PHASE STRUCTURE OF $\mathcal{N}=4$ SUPER YANG-MILLS AND TOPOLOGY CHANGE

Umut Gürsoy
École Polytechnique, ENS, Paris

In collaboration with,

Sean Hartnoll
Tim Hollowood
Prem Kumar

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Vacuum structure of $\mathcal{N}=4$ SYM at finite T

$\longleftrightarrow$ new dual geometries,
topology changes
properties of black-holes (?)

- Compute small λ effective action for the VEVs of scalars Φ_a, A_0
- Investigate saddle point solutions at large N
- Determine the phase structure

OUTLINE

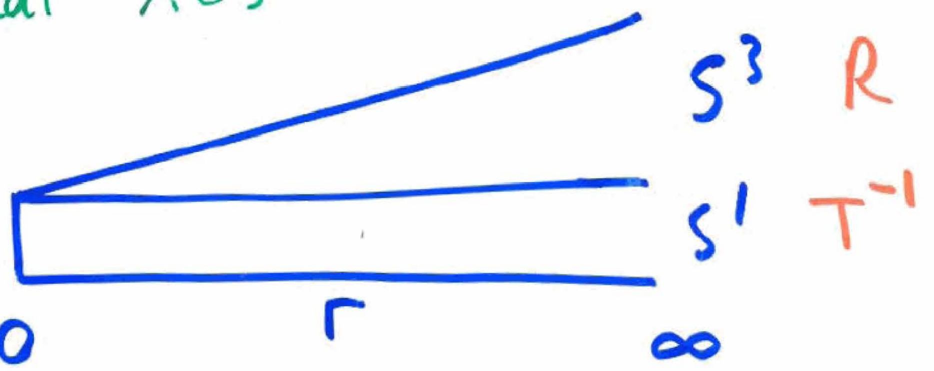
- Review of finite T AdS/CFT
- Polyakov loop as the order parameter for confinement - deconfinement transition
- $\mathcal{N}=4$ SYM : Polyakov loop + 6 scalars
  $V_{eff}(\theta, \vec{\varphi})$
- Implications for the dual geometries

king-Page vs. confinement/deconfinement phase transition

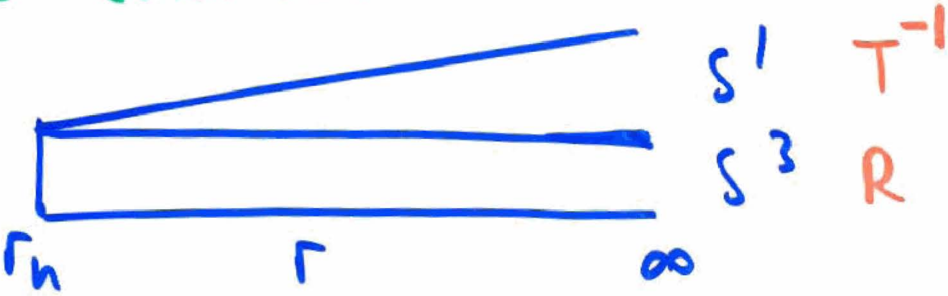
Hawking-Page '83

Two geometries with $S^1 \times S^3$ asymptotics on the boundary

① Thermal AdS



② AdS black hole



$$S_1 - S_2 < 0 \quad \text{for} \quad RT < \frac{3}{2\pi}$$

$$> 0 \quad \text{for} \quad RT > \frac{3}{2\pi}$$

- Weak coupling analogue (Witten '98)
(Sunborg '00, Aharony et al. '03)

- At small T, YM on $S^3 \times S^1$

The color charges confine kinematically

Number of states $\mathrm{tr}(M_1 M_2 \cdots M_s)|0\rangle$
independent of N
$\Rightarrow F \sim \mathcal{O}(1)$

- At high T, gluons in the deconfined phase $F \sim \mathcal{O}(N^2)$

The Polyakov loop as the order parameter

$$u = \frac{1}{N} \langle \operatorname{tr} P \, e^{\int_{S^1} A^0} \rangle$$

- $u = 0$ for $T < T_c$

  $\neq 0$ for $T > T_c$

- Center of the gauge group

  $\mathbb{Z}_N$ for $SU(N)$

  unbroken for $T < T_c$

  broken for $T > T_c$

$$T_c = -\frac{1}{\log(7 - 4\sqrt{3})} + \mathcal{O}(\lambda) \simeq 0.38$$

for $\mathcal{N}=4$ SYM

The phase structure of pure YM from the Polyakov loop

(Aharony et al. '03)

