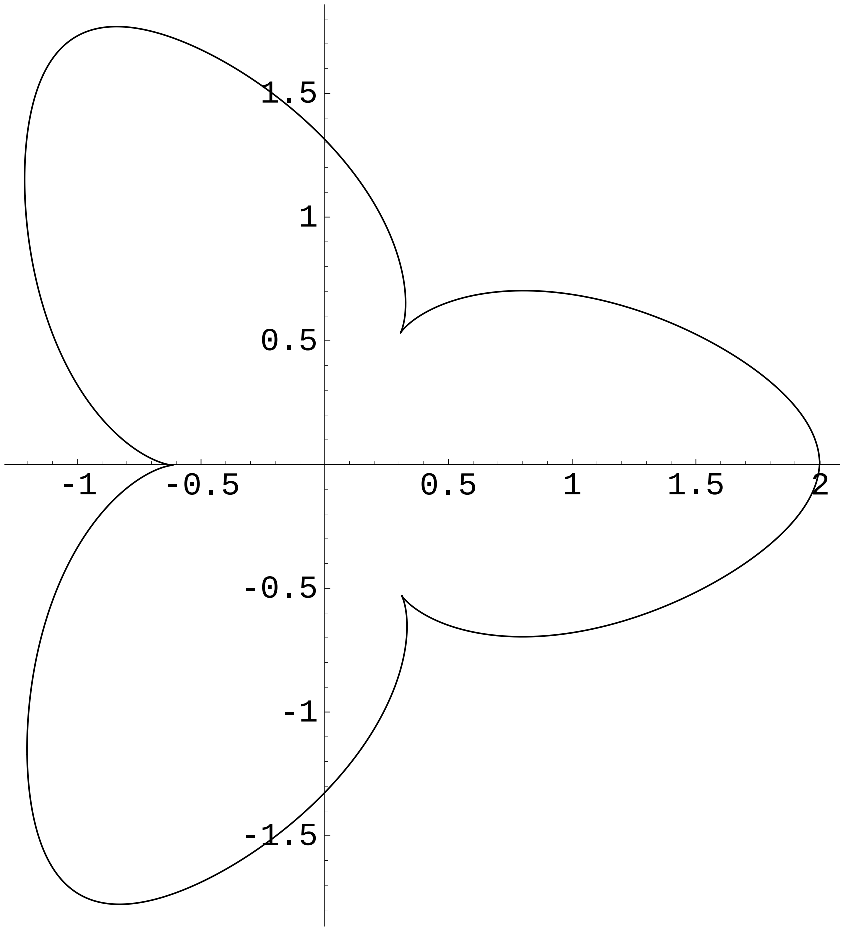
$$n_1 = \frac{u_l - u_c}{u_l + 1}\,, \qquad n_2 = \frac{u_l - u_c}{u_l - 1}\,, \qquad p = \sqrt{\frac{u_l - u_c}{2u_l}}$$

and $K(p)$, $E(p)$ and $\Pi(n, p)$ are the complete elliptic integrals of first, second and third kind respectively.

The usual *Spiky Strings* look like



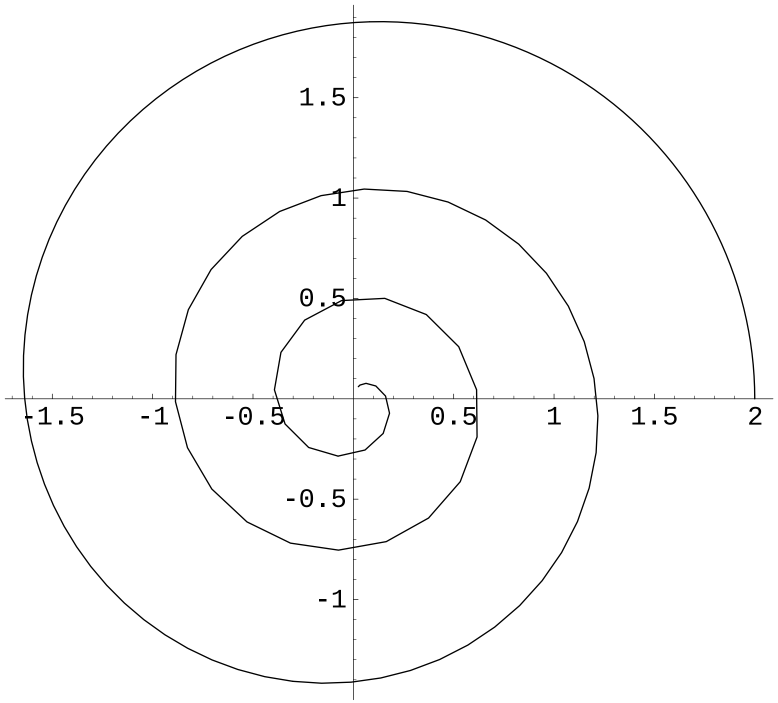
Whereas the *Dual Spiky Strings* look like



▶ Some interesting limits

- $\rho_c \ll 1$ ,    $\rho_l = fixed$

In this limit $\Delta\theta \gg 1$ and the periodicity condition can not be satisfied. The resulting configuration is a spiral



- $\rho_c < \rho_l \ll 1$

This limit corresponds to a small angular momentum spiky string close to the origin and reproduces the dual spikes found in the flat space and all the relations found there apply here.

