One entropy function to rule them all

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Plan & Motivation:

☞ Discuss black ring and black hole attractors in a unified way using Sen's entropy function.

- ❏ Starting point for considering higher derivative corrections to black hole/string entropy and checking micro-scopic vs. macroscopic entropy in detail.
- ❏ General framework for discussing various properties of attractors.

What are blackhole attractors?

- Context = Theory with gravity, gauge fields, neutral scalars
  - generically appear as (part of) low energy limit of string theory
- scalars (or moduli) encode geometry of compactified dimensions
- Attractor mechanism = scalars' values fixed at Blackhole's horizon
- independent of values at infinity
- So horizon area depends only on gauge charges $\Rightarrow$ Entropy depends only on charges
- works for Extremal $(T = 0)$ blackholes

Hand waving

- ☞ number of microstates of extremal blackhole determined by quantised charges
  - ❑ entropy can not vary continuously
- ☞ but the moduli vary continuously
- ☞ resolution: horizon area independent of moduli
  - ❑ moduli take on fixed values at the horizon determined by charges
- ☞ No mention of SUSY

Outline

- ☞ Go through examples of application of entropy function
  - ❑ discuss four dimensional spherically symmetric black holes
  - ❑ some simple black holes and black rings in 5d
    - → may be dimensionally reduced to previous case
  - ❑ Time permitting: more general black holes and black rings
- ☞ Study Lagrangians which generically appear as (the bosonic part of) certain low energy limits of string theory

Entropy function outline

- ☞ Only need near horizon geometry
- ☞ Equations of motion $\Leftrightarrow$ Extremising an Entropy function
- ☞ Entropy function at extremum = Entropy of Blackhole
- ☞ need to solve algebraic equations
- ☞ Argument is independent of SUSY
- ☞ in 4-d:
  - ❑ Assume extremal $(T = 0)$ $\leftrightarrow$ $AdS_2 \times S^2$ near horizon symmetries

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☞ Equations of motion $\Leftrightarrow$ Extremising an Entropy function

☞ Entropy function at extremum = Entropy of Blackhole

☞ need to solve algebraic equations

☞ Argument is independent of SUSY

☞ in 5-d:

❑ Assume $(AdS_2 \times S_2) \otimes U(1)$ near horizon symmetries
$\rightarrow$ $AdS_3 \times S_2$ near horizon symmetries = black-ring
$\rightarrow$ $AdS_2 \times S_3$ near horizon symmetries = black-hole

Entropy function outline

- ☞ Only need near horizon geometry
- ☞ Equations of motion $\Leftrightarrow$ Extremising an Entropy function
- ☞ Entropy function at extremum = Entropy of Blackhole
- ☞ need to solve algebraic equations
- ☞ Argument is independent of SUSY
- ☞ in 5-d:
  - ❑ More generally:
    $\rightarrow$ $AdS_2 \otimes U(1)^2$ near horizon symmetries

Step 1

- ☞ First we look at simple 4-dimensional spherically symmetric black holes
- ☞ Form the basis for generalisation to higher dimensions

Entropy Function (Sen)

Set up:

☞ Gravity, p-form gauge fields, massless neutral scalars

☞ $\mathcal{L}$ gauge and coordinate invariant - in particular there may be higher derivative terms

☞ Assume: Extremal $= AdS_2 \times S^2$ Near horizon geometry

☞ Entropy function:

- First we consider, f, the Lagrangian density evaluated at the horizon:

$$f[e^i, p^i, R_{AdS_2}, R_{S_2}, \varphi_s] = \int_H \sqrt{-g}\mathcal{L}$$

- The electric charges, conjugate to the electric fields, are defined as

$$q_i = \frac{\partial f}{\partial e^i}$$

Entropy Function (Sen)

Set up:

- ☞ Gravity, p-form gauge fields, massless neutral scalars
- ☞ $\mathcal{L}$ gauge and coordinate invariant - in particular there may be higher derivative terms
- ☞ Extremal $=$ $AdS_2 \times S^2$ Near horizon geometry
- ☞ Entropy function:
  - ❑ First we consider, f, the Lagrangian density evaluated at the horizon.
  - ❑ Now take the Legendre transform of f w.r.t the electric fields and their conjugate charges:

$$\mathcal{E} = 2\pi \left(q_i e^i - \int_H \sqrt{-g}\mathcal{L} \right)$$

$$\mathcal{E} = \mathcal{E}[q_i, p^i, R_{AdS_2}, R_{S_2}, \varphi_s]$$

Entropy Function (Sen)

$$\mathcal{E} = 2\pi \left(q_i e^i - \int_H \sqrt{-g}\mathcal{L} \right)$$

Results:

☞ equations of motion $\Leftrightarrow$Extremising $\mathcal{E}$

☞ Wald Entropy = Extremum of $\mathcal{E}$

☞ Fixing q_i and p^i fixes everything else completely

Caveats

Entropy function, $\mathcal{E}$, might have flat directions

$\Rightarrow$ The near horizon geometry is not completely determined by extremisation of $\mathcal{E}$
$\Rightarrow$ There may be a dependence of the near horizon geometry on the moduli

But since these are flat directions

✔ the entropy is still independent of the moduli

☞ Generalised attractor mechanism

Also note that we have assumed that a blackhole solution exists which may not always be the case.

Simple example: spherically symmetric case

$$
\begin{aligned}
\mathcal{L} &= R - h_{rs}(\vec{\Phi}) g^{\mu\nu} \partial_\mu \Phi_s \partial_\nu \Phi_r - f_{ij}(\vec{\Phi}) g^{\mu\rho} g^{\nu\sigma} F^{(i)}_{\mu\nu} F^{(j)}_{\rho\sigma} \\
&\quad - \frac{1}{2} \tilde{f}_{ij}(\vec{\Phi}) \epsilon^{\mu\nu\rho\sigma} F^{(i)}_{\mu\nu} F^{(j)}_{\rho\sigma}
\end{aligned}
$$

Ansatz: $AdS_2 \times S^2$ near horizon geometry

$$
\begin{aligned}
ds^2 &= v_1 \left(-r^2 dt^2 + dr^2/r^2 \right) + v_2 d\Omega_2^2 \\
A^i &= e^i r dt + p^i (1 - \cos\theta) d\phi \\
\Phi_r &= u_r \, (\text{const.})
\end{aligned}
$$

Simple example: spherically symmetric case

Ansatz: $AdS_2 \times S^2$ near horizon geometry

$$
\begin{aligned}
ds^2 &= v_1 \left(-r^2 dt^2 + dr^2/r^2\right) + v_2 d\Omega_2^2 \\
F^i_{rt} &= e^i \qquad\quad F^i_{\theta\phi} = p^i \sin\theta \\
\Phi_r &= u_r \,(\text{const.})
\end{aligned}
$$