Define $\theta = \frac{1}{\mathrm{Vol}_{S^3}} \int_{S^1 \times S^3} A^0$

homogenous mode of A^0

$$U = \frac{1}{N} \sum_{i=1}^{N} e^{i\theta_i}$$

Determine $V_{eff}(\theta)$ by integrating out

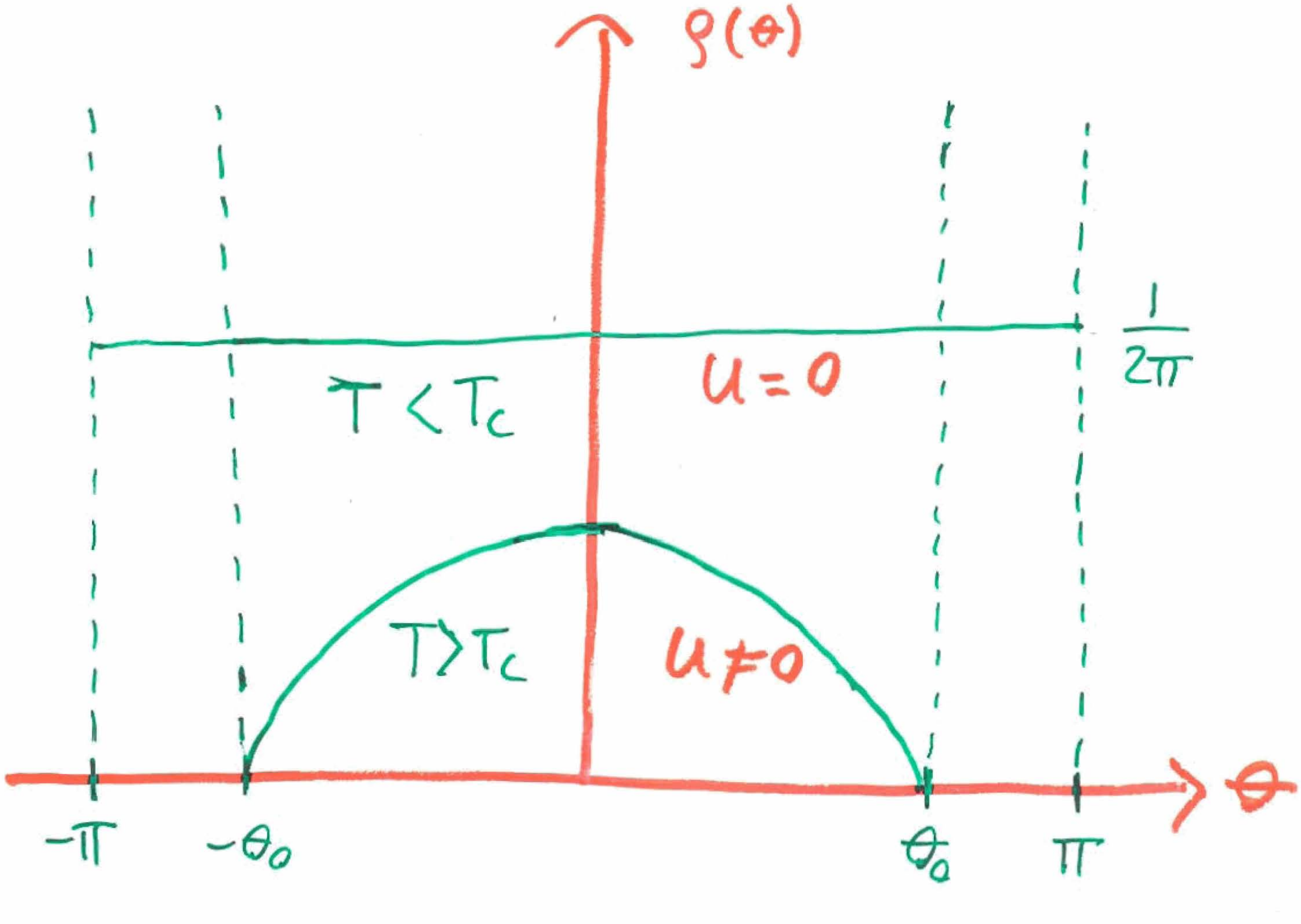
- off-diagonal components of θ_{ij}
- Kaluza-Klein modes on $S^1 \times S^3$

to quadratic order,

+ $O(\lambda)$ corrections

$$U = \frac{1}{N} \sum_{i=1}^{N} e^{i\theta_i} \xrightarrow[\text{large } N]{} \int_{-\pi}^{\pi} d\theta\, \rho(\theta)\, e^{i\theta}$$

