Instantons and Solitons in Holographic QCD

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1 TOPICS

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- Effective action on the Moduli Space of Monopoles
- References

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H. Hata, T. Sakai, S. Sugimoto and S. Yamato, arXiv:hep-th/0701280
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Introduction:

- Effective action of D4-branes in Type IIA

is a 4+1 SYM $\begin{cases} A_\mu & \mu=0,1,..,4 \\ \Phi_I & I=5,..,9 \\ \Psi & \end{cases}$

upon compactification over S^1 with anti-periodic b.c. on fermions, they become massive from the point of view of 4d theory. Zero modes of A_4 & Φ_I get massive at one-loop as susy is broken.

- End up with pure SU(N) YM theory in 4d.

- **Adding quarks & low energy SUGRA descrip.**

To include quarks, one adds up (Sakai-Sugimoto) N_f D8-$\overline{D8}$ branes to the above set up.

The low energy dynamics is then given by considering D8-$\overline{D8}$ branes as probes in the supergravity background of N_c D4-branes $(N_c \gg N_f)$.

- So the Effective $\mathcal{L}$ is DBI action

2 Euclidean 5d Action and 4d Instantons

Let us start with the effective action of D8-branes in the background of D4-branes:

$$S_{\rm YM} = -k \int d^4x\, dz\, \mathrm{tr}\left(\frac{1}{2}h(z)F_{\mu\nu}^2 + k(z)F_{\mu z}^2\right)$$

where

$$k = \frac{\lambda N_c}{216\pi^3}$$

and

$$h(z) = (1+z^2)^{-1/3}, \qquad k(z) = 1+z^2$$

where $\mu, \nu = 1, \ldots, 4$, and z is the 5th dimension. The action can be rewritten using auxiliary metric components

$$\boxed{S_{\rm YM} = -k \int \sqrt{g_5}\, d^4x\, dz\, \mathrm{tr}\left(g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + 2g^{\mu\nu}g^{zz}F_{\mu z}F_{\nu z}\right)}$$

where the metric components are

$$g_{\mu\nu} = \frac{1}{4}k(z)h(z)\delta_{\mu\nu}, \quad g_{zz} = \frac{1}{4}h(z)^2\delta_{zz}$$

The 5d Euclidean metric is

$$\boxed{ds^2 = \frac{1}{4}(1+z^2)^{2/3}\delta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{4}(1+z^2)^{-2/3}dz^2}$$

A set of classical solutions of the Euclidean 5d theory: set $F_{\mu z} = 0$, and require that all other fields to be independent of z then we have a 4d action with instantons as its classical minima. To get rid of that factor $\sqrt{g_{zz}}$, we do a coordinate transformation, $z \longrightarrow z'$, and write the metric as

$$\boxed{ds^2 = f(z')\,(\delta_{\mu\nu} dx^\mu dx^\nu) + dz'^2}$$

where

$$z'(z) = \frac{1}{2}\int \frac{dz}{(1+z^2)^{1/3}} = \frac{1}{2}\, z\, F\left(\frac{1}{2}, \frac{1}{3}; \frac{3}{2}; -z^2\right)$$

with F the hypergeometric function, and

$$f(z') = \frac{1}{4}(1 + z(z')^2)^{2/3}$$

Therefore, in this coordinate system $\sqrt{g_{z'z'}} = 1$, and the action reads

$$S_{\rm YM} = -k \int \sqrt{g_4}\, d^4x\, dz\, \mathrm{tr}\,(g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma})$$

The absolute minima of the 4d action can be worked out by adding and subtracting a topological term proportional to the instanton number:

$$\boxed{\begin{aligned} S &= -k \int \sqrt{g_4}\, d^4x\, \mathrm{tr}\,(g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}) \\ &= -\frac{k}{2} \int \sqrt{g_4}\, d^4x\, \mathrm{tr}\left(F_{\mu\nu} - \frac{1}{2\sqrt{g_4}} g_{\mu\rho} g_{\nu\sigma} \epsilon^{\rho\sigma\lambda\delta} F_{\lambda\delta}\right)^2 \\ &\quad - \frac{k}{2} \int d^4x\, \mathrm{tr}\left(\epsilon^{\rho\sigma\lambda\delta} F_{\rho\sigma} F_{\lambda\delta}\right) \end{aligned}}$$

Written in this form, it is now clear that the absolute minima of the 4d action are the instantons on a curved 4-dimensional space with the metric $g_{\mu\nu}$:

$$F_{\mu\nu} = \frac{1}{2\sqrt{g_4}} g_{\mu\rho} g_{\nu\sigma}\, \epsilon^{\rho\sigma\lambda\delta} F_{\lambda\delta}$$

with the convention $\epsilon^{123z} = 1$ and $\epsilon_{123z} \sim g_4$. However, since we are in four dimensions and since the metric in the remaining 4 coordinates is conformally flat, the above equations reduce to the instanton equations on flat space

$$F_{\mu\nu} = \frac{1}{2} \delta_{\mu\rho} \delta_{\nu\sigma}\, \epsilon^{\rho\sigma\lambda\delta} F_{\lambda\delta}$$

with completely known solutions.