Entropy function

Wish to calculate:

$$\mathcal{E} = 2\pi(q_i e^i - f) = 2\pi\left(q_i e^i - \int d\theta d\phi \sqrt{-g}\mathcal{L}\right)$$

Calculate the action:

$$f[\vec{e},\vec{p},\vec{u},v_1,v_2] = (4\pi)(v_1 v_2)\left(\frac{2}{v_2} - \frac{2}{v_1} + f_{ij}(u_r)\left(\frac{2e^i e^j}{v_1^2} - \frac{2p^i p^j}{v_2^2}\right)\right) - 8\pi^2 \tilde{f}_{ij} e^i p^j$$

Calculate the conjugate variables:

$$q_i = \frac{\partial f}{\partial e^i} = (16\pi)(v_1 v_2) f_{ij}(u_r)\left(\frac{e^j}{v_1^2}\right) - 8\pi^2 \tilde{f}_{ij} p^j$$

$\Rightarrow$

$$e^j = \left(\frac{v_1}{v_2}\right) f^{jk} \hat{q}_k$$

Entropy function

Wish to calculate:

$$\mathcal{E} = 2\pi(q_i e^i - f) = 2\pi \left(q_i e^i - \int d\theta d\phi \sqrt{-g} \mathcal{L} \right)$$

Calculate the action:

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Calculate the conjugate variables:

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$\Rightarrow$

$$e^j = \left(\frac{v_1}{v_2} \right) f^{jk} \hat{q}_k \qquad \hat{q}_k = \frac{1}{16\pi} \left(q_k + 8\pi^2 \tilde{f}_{kl} p^l \right)$$

Entropy function

Finally

$$\mathcal{E}[\vec{q},\vec{p},\vec{u},v_1,v_2] = 2\pi\left(8\pi(v_2 - v_1) + \left(\frac{v_1}{v_2}\right)V_{eff}\right)$$

Notation

$$V_{eff} = 8\pi(p^i f_{ij}(\vec{u})p^j + \hat{q}_i f^{ij}(\vec{u})\hat{q}_i)$$

Entropy function

Finally

$$\mathcal{E}[\vec{q},\vec{p},\vec{u},v_1,v_2] = 2\pi\left(8\pi(R_S^2 - R_{AdS}^2) + \left(\frac{R_{AdS}^2}{R_S^2}\right)V_{eff}\right)$$

Notation

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Entropy function

Finally

$$\mathcal{E}[\vec{q},\vec{p},\vec{u},v_1,v_2] = 2\pi\left(8\pi(R_S^2 - R_{AdS}^2) + \left(\frac{R_{AdS}^2}{R_S^2}\right)V_{eff}\right)$$

Roughly

$$V_{eff} \sim E^2 + B^2$$

Equations of Motion

Then the equations of motion are equivalent to extremising the entropy function:

$$\frac{\partial \mathcal{E}}{\partial \Phi_I} = 0 \;\; \Rightarrow \;\; \frac{\partial V_{eff}}{\partial \Phi_I} = 0$$
$$\frac{\partial \mathcal{E}}{\partial v_1} = 0 \;\; \Rightarrow \;\; 8\pi - v_2^{-1} V_{eff}(\Phi_I) = 0$$
$$\frac{\partial \mathcal{E}}{\partial v_2} = 0 \;\; \Rightarrow \;\; -8\pi + v_1 v_2^{-2} V_{eff}(\Phi_I) = 0$$

So

$$v_1 = v_2 = 8\pi V_{eff}$$

and

$$S_{BH} = 2\pi V_{eff}$$

5-d attractors

Consider 5-d Lagrangian with massless uncharged scalars coupled to $U(1)$ gauge fields with Chern-Simons terms:

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big(R - h_{ST}(\vec{\Phi})\partial_\mu X^S \partial^\mu X^T - f_{IJ}(\vec{\Phi})\bar{F}^I_{\mu\nu}\bar{F}^{J\,\mu\nu} - c_{IJK}\epsilon^{\mu\nu\alpha\beta\gamma}\bar{F}^I_{\mu\nu}\bar{F}^J_{\alpha\beta}\bar{A}^K_\gamma\Big)$$

- ☞ Lagrangian density is not gauge invariant $\longrightarrow$ Entropy function formalism does not apply
- ☞ similar to BTZ black hole with gravitational Chern-Simons and/or gauge Chern-Simons term
- ☞ compactify ψ (Sen, Sahoo)
- ✔ can apply formalism to dimensionally reduced action
- ☞ Related work: (Kraus, Larsen), (Dabholkar, Iizuka, Iqubal, Sen, Shigemori)

Dimensional reduction

Kaluza-Klein Ansatz:

$$ds^2 = w^{-1} g_{\mu\nu} dx^\mu dx^\nu + w^2 (d\psi + A^0_\mu dx^\mu)^2$$

$$\Phi^S = X^S(x^\mu)$$
$$\bar{A}^I = A^I_\mu dx^\mu + a^I(x^\mu)\left(d\psi + A^0_\mu dx^\mu\right)$$

☞ dimensionally reduce on ψ:

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g_4} \Big(R - h_{st}(\vec{\Phi}) \partial \Phi^s \partial \Phi^t - f_{ij}(\vec{\Phi}) F^i_{\mu\nu} F^{j\,\mu\nu} - \tilde{f}_{ij}(\vec{\Phi}) \epsilon^{\mu\nu\alpha\beta} F^i_{\mu\nu} F^j_{\alpha\beta} \Big)$$

where $F^i = (F^0, F^I)$, $\Phi^s = (w, X^S, a^I)$.

Gory Details

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g_4} \Big(R - h_{st}(\vec{\Phi}) \partial\Phi^s \partial\Phi^t - f_{ij}(\vec{\Phi}) F^i_{\mu\nu} F^{j\,\mu\nu}$$
$$- \tilde{f}_{ij}(\vec{\Phi}) \epsilon^{\mu\nu\alpha\beta} F^i_{\mu\nu} F^j_{\alpha\beta} \Big)$$

where $F^i = (F^0, F^I)$, $\Phi^s = (w, X^S, a^I)$ and

$$f_{ij} = \begin{array}{c} \\ 0 \\ I \end{array} \begin{array}{c} \begin{array}{cc} \qquad 0 \qquad & \qquad J \qquad \end{array} \\ \begin{pmatrix} \frac{1}{4}w^3 + w f_{LM} a^L a^M & w f_{JL} a^L \\ w f_{IL} a^L & w f_{IJ} \end{pmatrix} \end{array}$$

$$\tilde{f}_{ij} = \begin{array}{c} \\ 0 \\ I \end{array} \begin{array}{c} \begin{array}{cc} \qquad 0 \qquad & \qquad J \qquad \end{array} \\ \begin{pmatrix} 4c_{KLM} a^K a^L a^M & 4c_{JKL} a^K a^L \\ 6c_{IKL} a^K a^L & 12c_{IJK} a^K \end{pmatrix} \end{array}$$

$$h_{rs} = \mathrm{diag}\left(\tfrac{9}{2} w^{-2}, h_{RS}, 2w f_{IJ} \right)$$

Near-horizon dimensional reduction

- ☞ We consider a $5-$d near horizon geometry which reduces to $AdS_2 \times S^2$
- ☞ Starting with $AdS_2 \times S^2$

$$w_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + w_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