THE POLYAKOV LOOP EIGENVALUE DENSITY



! $\rho(\theta)$ determines the phase structure of the theory at large N

$\rho(\theta)$ is the MASTER FIELD

$\mathcal{N}=4$ Super Yang-Mills:

VEVs of the scalar fields, additional order parameters

$\rightarrow$ new phase transitions !

Compute $V_{eff}(\theta, \vec{\varphi})$ on 7D phase space

$\varphi_a = \langle \Phi_a \rangle \neq 0$ for generic λ, $RT \neq 0$

Phase structure determined by

$$\underline{\underline{\rho(\theta, \vec{\varphi}) \text{ at large } N}}$$

- $\mathcal{N}=4$ SYM on S^3

$$S_{bosonic} = \int tr \left\{ F^2 + D\Phi_a D\Phi_a + [\Phi_a, \Phi_b]^2 + \underline{\underline{\frac{1}{R^2}\Phi_a^2}} \right\}$$

conformal coupling to S^3

- At $T=0$ no moduli space

$$\boxed{\Phi_a = 0}$$

- For $T \neq 0$ this is modified

Determine $V_{eff}(\theta, \langle\Phi_a\rangle)$ by integrating out

- off-diagonal θ_{ij}, $\langle\Phi_a^{ij}\rangle$
- KK modes off Ψ, A_μ, Φ on $S^1 \times S^3$

The effective action to quadratic order

(Hollowood et al. '06)

$V_{eff}(\varphi_a^i, \theta^i)$:

$$\frac{\beta R \pi^2 N}{\lambda} \sum_i |\vec{\varphi}_i|^2$$

tree level
central attractive potential

$$+ \sum_{ij} \log \left| \sinh \beta \frac{|\vec{\varphi}_i - \vec{\varphi}_j| + i(\theta_i - \theta_j)}{2} \right|$$

Pairwise repulsive potential

Vandermonde from off-diagonal $\varphi_{ij}, \theta_{ij}$

$$+ \sum_{\ell=1}^{\infty} 2\ell(\ell+2) \left\{ \underbrace{\frac{\beta}{2} \sqrt{\frac{(\ell+1)^2}{R^2} + |\vec{\varphi}_i - \vec{\varphi}_j|^2}}_{\text{Casimir energy}} - \sum_{n=1}^{\infty} e^{-n\beta \sqrt{\frac{(\ell+1)^2}{R^2} + |\vec{\varphi}_i - \vec{\varphi}_j|^2}} \cdot \cos n(\theta_i - \theta_j) \right.$$

Kaluza-Klein modes on $S^1 \times S^3$

+ scalar KK's, fermion KK's

- Valid for $RT \ll \frac{1}{\lambda}$

PHYSICS OF V_{eff}

- At $\lambda = 0 \Rightarrow$ tree level term dominates
  $\langle \Phi_a \rangle = 0$
  $\Rightarrow$ Aharony et al.'s results carry over

- <u>At $\lambda \neq 0$</u>:

i) Low temperatures, $RT \ll 1$

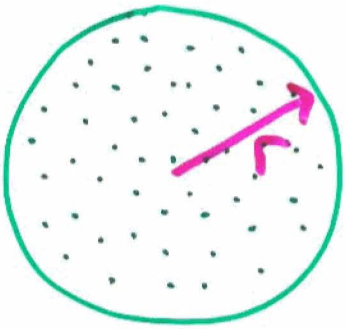
- θ-distribution and $\vec{\varphi}$-distribution Seperately determined
- pairwise repulsion for $\theta \Rightarrow \rho(\theta) = \text{const}$
- KK terms negligible

$$V_{eff} \to \frac{NRT}{\lambda}\pi^2 \sum_i |\vec{x}_i|^2 - \frac{\beta}{2}\sum_{ij} |\vec{x}_i - \vec{x}_j|$$

