Field Equations at Horizon

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- ## Abstract

There is a very simple approach to derive the entropy of horizons, developed by T. Padmanabhan: It is possible to interpret gravitational field equations near any spherically symmetric horizon as a thermodynamic identity $T\,dS - dE = P\,dV$.

We study this approach further in case of BTZ black hole in three dimensions and show that it's entropy correction due to gravitational Chern-Simons term, derived in this manner, is in full agreement with standard results.

We also observe that some class of topological black holes could be studied in this manner very simply!

• Introduction

Padmanabhan's observation in 4-dimension (gr-qc/0204019) would be a good beginning:

> " ... it is possible to write Einstein's equations for a spherically symmetric spacetime in the form $T\,dS - dE = P\,dV$ near any horizon of radius a with $S = (4\pi a^2)/4$, $E = a/2$ and temperature T determined from the surface gravity at the horizon. The pressure P is provided by the source of the Einstein's equation and dV is the change in the volume when the horizon is displaced infinitesimally. "

To be more precise let us take a look at a D-dimensional generalization of this observation.

- **Einstein-Hilbert Theory**

Consider Einstein-Hilbert gravity with arbitrary matter sector in a D-dimensional space-time ($\hbar = c = G_D = 1$):

$$I = \frac{1}{16\,\pi} \int d^D x \sqrt{-g}\, R + I_{matter}$$

R is the Ricci scaler, and cosmological constant could be included in the matter part. The Equation of motion is:

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. Now consider a class of static and spherically symmetric metrics:

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2_{D-2}$$

where $d\Omega^2_{D-2}$ is the metric of $(D-2)$-dimensional sphere.

With this choice, the rr-component of the equation of motion ($G^r{}_r = 8\pi\, T^r{}_r$) becomes:

$$\frac{(D-2)}{2\,r} f'(r) - \frac{(D-2)(D-3)}{2\,r^2}\left(1 - f(r)\right) \;=\; 8\pi\, T^r{}_r(r)$$

Suppose that the function $f(r)$ has a simple zero ar $r = r_H$ and $f'(r_H)$ is finite, so that space-time has a spherical horizon with radius r_H, and non-vanishing temperature $T = f'(r_H)/4\pi$. Let us evaluate the above equation at $r = r_H$

$$\frac{(D-2)}{2\,r_H} f'(r_H) - \frac{(D-2)(D-3)}{2\,r_H^2} \;=\; 8\pi\, T^r{}_r(r_H)$$

and then multiply both sides by a suitable factor $d\,V/8\pi$ where

$$\begin{aligned} dV &= \textit{area of horizon} \times \textit{virtual radial displacement of the horizon} \\ &= A_{D-2} r_H^{(D-2)} \times dr_H \end{aligned}$$

and A_{D-2} is the area of a unit $(D-2)$-sphere. The result is:

$$\frac{f'(r_H)}{4\,\pi} d\left(\frac{A_{D-2} r_H^{(D-2)}}{4}\right) - d\left(\frac{(D-2) A_{D-2} r_H^{(D-3)}}{16\,\pi}\right) = T^r{}_r(r_H) dV$$

By identifying the radial pressure $P = T^r{}_r(r_H)$ we can rewrite this equation in the form of the first law of thermodynamics!

$$T\,dS - dE = P\,dV$$

where the entropy is one quarter of the horizon's area:

$$S = \frac{A_{D-2} r_H^{(D-2)}}{4}$$

and the energy of horizon is given by

$$E = \frac{r_H^{(D-3)}}{\omega_D} \; ; \quad \omega_D \equiv \frac{16\,\pi}{(D-2) A_{D-2}}$$

The meaning of this local energy is not clear, except for some simple cases.

As an explicit example consider the vacuum solution in a $D \geq 4$ space-time (the Schwarzschild solution) for which:

$$T^{\mu}{}_{\nu} = 0 \qquad \Rightarrow \qquad f(r) = 1 - \frac{\omega_D \, M}{r^{(D-3)}}$$

where M is the mass of the black hole and $r_H = (\omega_D \, M)^{\frac{1}{(D-3)}}$. For this solution $P = 0$ and the energy of horizon is $E = M$.

In general E is not the total mass. Adding cosmological constant or other types of matter will change the radial pressure and the radius of horizon, but the energy formula and the Bekenstein-Hawking formula for entropy remain unchanged!

The familiar form of the first law could also be derived. For example, for a charged black hole it is easy to show that

$$dE + P \, dV \;=\; dM - \Phi \, dQ$$

- **Higher Order Corrections**

Next development was made by A. Paranjape, S. Sarkar and T. Padmanabhan (hep-th/0607240):

They observed that the equivalence of the equation of motion and the thermodynamic identity $T\,dS = dE + P\,dV$ goes beyond Einstein gravity and is applicable even in the more general Lanczos-Lovelock gravity as well!

For example let us look at the lowest order correction in Lanczos-Lovelock theory.