3 String-like Solutions and Monopoles

We discuss a class of solutions of the 5d theory which are independent of t and the z direction, and hence resemble vortex solutions.

The 5d action now includes adjoint scalars ϕ. The $U(1)$ part of ϕ gets stabilized at antipodal points of S^1. So the 5d action is

$$\begin{aligned} S_{\mathrm{YM}} &= -k \int \sqrt{g_5}\, d^5x \, \mathrm{tr} \left(g^{ML} g^{NK} F_{MN} F_{LK} \right. \\ &+ \left. 2 g^{MN} D_M \phi D_N \phi \right) \end{aligned}$$

In looking for solitons of the 5d model, the simplest choice is to look for a static field configuration and set $\partial_0 = A_0 = 0$, so the action reads

$$\begin{aligned} S_{\mathrm{YM}} &= -k \int \sqrt{-g_{00}}\, dt \sqrt{g_4}\, d^3x dz \, \mathrm{tr} \left(g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \right. \\ &+ \left. 2 g^{\alpha\beta} D_\alpha \phi D_\beta \phi \right) \end{aligned}$$

with $\alpha, \beta, \ldots = 1, 2, 3, z$, and the metric

$$ds^2 = \frac{1}{4}(1+z^2)^{2/3} \delta_{ij} dx^i dx^j + \frac{1}{4}(1+z^2)^{-2/3} dz^2$$

If $\sqrt{-g_{00}}$ was absent, we could have argued that the instantons (with $\phi = 0$) are sitting at the minima of the action. However, it is not possible to get rid of $\sqrt{-g_{00}}$ by a coordinate transformation, and thus instantons are not solutions to the field equations. By varying the action, the field equations read

$$\hat{D}_\alpha\left(\sqrt{-g_{00}}\,F^{\alpha\beta}\right)+i\sqrt{-g_{00}}\left[\phi\,,\hat{D}^\beta\phi\right]=0\,,$$

$$\hat{D}_\alpha\left(\sqrt{-g_{00}}\,\hat{D}^\alpha\phi\right)=0$$

with $\hat{D}_\alpha$ the 4d connection. Now for instantons we have

$$F_{\alpha\beta}=\frac{1}{2\sqrt{g_4}}g_{\alpha\gamma}g_{\beta\eta}\epsilon^{\gamma\eta\delta\kappa}F_{\delta\kappa}$$

together with $\phi=0$. These clearly do not satisfy field eqs; with the $\sqrt{-g_{00}}$ factor inside the covariant derivative the field equations do not reduce to the Bianchi identities.

Although $\sqrt{-g_{00}}$ cannot be set to 1 by a coordinate transformations, we can get rid of that by dimensionally reducing the action one step further. In fact, since

$$\sqrt{-g_{00}}\cdot\sqrt{g_{zz}}=\frac{1}{4}$$

we observe that if we reduce action to 3 dimensions there is a chance of reducing the field equations to some first order differential equations. Let us first discuss that without the scalars ϕ it is not possible to get to the monopole equations.

3.2 The case with $\phi \neq 0$

The 5d equations of motion read

$$\tilde{D}_M F^{MN} + i[\phi\,, D^N \phi] = 0$$
$$D_M D^M \phi = 0$$

requiring $\partial_0 = A_0 = 0$, the above equations to 4 dimensions

$$\hat{D}_\alpha F^{\alpha\beta} + \Gamma^0_{z0} F^{z\beta} + i[\phi\,, D^\beta \phi] = 0$$
$$D_\alpha D^\alpha \phi = 0$$

Splitting the indices to z and i

$$D_i F^{iz} + i[\phi\,, D^z \phi] = 0$$
$$D_j F^{ji} + i[A_z, F^{zi}] + \partial_z F^{zi} + \Gamma^j_{jz} F^{zi} + i[\phi\,, D^i \phi] = 0$$

We reduce these equations to 3 dimensions. We notice that there is a consistent ansatz of the form $\partial_z A_i = 0$ and $A_z = 0$, which also implies $F_{zi} = 0$. In this case

$$[\phi\,, \partial^z \phi] = 0$$
$$D_j F^{ji} + i[\phi\,, D^i \phi] = 0$$

In 3 dimensions, the last equation is solved by solutions to the Bogomolny equations:

$$F_{ij} = \frac{1}{\sqrt{g_3}} g_{im} g_{jn} \epsilon^{mnk} D_k \phi$$

plugging back the metric components this equation becomes

$$F_{ij} = \frac{1}{2}(1+z^2)^{1/3}\delta_{im}\delta_{jn}\epsilon^{mnk}D_k\phi$$

Since $\partial_z A_i = 0$, the left hand side is z-independent. Define

$$\tilde{\phi} = \frac{1}{2}(1+z^2)^{1/3}\phi$$

which is to be z-independent. Written in terms of $\tilde{\phi}$, the monopole equation becomes:

$$F_{ij} = \delta_{im}\delta_{jn}\epsilon^{mnk}D_k\tilde{\phi}$$

These monopole configurations can be seen that minimize the energy density. The reduced 3d action can be read:

$$S_{\rm YM} = -k\int\sqrt{g_3}\,dt\,d^3x\,dz\,\mathrm{tr}\left(g^{im}g^{jn}F_{ij}F_{mn} + 2g^{ij}D_i\phi D_j\phi\right)$$