$\vec{x} = \beta \vec{u}$ dimensionless

Central attraction vs. pairwise repulsion

$\Rightarrow$ N eigenvalues distributed on S^5



$$\boxed{r = \frac{\lambda}{\pi^3 RT}\,\frac{1024}{945}}$$

Full S^5 is covered as $N \to \infty$

Full distribution at $N = \infty$: $\boxed{S^1 \times S^5}$

$$\underline{\underline{\rho(\theta, \vec{x}) \propto \delta(|\vec{x}| - r)}}$$

ii) Intermediate temperatures $1 \lesssim RT \ll \sqrt{\lambda}$

- One-loop repulsive term competes with tree-level attraction iff $\vec{x} = \mathcal{O}(\lambda)$
- Define $\vec{x} = \vec{\tilde{x}} \lambda$, expand V_{eff} in $\tilde{x}$

$$V_{eff} = \lambda^0 \underbrace{S^{(0)}(\theta)}_{\text{Aharony et al.'s action determines } \theta\text{-distrib.}} + \lambda \Big(\underbrace{S^{(1)}(\theta, \tilde{x})}_{\text{given } \theta\text{-distb. determines } \vec{x}\text{-distb.}} + S_{2\text{-loop}}(\theta) \Big) + \mathcal{O}(\lambda)$$

- From $S^{(1)}(\theta, \tilde{x})$ one obtains

$$\rho(\tilde{x}) \propto \delta(|\tilde{x}| - r(\theta))$$

with

$$\boxed{r(\theta) = \frac{c}{RT}\rho(\theta)}$$

$$\rho(\theta, x) \propto \rho(\theta)\, \delta\!\left(|\vec{x}| - \frac{c\lambda}{RT}\rho(\theta)\right)$$

S^5 fibered over $\underbrace{D_\theta}_{\text{support of } \theta\text{-distribution}}$

THIS IMPLIES TOPOLOGY CHANGE AT $T = T_c$

Below T_c S^5 fibered over S^1

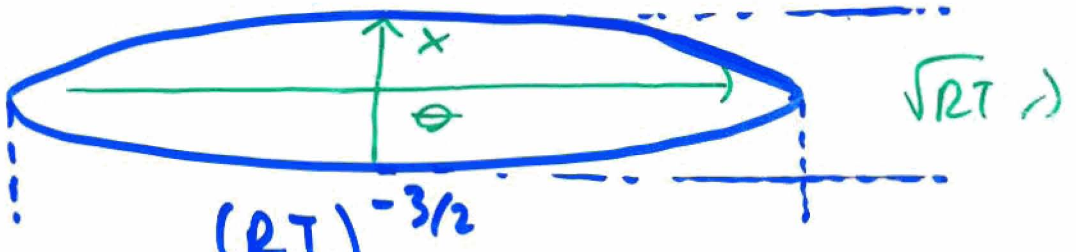
Above T_c S^5 fibered over segment $[-\theta_0, \theta_0]$