- **Gauss-Bonnet Correction**

The Einstein-Hilbert theory with Gauss-Bonnet correction in D-dimension is given by:

$$I \;=\; \frac{1}{16\,\pi}\int d^Dx\sqrt{-g}\left[R+\alpha\left(R^2-4R_{ab}R^{ab}+R_{abcd}R^{abcd}\right)\right]+I_{matter}$$

So the Einstein equation will be modified as:

$$G_{\mu\nu}+\alpha H_{\mu\nu} \;=\; 8\pi\,T_{\mu\nu}$$

For a spherically symmetric metric, the rr-component of the equation of motion, evaluated at a typical horizon $r=r_H$ is given by

$$\frac{(D-2)}{2\,r_H}f'(r_H)\left[1+\frac{2\hat{\alpha}}{r_H^2}\right]-\frac{(D-2)(D-3)}{2\,r_H^2}\left[1+\frac{\hat{\alpha}}{r_H^2}\left(\frac{D-5}{D-3}\right)\right]=8\pi\,T^r{}_r(r_H)$$

where $\hat{\alpha}\equiv\alpha\,(D-3)(D-4)$.

As before multiplying both sides by a suitable factor $dV/8\pi$, it becomes the first law of black hole thermodynamics, from which entropy and energy of horizon could be read:

$$S = \frac{A_{D-2}r_H^{(D-2)}}{4}\left[1 + \frac{2\hat{\alpha}}{r_H^2}\left(\frac{D-2}{D-4}\right)\right]$$

$$E = \frac{r_H^{(D-3)}}{\omega_D}\left[1 + \frac{\hat{\alpha}}{r_H^2}\right]$$

These expressions were previously derived in literature by other approaches (for zero cosmological constant). It should be emphasized again that these corrections are independent of matter content of the theory or cosmological constant background!

- **Advantages**

In General Relativity, mass is a global concept which is only defined for space-times with well defined asymptotic behavior, *i.e.*, asymptotically flat and asymptotically AdS spaces.

standard methods for entropy calculation (*e.g.*, Wald formula) are based on mass definitions, so they are not applicable for space-times which are not asymptotically flat or AdS, or space-times with multiple horizons such as Schwarzschild-dS.

Padmanabhan's method is a local approach which could be applied to each horizon in a multiple horizon space-time, without worrying about asymptotic behavior of the space-time!

- **Other Generalizations**

This approach is generalized to:

stationary axially-symmetric horizons and time dependent evolving horizons (T. Padmanaban et al. gr-qc/0701002)

apparent horizon of FRW universe (Rong-Gen Cai et al. gr-qc/0611071, hep-th/0609128)

the $f(R)$ theory of gravity (Rong-Gen Cai et al. gr-qc/0612089) in which horizon thermodynamics is a non-equilibrium one.

the BTZ black hole in 3-dimensions (M. Akbar hep-th/0702029)

- ## The BTZ Black Hole

Consider the Einstein-Hilbert theory with non-zero cosmological constant Λ in a $(1+2)$-dimensional space-time:

$$I = \frac{1}{16\,\pi}\int d^3x\sqrt{-g}\,\Big[R - 2\Lambda\Big]$$

The equation of motion is given by:

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu} = -\Lambda g_{\mu\nu}$$

Now consider a solution of the form:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\Big(d\phi + N(r)dt\Big)^2$$

With a negative cosmological constant, the only black hole solution is:

$$\Lambda = -\frac{1}{l^2} \qquad \Rightarrow \qquad \begin{aligned} f(r) &= -8M + \frac{r^2}{l^2} + \frac{16\,J^2}{r^2} \\ N(r) &= -\frac{4\,J}{r^2} \end{aligned}$$

where M and J are mass and angular momentum. This constant curvature (no singularity!) geometry is called BTZ black hole. This black hole is asymptotically AdS, and has two horizon at $r_\pm$ so that:

$$M = \frac{r_+^2 + r_-^2}{8\,l^2} \quad ; \qquad J = \frac{r_+\,r_-}{4\,l}$$

Entropy of the BTZ solution, associated to it's outer horizon is given by Bekenstein-Hawking formula: $S_+ = (2\pi r_+)/4$.

The BTZ solution is also a solution of the field equations of TMG which is constructed by adding a gravitational Chern-Simons term to the Einstein-Hilbert action.

$$I_{GCS} = -\frac{\beta_S}{64\,\pi}\int d^3x\,\epsilon^{\mu\nu\lambda}\left(R_{ab\mu\nu}\,\omega^{ab}{}_{\lambda} + \frac{2}{3}\omega^{b}{}_{c\mu}\omega^{c}{}_{a\nu}\omega^{a}{}_{b\lambda}\right)$$

Equation of motion of TMG is given by:

$$G_{\mu\nu} + \beta_S C_{\mu\nu} = 8\pi\,T_{\mu\nu}$$

where the Cotton tensor is traceless and covariantly conserved:

$$C^{\mu}{}_{\nu} = \frac{1}{\sqrt{-g}}\epsilon^{\mu\rho\sigma}\nabla_{\rho}\Big(R_{\nu\sigma} - \frac{1}{4}g_{\nu\sigma}R\Big)$$

What is the entropy correction of BTZ black hole due to GCS term? The answer is well known, so we could test Padmanabhan's approach once again.