- $\rho_l \gg 1 \ , \quad \rho_c = fixed$

This limit corresponds to spiky strings with large angular momentum and a fixed number of spikes. In this limit we have

$$\Delta\theta \approx \frac{\pi}{2} \, \{\frac{\sqrt{u_c + 1}}{\sqrt{u_c - 1}} - 1\}$$

$$J \approx n \, \frac{\sqrt{\lambda}}{4} \, u_l$$

$$E - \omega J \approx \frac{2n}{2\pi} \, \sqrt{\lambda} \, \frac{2E(1/\sqrt{2}) - K(1/\sqrt{2})}{\sqrt{u_c - 1}} \, {u_l}^{1/2}$$

We further assume that $\rho_c \gg 1$ (or $\omega \approx 1$), to avoid spirals, but still $\rho_c \ll \rho_l$ such that the number of spikes doesn't blow up and $(E-J)/n$ remains large. In this limit a semi classical analysis for each spike remains valid. We will have

$$\Delta\theta \approx \pi \ e^{-2\rho_c}$$

$$J \approx n \ \frac{\sqrt{\lambda}}{8} \ e^{2\rho_l}$$

$$E - J \approx n \ \frac{\sqrt{\lambda}}{\pi} \ [2E(1/\sqrt{2}) - K(1/\sqrt{2})] \ e^{(\rho_l - \rho_c)}$$

For *Dual Spiky Strings* and in the large angular momentum limit we finally find

$$E \approx J + 1.34\ \sqrt{\lambda}\ \frac{n}{2\pi}\ \left(\frac{4\pi}{n}\frac{J}{\sqrt{\lambda}}\right)^{1/2}$$

The anomalous part is similar to what is found for *circular pulsating* strings.

For *Spiky Strings* and in the large angular momentum it was found that [Kruczenski; 2004]

$$E \approx J + \sqrt{\lambda}\ \frac{n}{2\pi}\ \ln\left(\frac{4\pi}{n}\frac{J}{\sqrt{\lambda}}\right)$$

The anomalous part has the usual behavior for *folded spinning* strings.

## ■ Discussion ■

What is the corresponding operator to the "*Dual Spiky String*" in SYM ?

Semi classical string configurations in AdS have led to the picture that spikes on string represent fields in SYM.

A profile in the radius of AdS, which should generically be accompanied by rotation to give a string solution, are represented by covariant derivatives $D_+ = D_1 + iD_2$ which also carry spin.

The logarithmic behavior of anomalous dimension for spinning strings is believed to be caused by the large number of derivatives as compared to fields.

• The large *J Spiky Strings*, and near the boundary of AdS, can be considered as a number of folded spinning strings. These configurations are therefore conjectured to correspond to *twist n operators* with the following schematic form [Kruczenski; 2004]

$$\mathcal{O}_{\mathcal{S}} \sim \mathbf{Tr}\{\Pi_i^n (D_+)^{J/n}\ \Phi_i\}$$

This can explain the logarithmic anomalous part found in the semi classical analysis.

• The large *J Dual Spiky Strings*, and near the boundary, look like portions of circular rotating strings. Moreover, the profile in $\rho$ in addition to rotation, induce a pulsating motion for the string.

In fact if we define $\eta \equiv \omega - 1$ and keep $\eta$ small, we will still have a finite number of spikes and a valid semi classical limit for each spike if we keep $\eta J$ fixed and large. In this limit we have

$$E \approx J + \eta J + 1.34 \sqrt{\frac{2}{\pi}}\, n\, \lambda^{1/4} \sqrt{\eta J}$$

Comparing this relation with that for a pulsating string we see that here, $\eta J$ is replacing the oscillator number for pulsation.

For smaller $\eta$, we will have a larger $\rho_c$ and hence the portion of string near the boundary becomes more circular. This will reduce the pulsation-like movement of the string and gives a smaller oscillation number.

One might then guess that the *Dual Spiky Strings* in AdS in the large angular momentum limit are schematically represented by operators of the form

$$\mathcal{O}_{\mathcal{DS}} \sim \mathbf{Tr}\{\Pi_i^n (D_+)^{J/n} (D_+ D_-)^{\eta J/2n}\ \Phi_i\}$$

The $D_+$ operators are responsible for the profile in $\rho$ as well as the rotation. The $D_+ D_-$, on the other hand, contribute to the dimension but not to the angular momentum. The fields $\Phi$ as before represent the spikes on the string.

$$E \approx J + \eta J + 1.34\ \sqrt{\frac{2}{\pi}}\ n\ \lambda^{1/4}\ \sqrt{\eta J}$$

- *Dual Spikes* on sphere

We also found *Dual Spiky Strings* on sphere and found that no large angular momentum limit for such solutions exists. This means that a semi classical analysis for such solutions can not be trusted. However, we found Spiral configurations in certain limits (similar to the spirals on AdS) which have infinite winding number around the equator of sphere. These were used to build Single Spike configurations in [Riei, Kruczenski; 2007, Bobev, Rashkov; 2007]. These might be considered as "*Dual*" configurations to *Giant Magnons* [Hofman, Maldacena; 2006] where the winding number is replacing angular momentum as a large charge.

## ■ Elliptic Integrals ■

The Elliptical integrals of first, second and third kind, $F$, $E$ and $\Pi$ are defined as

$$F(\alpha; q) = \int_0^\alpha d\theta \frac{1}{(1 - q \sin^2 \theta)^{\frac{1}{2}}}$$

$$E(\alpha; q) = \int_0^\alpha d\theta (1 - q \sin^2 \theta)^{\frac{1}{2}}$$

$$\Pi(\alpha; n, q) = \int_0^\alpha d\theta \frac{1}{(1 - n \sin^2 \theta)(1 - q^2 \sin^2 \theta)^{\frac{1}{2}}}$$