and $r_{S^5} \to 0$ as $\theta \to \mp \theta_0$

$\Rightarrow$ $\underline{S^1 \times S^5 \to S^6 \text{ at } T = T_c}$

• Precise form of the ellipsoid $\tilde{S}^6$

$$\frac{\pi^2 |\vec{x}|^2}{4 C^2 RT \lambda^2} + 2\pi^2 (RT)^3 \theta^2 = 1$$

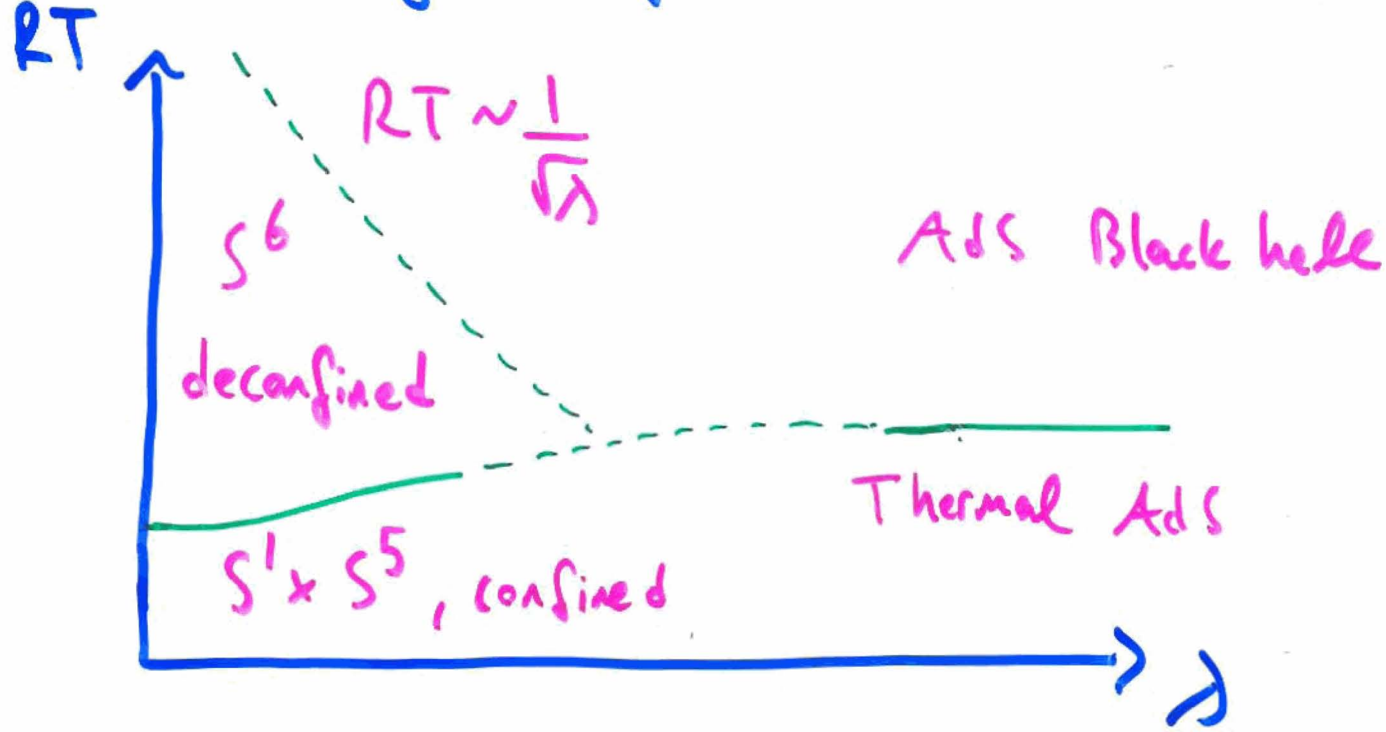


• As $RT \to \frac{1}{\sqrt{\lambda}}$, $\tilde{S}^6 \to$ full S^6 with $SO(7)$ sym.

• Applicability off $S^{(1)}$ fails at $RT \sim \frac{1}{\sqrt{\lambda}}$ as $|\tilde{x}| \sim |\theta|$

SUMMARY SO FAR

Phase diagram of $\mathcal{N}=4$ SYM on S^3



For $RT \sim \frac{1}{\sqrt{\lambda}}$

$\vec{x}$ - distribution backreacts on θ-distrb:

<u>Solve the full system</u>

iii) High temperatures, $\frac{1}{\sqrt{\lambda}} \simeq RT \ll \frac{1}{\lambda}$

At large N, the saddles are determined by the full coupled system:

$$P\vec{x} = \int d^6x'\, d\theta'\, \rho(\vec{x}', \theta')\, \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^2 + (\theta-\theta')^2}$$

$$Q\,\theta = \int d^6x'\, d\theta'\, \rho(\vec{x}', \theta')\, \frac{\theta-\theta'}{|\vec{x}-\vec{x}'|^2 + (\theta-\theta')^2}$$

$$P = \pi^2 RT\left(\frac{1}{\lambda} + 2(RT)^2\right), \quad Q = 4\pi^2 (RT)^3$$