The rr-component of the equation of motion, $G^{\ r}_{\ r} + \beta_S C^{\ r}_{\ r} = 8\pi\, T^{\ r}_{\ r}$ is

$$\frac{f'(r)}{4\, r}\left[2 + \beta_S\, r\left(4N' + r\, N''\right)\right] + \cdots \ = \ 8\pi\, T^{\ r}_{\ r}(r)$$

where dots denotes all other terms in the left hand side which have no contribution to entropy. Evaluating the above equation at a typical horizon $r = r_H$ of BTZ black hole yields:

$$\frac{f'(r_H)}{2\, r_H}\left[1 + \beta_S\, \frac{4\, J}{r_H^2}\right] + \cdots \ = \ 8\pi\, T^{\ r}_{\ r}(r_H)$$

where we are using $N = -4\ J/r^2$. Multiplying both sides by $2\pi\, r_H\, dr_H/8\pi$ we can easily find the entropy of the horizon corrected by GCS term:

$$S \ = \ \frac{2\pi\, r_H}{4}\left[1 - \beta_S\, \frac{4\, J}{r_H^2}\right]$$

for example entropy of outer horizon r_+ become:

$$S_+ = \frac{2\pi\, r_+}{4} + \beta\, \frac{2\pi\, r_-}{4}$$

and of inner horizon r_-:

$$S_- = \frac{2\pi\, r_-}{4} + \beta\, \frac{2\pi\, r_+}{4}$$

where $\beta = -\beta_S/l$. These are well known results.

It is interesting to note that the same procedure can also be followed in a positive cosmological constant background, so our entropy formula is also applicable in case of Kerr-dS3 solution.

- **Topological Black Holes**

Let us return to the Einstein-Hilbert gravity with arbitrary matter sector in a D-dimensional space-time ($\hbar = c = G_D = 1$):

$$I = \frac{1}{16\,\pi}\int d^D x\sqrt{-g}\, R + I_{matter}$$

Now consider a class of black holes with a non-spherical horizon with the following metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\, h_{ij}(y)dy^i dy^j$$

where the coordinates are labeled as $x^\mu = (t, r, y^i)$, $i = 1, \cdots, (D-2)$. The horizon metric h_{ij} is a function of the coordinates y^i only.

Again, the rr-component of the equation of motion ($G^{\,r}{}_{r} = 8\pi\, T^{\,r}{}_{r}$) becomes:

$$\frac{(D-2)}{2\,r} f'(r) - \frac{(D-2)(D-3)}{2\,r^2}\Big(k(h) - f(r)\Big) \;=\; 8\pi\, T^{\,r}{}_{r}(r) \;\;,$$

where

$$k(h) \equiv \frac{\mathcal{R}(h)}{(D-2)(D-3)} \;\;,$$

and $\mathcal{R}(h) = h^{ij}\mathcal{R}_{ij}(h)$ is the Ricci tensor for the horizon metric. k is a constant in the case of constant curvature surfaces for example.

Evaluating the above equation at $r = r_H$, and then multiplying both sides by a suitable factor $d\;V/8\pi$, we end up with the first law of thermodynamics.

Note that here

$$dV = A_{D-2}\, r_H^{(D-2)} \times dr_H$$

as before, but A_{D-2} is the area of a unit $(D-2)$-surface with the metric $h_{ij}(y)$.

Finally the entropy of this type of horizon, is also equal to one quarter of it's area:

$$S = \frac{A_{D-2} r_H^{(D-2)}}{4}$$

and it's associated energy is given by

$$E = \frac{k\, r_H^{(D-3)}}{\omega_D} \quad ; \quad \omega_D \equiv \frac{16\,\pi}{(D-2)A_{D-2}}$$

So for topological black holes, the horizon's curvature enters in the energy formula only, and has no direct contribution to the entropy.

As an explicit example consider some class of D-dimensional solutions for the Einstein's equation with a negative cosmological constant (D. Birmingham, hep-th/9808031):

$$T_{\mu\nu} = \frac{(D-1)(D-2)}{16\,\pi\,l^2} g_{\mu\nu} \qquad \Rightarrow \qquad f(r) = k - \frac{\omega_D\, M}{r^{(D-3)}} + \frac{r^2}{l^2}$$

here k is constant and M is the mass parameter:

$$M \;=\; \frac{r_H^{(D-3)}}{\omega_D}\left(k + \frac{r_H^2}{l^2}\right)$$

so our local energy E is just a part of the mass which is not associated with the cosmological constant. Anyway it is easy to check that

$$dE + P\,dV \;=\; dM$$

Many Thanks