Near-horizon dimensional reduction

☞ We consider a $5-$d near horizon geometry which reduces to $AdS_2 \times S^2$

☞ We add an extra-dimension

$$w_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + w_2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) + w_3 d\psi^2$$

Near-horizon dimensional reduction

☞ We consider a $5-$d near horizon geometry which reduces to $AdS_2 \times S^2$

☞ Can add 5-d rotation and a Hopf-fibration

$$w_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+w_2\left(d\theta^2+\sin^2\theta d\phi^2\right)$$
$$+w_3\left(d\psi+e^0 r dt+\cos\theta d\phi\right)^2$$

Near-horizon dimensional reduction

☞ We consider a $5-$d near horizon geometry which reduces to $AdS_2 \times S^2$

☞ More generally we could have a Taub-Nut charge

$$w_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + w_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$
$$+ w_3 \left(d\psi + e^0 r dt + p^0 \cos\theta d\phi \right)^2$$

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$$+w_3\left(d\psi+e^0 r dt+p^0\cos\theta d\phi\right)^2$$

- For black rings we set $p^0=0$.

Near-horizon dimensional reduction

- We consider a $5-$d near horizon geometry which reduces to $AdS_2 \times S^2$

- As is usual with Kalusa-Klein reduction it is convenient to choose the following parameterisation

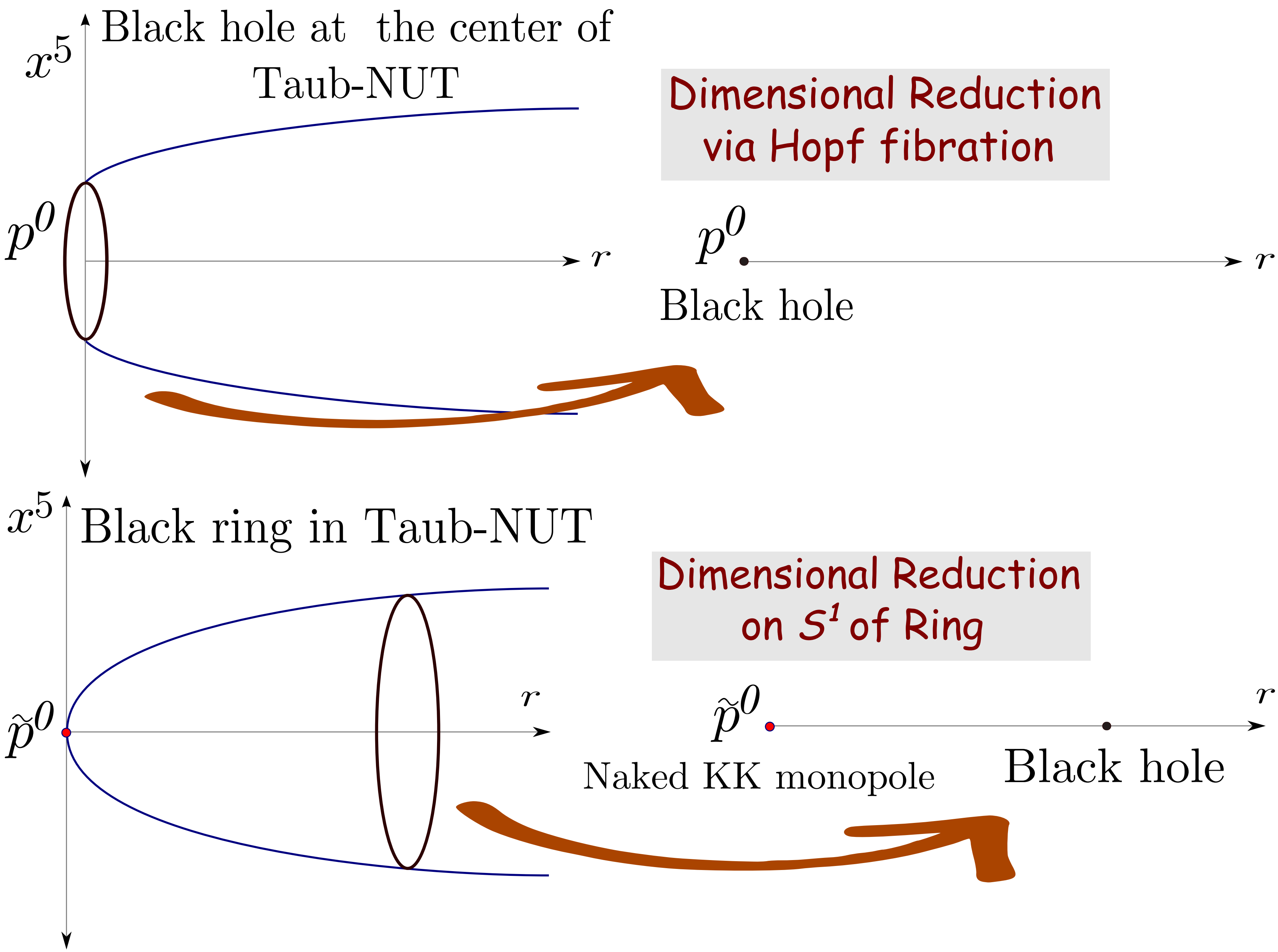
$$w^{-1}\left[v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right]$$
$$+w^2\left(d\psi+e^0 rdt+p^0\cos\theta d\phi\right)^2$$

5-d near horizon ansatz

$$
\begin{aligned}
ds^2 &= w^{-1}\left[v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right] \\
&\quad +w^2\left(d\psi+e^0 r dt+p^0\cos\theta d\phi\right)^2, \\
A^I &= e^I r dt+p^I\cos\theta d\phi+a^I\left(d\psi+e^0 r dt+p^0\cos\theta d\phi\right), \\
\Phi^S &= u^S,
\end{aligned}
$$

where the coordinates, θ, ϕ and ψ, have periodicity π, 2π, and $4\pi/\tilde{p}^0$