Two solutions with $SO(6)$ symmetry

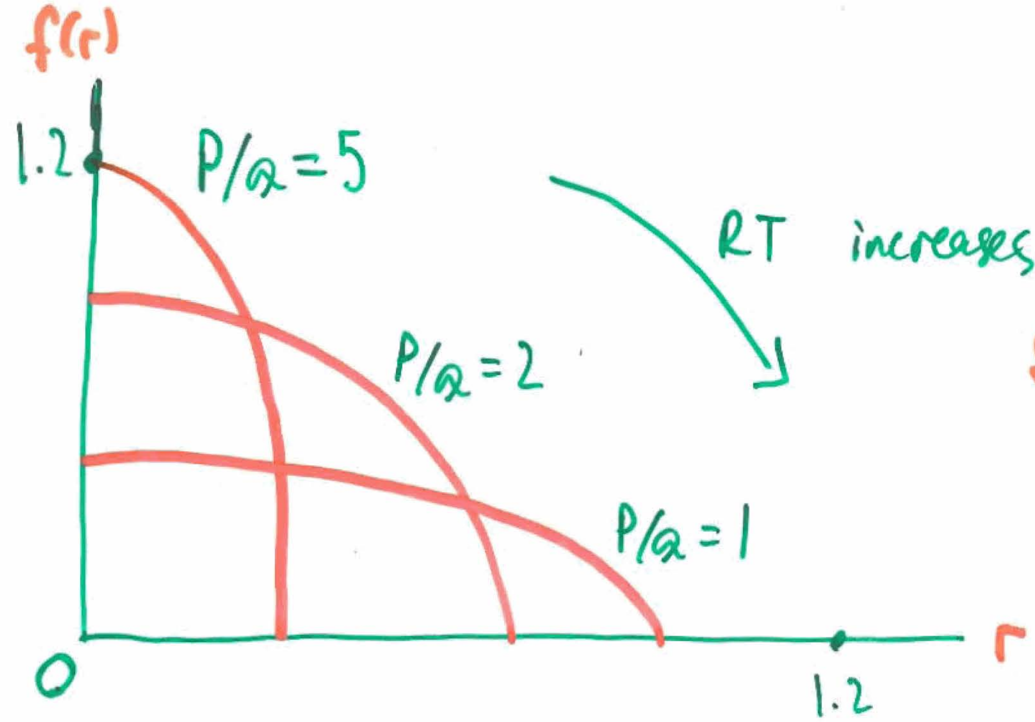
A) Round S^5: $\rho(\theta) \propto \delta(\theta)$

$\rho(\vec{x}) \propto \delta\left(|\vec{x}| - \frac{1}{\sqrt{2P}}\right)$

- Determined analytically
- Unstable at intermediate T

B) Ellipsoid $\tilde{S}^6$

- Extension of $\tilde{S}^6$ at intermediate temperatures into high T
- Determined numerically
  Minimize action in Monte-Carlo



$\delta(\theta - f(r))$

- Numerically: $\dfrac{a}{b} = -0.81 + 1.51\,\dfrac{P}{Q}$

an ellipsoid

- One can follow $\tilde{S}^6$ numerically from intermediate T into high T

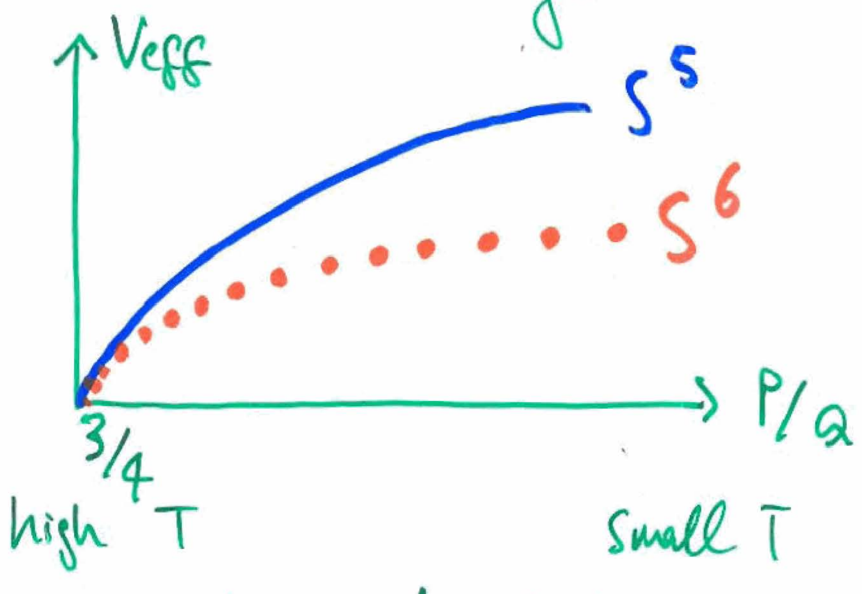
• A SECOND TOPOLOGY CHANGE AT $RT = \frac{1}{\sqrt{\lambda}}$