Hence, for the energy density we have

$$\begin{aligned} E =& \frac{k}{4}\int\sqrt{g_3}\,d^3x\,\mathrm{tr}\left(g^{im}g^{jn}F_{ij}F_{mn} + 2g^{ij}D_i\phi D_j\phi\right) \\ =& \frac{k}{4}\int\sqrt{g_3}\,d^3x\,\mathrm{tr}\left(F_{ij} - \frac{1}{\sqrt{g_3}}g_{im}g_{jn}\epsilon^{mnk}D_k\phi\right)^2 \\ &+ \frac{k}{2}\int d^3x\,\mathrm{tr}\left(\epsilon^{ijk}F_{ij}D_k\phi\right) \end{aligned}$$

The last term is proportional to the winding number. The energy density functional, in each topological sector, is minimized if the fields satisfy the

Bogomolny Eqs. The energy density (energy per unit invariant length in the z direction) is

$$\frac{E}{\sqrt{g_{zz}}} = 2k \int d^3x \, \mathrm{tr}\Big(\epsilon^{ijk} F_{ij} D_k \tilde{\phi}\Big)$$

which is finite and proportional to the winding number associated to the behaviour of $\tilde{\phi}$ on the boundary of R^3. These monopole solutions, viewed from the four dimensions, look like strings extended along the z direction.

Slow motion of monopole Strings

Effective action of collective modes

In the moduli space approx. of monopoles motion, we let the collective coordinates be slowly varying functions of t & z

$$x^a \longrightarrow x^a(t, z)$$

- Such that δA_i & $\delta \Phi$ are small and satisfy the linearized monopole EQS.

- Neglecting the higher order derivative terms the field EQS. are reduced to:

$$D_i F^{i0} + [\Phi, D^0 \Phi] = 0$$

$$D_i F^{iz} + [\Phi, D^z \Phi] = 0$$

The variation of fields

$$\delta_a \Phi = \partial_a \Phi + [\Phi, \Omega_a]$$

$$\delta_a A_i = \partial_a A_i + D_i \Omega_a$$

Ω_a are gauge parameters introduced to ensure that δA & $\delta\Phi$ satisfy the same gauge fixing as that of monopole fields, i.e.,

$$D_i \delta A^i + [\Phi, \delta\Phi] = 0$$

Further note that in the m.s.a δA^i & $\delta\Phi$ are tangent to the moduli space. So they satisfy the linearized monopole Eqs.

$$F_{0i} = \partial_0 A_i - \partial_i A_0 + [A_0, A_i] = \partial_a A_i \dot{x}^a - D_i A_0$$

$$= \delta_a A_i \dot{x}^a - \dot{x}^a D_i \Omega_a - D_i A_0$$

set $\underline{A_0 = -\dot{x}^a \Omega_a}$ $\Rightarrow$ $\boxed{F_{0i} = \delta_a A_i \dot{x}^a}$

$$F_{zi} = \delta_a A_i x'^a - x'^a D_i \Omega_a - D_i A_z$$

set $\underline{A_z = -x'^a \Omega_a}$ $\Rightarrow$ $\boxed{F_{zi} = \delta_a A_i x'^a}$

Similarly we can check
$$\boxed{\begin{aligned} D_0 \Phi &= \delta_a \Phi \dot{x}^a \\ D_z \Phi &= \delta_a \Phi x'^a \end{aligned}}$$

Substituting into the action we get:

$$S = \int dt\,dz \sqrt{g_{00} g_{zz}}\, G_{ab} \left(g^{00} \dot{X}^a \dot{X}^b + g^{zz} X'^a X'^b \right)$$

where

$$G_{ab} = \int \sqrt{g_3}\, d^3x \; \mathrm{tr}\left(g^{ij} \delta_a A_i \delta_b A_j + \delta_a \Phi \delta_b \Phi \right)$$

is the metric on the moduli space.

For 1-monopole solutions with $SU(2)$ gauge group G_{ab} is

$$ds^2 = M_{mono} \left[(1+z^2)^{1/3} \delta_{ij}\, dX^i dX^j + \frac{1}{(1+z^2)^{1/3} v^2} d\chi^2 \right]$$

- Solitonic Solutions to the Effective action

Set $\dot{x}^a = 0$, $\chi = 0$. The Eq. of motion

$$\frac{d}{dz}\left[(1+z^2)\frac{dx^i}{dz}\right] = 0$$

which have solutions

$$\frac{dx^i}{dz} = \frac{c^i}{(1+z^2)}$$

with constant c^i.

The Energy functional

$$E = c^2 \int_{-\infty}^{+\infty} \frac{dz}{(1+z^2)} = \pi c^2$$