Geometry of 4-d/5-d lift



Back to the entropy function

☞ After dimensional reduction, we can just read off the 5-d entropy function from the $4-d$ result

Gory Details Again

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big(R - h_{ST}(\vec{\Phi})\partial_\mu X^S\partial^\mu X^T - f_{IJ}(\vec{\Phi})\bar{F}^I_{\mu\nu}\bar{F}^{J\,\mu\nu}$$
$$-c_{IJK}\epsilon^{\mu\nu\alpha\beta\gamma}\bar{F}^I_{\mu\nu}\bar{F}^J_{\alpha\beta}\bar{A}^K_\gamma\Big)$$
$$\rightarrow \quad \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\Big(R_{(4)} - h_{st}(\vec{\Phi})\partial\Phi^s\partial\Phi^t - f_{ij}(\vec{\Phi})F^i_{\mu\nu}F^{j\,\mu\nu}$$
$$-\tilde{f}_{ij}(\vec{\Phi})\epsilon^{\mu\nu\alpha\beta}F^i_{\mu\nu}F^j_{\alpha\beta}\Big)$$

$$F^i = (F^0, F^I),\ \Phi^s = (w, X^S, a^I),$$

$$f_{ij} = \begin{pmatrix} \frac{1}{4}w^3 + wf_{IJ}a^Ia^J & wf_{IJ}a^J \\ wf_{IJ}a^J & wf_{IJ} \end{pmatrix}$$

$$\tilde{f}_{ij} = \begin{pmatrix} \frac{1}{3}c_{KLM}a^Ka^La^M & \frac{1}{2}c_{JKL}a^Ka^L \\ \frac{1}{2}c_{IKL}a^Ka^L & c_{IJK}a^K \end{pmatrix}$$

$$h_{rs} = \mathrm{diag}\left(\ \frac{9}{2}w^{-2},\quad h_{RS},\quad 2wf_{IJ}\ \right)$$

Back to the entropy function

☞ After dimensional reduction, we can just read off the 5-d entropy function from the $4-d$ result

$$\mathcal{E} = 2\pi(Nq^i e_i - f) = \frac{4\pi^2}{\tilde{p}^0 G_5}\left\{ v_2 - v_1 + \frac{v_1}{v_2} V_{eff} \right\}$$

$$\begin{aligned} V_{eff} &= f^{ij}\hat{q}_i\hat{q}_j + f_{ij}p^i p^j \\ \hat{q}_i &= q_i - \tilde{f}_{ij}p^j \end{aligned}$$

☞ Choose $Nq_i = \frac{\partial f}{\partial e^i}$ for convenience ($N = 4\pi/G_5$)

☞ As before it is easy to solve for v_1 and v_2 to get

$$\mathcal{E} = \frac{4\pi^2}{\tilde{p}^0 G_5} V_{eff}|_{\partial V=0},$$

$$v_1 = v_2 = V_{eff}\big|_{\partial V=0},$$

Now we "just" need to solve:

$$\partial_{\{w,\vec{a},\vec{X}\}} V_{eff} = 0.$$

As a check, we note that, even before extremising V_{eff}, this result agrees with the Hawking-Bekenstein entropy since,

$$S = \frac{A_H}{4G_5} = \frac{\left(\frac{16\pi^2}{\tilde{p}^0} v_2\right)}{4G_5} = \mathcal{E}.$$

☞ To get black rings or black holes we fix p^0 and $\tilde{p}^0$ appropriately.

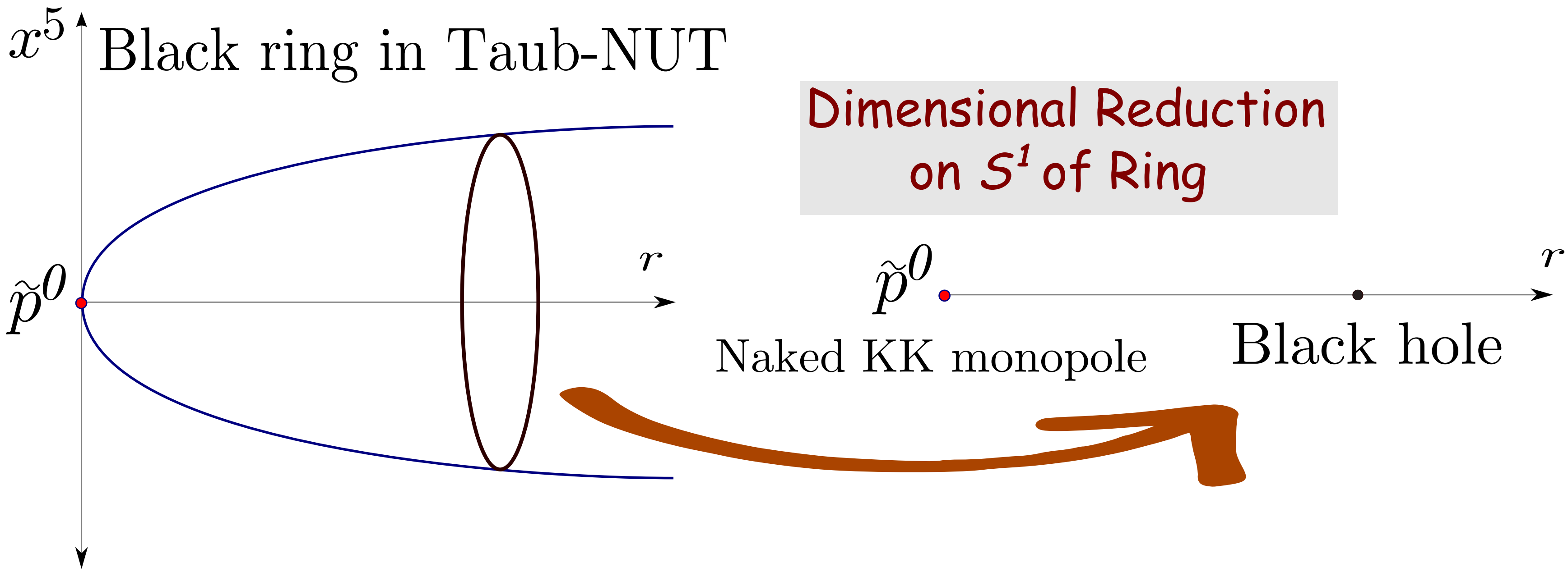
More gory details: Effective potential

$$
\begin{aligned}
V_{eff} &= \tfrac{1}{4}w^3(p^0)^2 + 4w^{-3}(q_0 - \tilde{f}_{0j}(\vec{a})p^j - a^I(q_I - \tilde{f}_{Ij}(\vec{a})p^j))^2 \\
&\quad + w f_{IJ}(\vec{X})(p^I + a^I p^0)(p^J + a^J p^0) \\
&\quad + w^{-1} f^{IJ}(\vec{X})(q_I - \tilde{f}_{Ik}(\vec{a})p^k)(q_J - \tilde{f}_{Jl}(\vec{a})p^l),
\end{aligned}
$$