Second order quantum phase transition

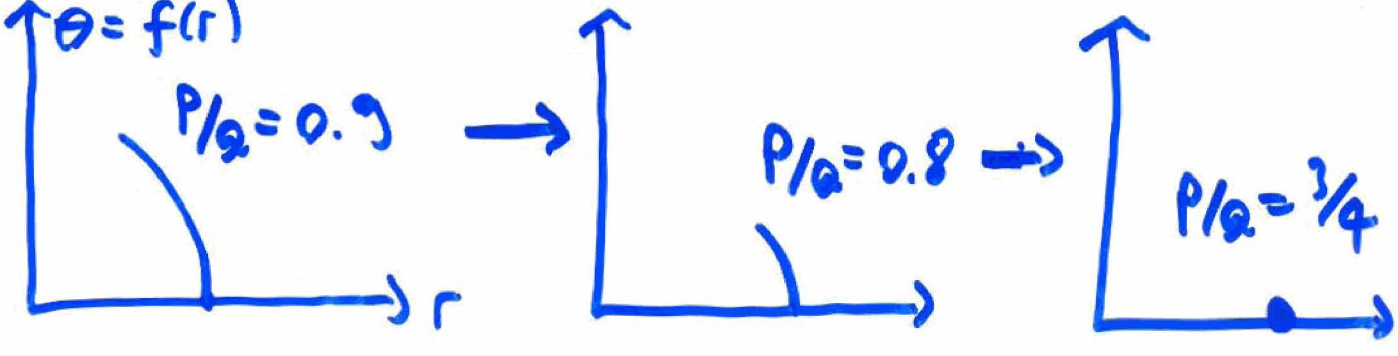
$\boxed{S^6 \to S^5}$ at $\frac{P}{Q} = \frac{3}{4}$

• <u>Numerical evidence</u>

① Actions converge



② Eigenvalue distributions converge



- **Analytic evidence**

(1) S^6 solution collapses onto S^5 solution
S^5 solves the EOM for S^6 at $\boxed{\frac{p}{q} = \frac{3}{4}}$

(2) S^5 solution develops a tachyonic direction in θ-fluctuations at $\frac{3}{4}$

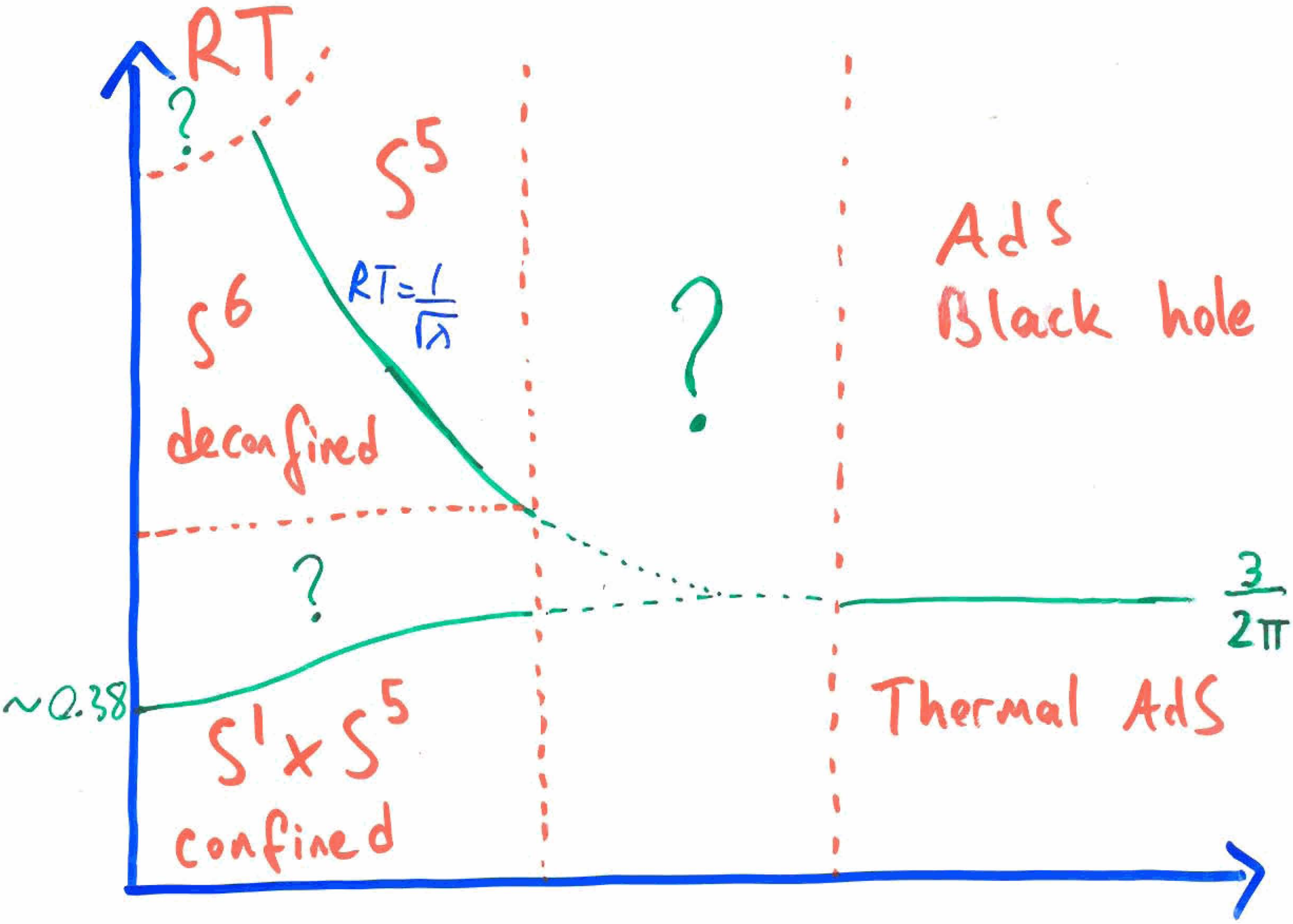
$\theta = 0$ forced to expand into
$\theta \in [-\theta_0, \theta_0]$

There exists a second phase boundary at $\frac{p}{q} = \frac{3}{4}$

or $RT = \frac{1}{\sqrt{\lambda}}$

SUMMARY OF RESULTS

Phase diagram of $\mathcal{N}=4$ SYM on S^3



- Various topology changes
- S^5 phase continued to AdS Black hole?

  ⇒ IF SO high T black holes from perturbative QFT

DUAL SPACE-TIME INTERPRETATION

At zero temperature (Berenstein '04; Lin, Lunin, Maldacena '04; Berenstein, Cotta '07)

Geometry of space-time at the locus where S^3 degenerates $\Longleftrightarrow$ Eigenvalue distribution of Φ^a_{ij}'s.

Can we extend to $T > 0$?

- In thermal AdS indeed $ds^2 \xrightarrow[r=0]{} S^1 \times S^5$
- In Black hole S^3 never degenerates — But S^1 is contractible $\Rightarrow$ S^5 topology ?
- How to see S^3 if not degenerate ?
  - Turn on KK modes $\langle \vec{u}_\ell \rangle$ in a uniform fashion at $N = \infty$?
  - Compute $\langle A_i A^i \rangle$?

FURTHER DISCUSSION AND OUTLOOK

- Computed saddle points in a subset of full configuration space:

$$[\varphi_a, \varphi_b, \Phi] = 0$$

Non-trivial phases from off-diagonal modes ?

- Looked at solutions with $SO(6)$ symmetry

There are more: $S^1 \times S^3$, $S^2 \times S^2$, $S^1 \times S^1 \times S^1$, S^4, $S^1 \times S^2$, S^3, etc.

Break $SO(6)_R$

→ minimize V_{eff} with chemical potentials turned on ?

- $\tilde{S}^6$ ellipsoid phase
  → a new stringy geometry ?

- Understand how Berenstein et al.'s argument extends to finite T
- Computation of thermodynamic quantities for Black holes from perturbative YM
  $V_{eff}(S^5 \text{ sol.}) = F_{BH}$
  requires identification of parameters
- Computation of Polyakov – Maldacena loop
  $$\mathcal{U} = \frac{1}{N} \langle \mathrm{Tr}\, P\, e^{\int A^0 + \Phi} \rangle$$
  Compare with large λ computations