Two Examples

We now consider the black-ring and black-hole examples

Black ring



x^5
Black ring in Taub-NUT
Dimensional Reduction on S^1 of Ring
r
$\tilde{p}^0$
$\tilde{p}^0$
r
Naked KK monopole
Black hole

Black ring ($p^0 = 0$)

Ansatz:

$$\begin{aligned} ds^2 &= w^{-1}\left[v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right] \\ &\quad +w^2\left(d\psi+e^0 r dt\right)^2 \\ A^I &= e^I r dt + p^I \cos\theta d\phi + a^I\left(d\psi + e^0 r dt\right) \\ \Phi^S &= u^S \end{aligned}$$

Already know

$$\mathcal{E} = \frac{4\pi^2}{\tilde{p}^0 G_5} V_{eff}|_{\partial V=0}$$

and $v_1 = v_2$. Still need to solve

$$\partial_{\{w,\vec{a},\vec{u}\}} V_{eff} = 0$$

Finding the solution

We need to solve

$$\partial_{\{w,\vec{a},\vec{u}\}} V_{eff} = 0$$

☞ Easiest to solve for the axions/gauge fields first:

$$\partial_{\vec{a}} V_{eff} = 0 \;\; \Rightarrow \;\; F^I_{tr} = 0$$
$$\Rightarrow \;\; V_{eff} = w f_{IJ} p^I p^J + (4w^{-3})(\hat{q}_0)^2.$$

☞ Solving $\partial_w V_{eff} = 0 \Rightarrow$

$$\mathcal{E} = \frac{8\pi^2}{\tilde{p}^0 G_5} \sqrt{\hat{q}_0 (\tfrac{4}{3} V_M)^{\frac{3}{2}}} \qquad V_M = f_{IJ} q^I q^J$$

and

$$e_0^2 w^2 = v_1 w^{-1}$$

☞ which means that we our fibration $(AdS_2 \times S^2) \otimes U(1)$ is actually $AdS_3 \times S^2$

☞ more precisely: $(AdS_3/\mathbb{Z}_{\tilde{p}^0}) \times S^2$

Example: 11-d supergravity on T^6

☞ 11-d supergravity on $T^6 \longrightarrow$ 5-d $U(1)^3$ supergravity

- ❑ $2f_{IJ} = h_{IJ} = \frac{1}{2}\mathrm{diag}\left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\right), \quad c_{IJK} = |\epsilon_{IJK}|/24$
- ❑ $X^1X^2X^3 = 1$

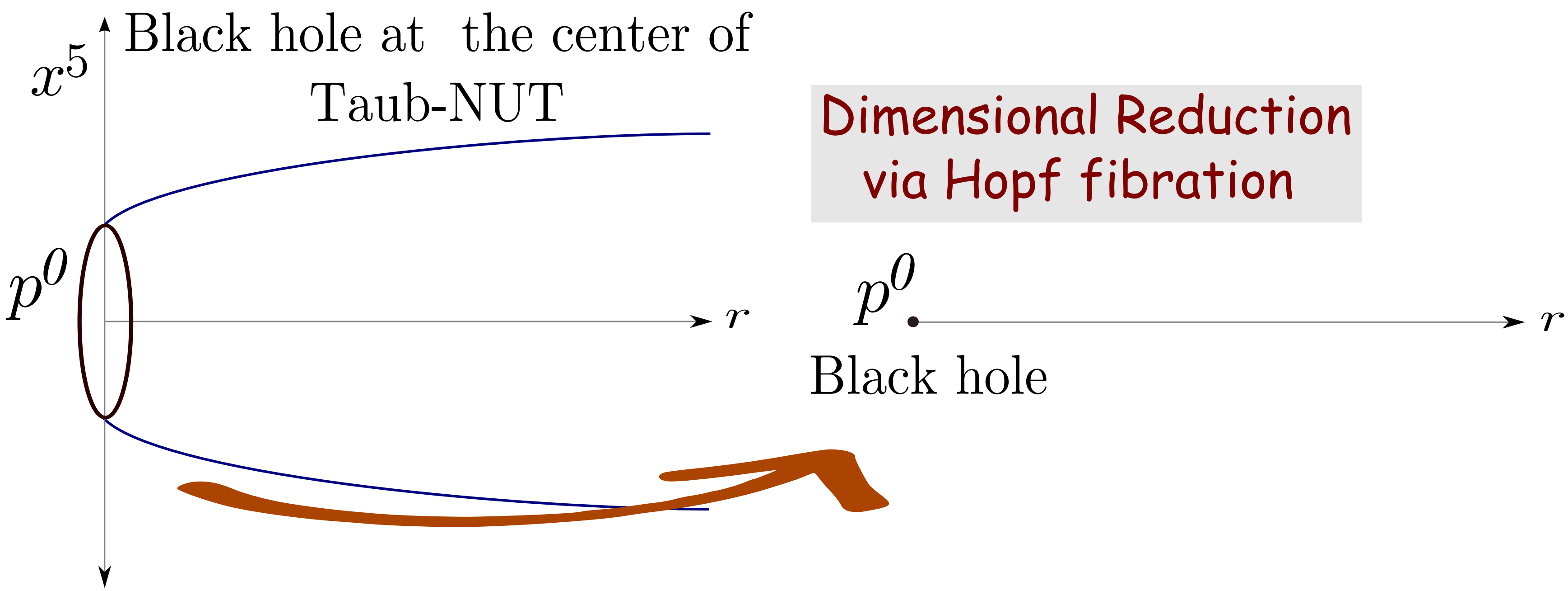
☞ We get the magnetic potential

$$V_M = f_{ij}p^ip^j = \frac{1}{4}\left(\frac{(p^1)^2}{(X^1)^2} + \frac{(p^2)^2}{(X^2)^2} + (p^3)^2(X^1)^2(X^2)^2\right)$$

☞ Extremising gives

- ❑ $(X^1)^3 = \frac{(p^1)^2}{p_2p_3}, \qquad (X^2)^3 = \frac{(p^2)^2}{p^3p^1}$
- ❑ $V_M = \frac{3}{4}(p^1p^2p^2)^{\frac{2}{3}}.$

Non-rotating black hole



Non-rotating black hole

- ☞ This is in some sence dual to the black ring case: $p^0 \leftrightarrow e^0$, $p^i \leftrightarrow \hat{q}^i$

Ansatz:

$$\begin{aligned} ds^2 &= w^{-1}\left[v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right] \\ &\quad +w^2\left(d\psi+p^0\cos\theta d\phi\right)^2 \\ A^I &= e^I r dt + p^I\cos\theta d\phi + a^I\left(d\psi+p^0\cos\theta d\phi\right) \\ \Phi^S &= u^S \end{aligned}$$

Need to solve

$$\partial_{\{w,\vec{a},\vec{u}\}}V_{eff}=0$$

Easiest to solve for the gauge fields/axions a^I first $\Rightarrow$

$$F^I_{\theta\phi}=0$$

which gives

$$V_{eff} = (\frac{1}{4}w^3)(p_0)^2 + w^{-1} f^{IJ} \hat{q}_I \hat{q}_J.$$

Solving $\partial_w V_{eff} = 0 \Rightarrow$

$$\mathcal{E} = \frac{4\pi^2}{G_5}\sqrt{p_0(\tfrac{4}{3}V_E)^{\frac{3}{2}}}. \qquad V_E = f^{IJ}\hat{q}_I \hat{q}_J$$

and

$$p_0^2 w^2 = v_2 w^{-1}$$

which means that we our fibration $(AdS_2 \times S^2) \otimes U(1)$ is actually $AdS_2 \times S^3$

☞ $AdS_2 \times \left(S^3/\mathbb{Z}_{p^0}\right)$

Non-supersymmetric solutions of (very) special geometry

- In 4 dimensional $\mathcal{N}=2$ special geometry we can write V_{eff} or the "blackhole potential function"

$$V_{eff}=|Z|^2+|DZ|^2.$$

  - BPS solutions: each term of the potential is separately extremised
  - non-BPS solutions: V_{eff} extremised but $DZ\neq 0$

- For the black holes and rings with very special geometry we get

$$V=Z^2+(DZ)^2.$$

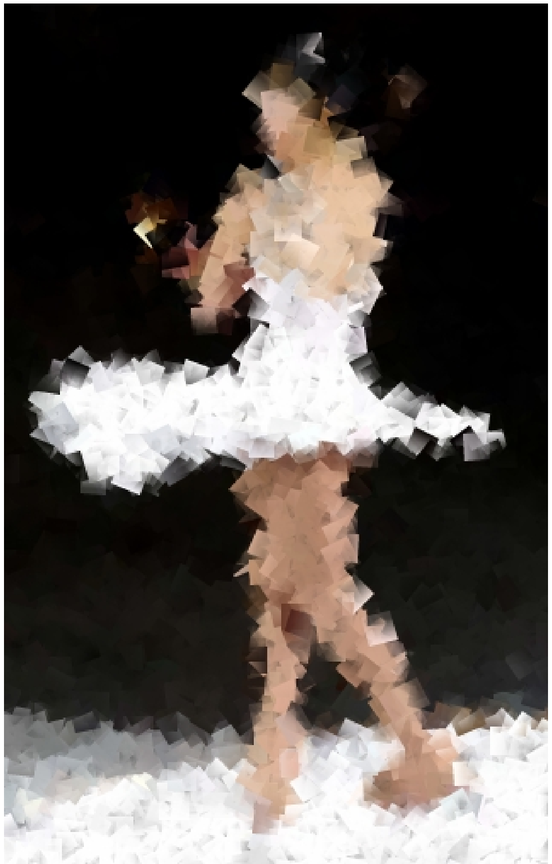
  which may also have both BPS and non-BPS extrema.

  - Black holes: $Z_E=X^I q_I$
  - Black rings: $Z_M=X_I p^I$

Less Symmetry

- ☞ Again in will be helpful to consider $4-d$ blackholes
- ☞ in 4-d rotation leads to less symmetric attractor blackholes
- ☞ $AdS_2 \times S_2 \longrightarrow AdS_2 \times U(1)$

Rotating attractors in 4-d



What is the generalisation of an $AdS_2 \times S^2$ near horizon geometry for rotating blackholes?

- ☞ Take a hint from the near horizon geometry of extremal Kerr Blackholes (Bardeen, Horowitz)
  - ❑ $SO(2,1) \times U(1)$

Recall: $SO(2,1) \times S^2$ Ansatz

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 d\theta^2 + v_2 \sin^2\theta \, d\phi^2$$

$$\varphi_s = u_s$$

$$\frac{1}{2} F^{(i)}_{\mu\nu} dx^\mu \wedge dx^\nu = e^i dr \wedge dt + \frac{p^i \sin\theta}{4\pi} d\theta \wedge d\phi$$

$SO(2,1) \times U(1)$ **Ansatz**

$$ds^2 = v_1(\theta)\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \beta^2 d\theta^2 + v_2(\theta)\sin^2\theta\,(d\phi - \alpha r dt)^2$$

$$\varphi_s = u_s(\theta)$$
$$A^i = e^i r dt + b^i(\theta)(d\phi - \alpha r\, dt)$$

Horizon has spherical topology $\Rightarrow v_2(\theta)$ at poles ~ 1

$$p^i = \int d\theta d\phi F^{(i)}_{\theta\phi} = 2\pi(b^i(\pi) - b^i(0))\,.$$

$SO(2,1) \times U(1)$ Ansatz

$$ds^2 = v_1(\theta)\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \beta^2 d\theta^2 + v_2(\theta)\sin^2\theta\,(d\phi - \alpha r dt)^2$$

$$\varphi_s = u_s(\theta)$$

$$\frac{1}{2}F^{(i)}_{\mu\nu}dx^\mu \wedge dx^\nu = (e^i - \alpha b^i(\theta))dr \wedge dt + b^{i\prime}(\theta)d\theta \wedge (d\phi - \alpha r\, dt)$$

Horizon has spherical topology $\Rightarrow v_2(\theta)$ at poles ~ 1

$$p^i = \int d\theta d\phi F^{(i)}_{\theta\phi} = 2\pi(b^i(\pi) - b^i(0))\,.$$

$SO(2,1) \times U(1)$ Ansatz

$$ds^2 = \Omega^2 e^{2\psi}\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \beta d\theta^2 + e^{-2\psi}(d\phi - \alpha r dt)^2$$

$$\varphi_s = u_s(\theta)$$

$$\tfrac{1}{2}F^{(i)}_{\mu\nu}dx^\mu \wedge dx^\nu = (e^i - \alpha b^i(\theta))dr \wedge dt + b^{i\prime}(\theta)d\theta \wedge (d\phi - \alpha r\, dt)$$

Horizon has spherical topology $\Rightarrow e^{-2\psi}$ at poles $\sim \sin^2\theta$

$$p^i = \int d\theta d\phi F^{(i)}_{\theta\phi} = 2\pi(b^i(\pi) - b^i(0)).$$

Symmetries

☞ One way to see that the ansatz has $SO(2,1) \times U(1)$ symmetries is to check that it is invariant under the Killing vectors, ∂_ϕ and

$$L_1 = \partial_t, \quad L_0 = t\partial_t - r\partial_r, \quad L_{-1} = \frac{1}{2}\left(\frac{1}{r^2} + t^2\right)\partial_t - (tr)\partial_r + \frac{\alpha}{r}\partial_\phi \,.$$

☞ can also be seen by thinking of ϕ as a compact dimension and find that the resulting geometry has a manifest $SO(2,1)$ symmetry with the conventional generators.

5-d Ansatz

For the five dimensional black-rings and black-holes we take a $SO(2,1)\times U(1)^2$ ansatz which will give us the $4d$ one after dimensional reduction:

$$\begin{aligned}
ds^2 =& w^{-1}(\theta)\Omega^2(\theta)e^{2\Psi(\theta)}\left(-r^2dt^2+\frac{dr^2}{r^2}+\beta^2 d\theta^2\right)\\
&+w^{-1}(\theta)e^{-2\Psi(\theta)}(d\phi+e_\phi r dt)^2\\
&+w^2(\theta)(d\psi+e_0 r dt+b_0(\theta)d\phi)^2\\
A^I =& e^I r dt+b^I(\theta)(d\phi+e_\phi r dt)\\
&+a^I(\theta)(d\psi+e_0 r dt+b_0(\theta)d\phi)\\
\phi^S =& u^S(\theta).
\end{aligned}$$

Entropy function:

We define

$$f[\alpha, \beta, \vec{e}, \Omega(\theta), \psi(\theta), \vec{u}(\theta), \vec{b}(\theta)] := \int d\theta d\phi \sqrt{-g}\mathcal{L}$$

☞ The equations of motion are:

$$\frac{\partial f}{\partial \alpha} = J \qquad \frac{\partial f}{\partial \beta} = 0 \qquad \frac{\partial f}{\partial e^i} = q_i \qquad \frac{\delta f}{\delta b^i(\theta)} = 0$$

$$\frac{\delta f}{\delta \Omega(\theta)} = 0 \quad \frac{\delta f}{\delta \psi(\theta)} = 0 \quad \frac{\delta f}{\delta u_s(\theta)} = 0$$

Entropy function:

Equivalently we let

$$\mathcal{E}[J,\vec{q},\vec{b}(\theta),\beta,v_1(\theta),v_2(\theta),\vec{u}(\theta)] = 2\pi\,(J\alpha+\vec{q}\cdot\vec{e}-f)$$

☞ The equations of motion:

$$\frac{\partial\mathcal{E}}{\partial\alpha}=0 \qquad \frac{\partial\mathcal{E}}{\partial\beta}=0 \qquad \frac{\partial\mathcal{E}}{\partial e^i}=0 \qquad \frac{\delta\mathcal{E}}{\delta b^i(\theta)}=0$$

$$\frac{\delta\mathcal{E}}{\delta v_1(\theta)}=0 \quad \frac{\delta\mathcal{E}}{\delta v_2(\theta)}=0 \quad \frac{\delta\mathcal{E}}{\delta u_s(\theta)}=0$$

Examples

- ☞ Kerr, Kerr-Newman, constant scalars (non-dyonic)

- ☞ Dyonic Kaluza Klein blackhole (5-d$\circlearrowleft\longrightarrow 4$-d).
  -(Rasheed)

- ☞ Blackholes in toroidally compactified heterotic string theory
  -(Cvetic,Youm;Jatkar,Mukherji,Panda)

Two derivative Lagrangians

$$\mathcal{L} = R - h_{rs}(\vec{\Phi}) g^{\mu\nu} \partial_\mu \Phi_s \partial_\nu \Phi_r - f_{ij}(\vec{\Phi}) g^{\mu\rho} g^{\nu\sigma} F^{(i)}_{\mu\nu} F^{(j)}_{\rho\sigma}$$

$$\begin{aligned}
\mathcal{E} &\equiv 2\pi (J\alpha + \vec{q}\cdot\vec{e} - \int d\theta d\phi \sqrt{-\det g}\,\mathcal{L}) \\
&= 2\pi J\alpha + 2\pi\vec{q}\cdot\vec{e} - 4\pi^2 \int d\theta \Big[2\Omega^{-1}\beta^{-1}\Omega'^2 - 2\Omega\beta - 2\Omega\beta^{-1}\psi'^2 \\
&\quad + \frac{1}{2}\alpha^2\Omega^{-1}\beta e^{-4\psi} - \beta^{-1}\Omega h_{rs}(\vec{u}) u'_r u'_s \\
&\quad + 2 f_{ij}(\vec{u}) \left\{ \beta\Omega^{-1} e^{-2\psi} (e^i - \alpha b^i)(e^j - \alpha b^j) - \beta^{-1}\Omega e^{2\psi} b^{i\prime} b^{j\prime} \right\} \Big] \\
&\quad + 8\pi^2 \left[\Omega^2 e^{2\psi} \sin\theta (\psi' + 2\Omega'/\Omega) \right]_{\theta=0}^{\theta=\pi} .
\end{aligned}$$

Equations of motion

Notation:

$$\chi^i = e^i - \alpha b^i$$

Ω equation:

$$-4\beta^{-1}\Omega''/\Omega + 2\beta^{-1}(\Omega'/\Omega)^2 - 2\beta - 2\beta^{-1}(\psi')^2 - \frac{1}{2}\alpha^2\Omega^{-2}\beta e^{-4\psi}$$
$$-\beta^{-1}h_{rs}u_r'u_s' + 2f_{ij}\left\{-\beta\Omega^{-2}e^{-2\psi}\chi^i\chi^j - \alpha^{-2}\beta^{-1}e^{2\psi}\chi^{i\prime}\chi^{j\prime}\right\} = 0,$$

ψ equation:

$$4\beta^{-1}(\Omega\psi')' - 2\alpha^2\Omega^{-1}\beta e^{-4\psi}$$
$$+2f_{ij}\left\{-2\beta\Omega^{-1}e^{-2\psi}\chi^i\chi^j - 2\alpha^{-2}\beta^{-1}\Omega e^{2\psi}\chi^{i\prime}\chi^{j\prime}\right\} = 0,$$

u_s equation:

$$2\left(\beta^{-1}\Omega h_{rs}u'_s\right)' + 2\partial_r f_{ij}\left\{\beta\Omega^{-1}e^{-2\psi}\chi^i\chi^j - \alpha^{-2}\beta^{-1}\Omega e^{2\psi}\chi^{i\prime}\chi^{j\prime}\right\}$$
$$-\beta^{-1}\Omega\left(\partial_r h_{ts}\right)u'_t u'_s = 0,$$

b equation:

$$-\alpha\beta f_{ij}\Omega^{-1}e^{-2\psi}\chi^j - \alpha^{-1}\beta^{-1}\left(f_{ij}\Omega e^{2\psi}\chi^{j\prime}\right)' = 0$$

β equation:

$$\int d\theta I(\theta) = 0$$

where

$$I(\theta) = -2\Omega^{-1}\beta^{-2}(\Omega')^2 - 2\Omega + 2\Omega\beta^{-2}(\psi')^2 + \frac{1}{2}\alpha^2\Omega^{-1}e^{-4\psi}$$
$$+\beta^{-2}\Omega h_{rs}u'_r u'_s + 2f_{ij}\left\{\Omega^{-1}e^{-2\psi}\chi^i\chi^j + \alpha^{-2}\beta^{-2}\Omega(\theta)e^{2\psi(\theta)}\chi^{i\prime}\chi^{j\prime}\right\}$$

Charges:

$$q_i = 8\pi \int d\theta \,\left[f_{ij} \beta \Omega^{-1} e^{-2\psi} \chi^j \right] \, ,$$

$$J = \;\; 2\pi \int_0^\pi d\theta \left\{ \alpha \Omega^{-1} \beta e^{-4\psi} - 4\beta f_{ij} \Omega^{-1} e^{-2\psi} \chi^i b^j \right\}$$

Solutions

- Equations can be solved for some simple cases
  - Kerr, Kerr-Newmann, constant scalars
- Check known solutions fitted into the frame work:
  - KK blackholes, Toroidal compactification of Heterotic string theory

Kaluza-Klein Blackholes

$$\mathcal{L} = R - 2(\partial\varphi)^2 - e^{2\sqrt{3}\varphi}F^2$$

- ☞ Charges = Q, P, J
- ☞ 2 types of extremal blackholes
  - ❏ Both have $SO(2,1) \times U(1)$ near horizon geometry
  - ❏ non-SUSY

1. Ergo branch
   - ☞ $|J| > PQ$
   - ☞ Ergo-sphere
   - ☞ $S = 2\pi\sqrt{J^2 - P^2Q^2}$
   - ☞ $\mathcal{E}$ has flat directions

2. Ergo-free branch
   - ☞ $|J| < PQ$
   - ☞ no Ergo-sphere
   - ☞ $S = 2\pi\sqrt{P^2Q^2 - J^2}$
   - ☞ $\mathcal{E}$ has no flat directions

Blackholes in Heterotic String Theory on T^6

☞ Charges $= Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3, P_4, J,$

❑ (Actually 56 P's and Q's)

☞ Duality invariant quartic

$$\begin{aligned} D &= (Q_1Q_3 + Q_2Q_4)(P_1P_3 + P_2P_4) \\ &\quad -\frac{1}{4}(Q_1P_1 + Q_2P_2 + Q_3P_3 + Q_4P_4)^2 \end{aligned}$$

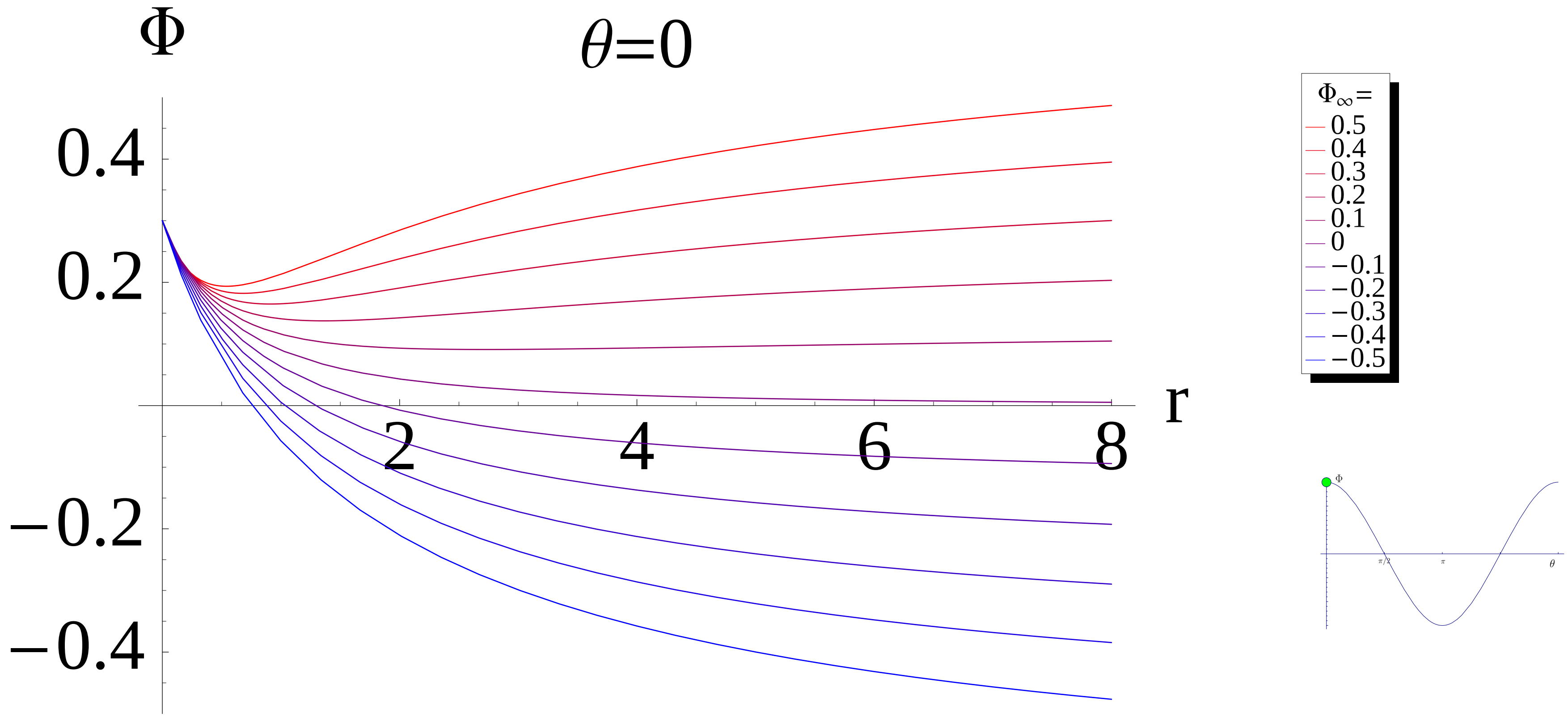
1. Ergo branch

   ☞ Ergo-sphere
   ☞ $S = 2\pi\sqrt{J^2 + D}$
   ☞ $\mathcal{E}$ has flat directions

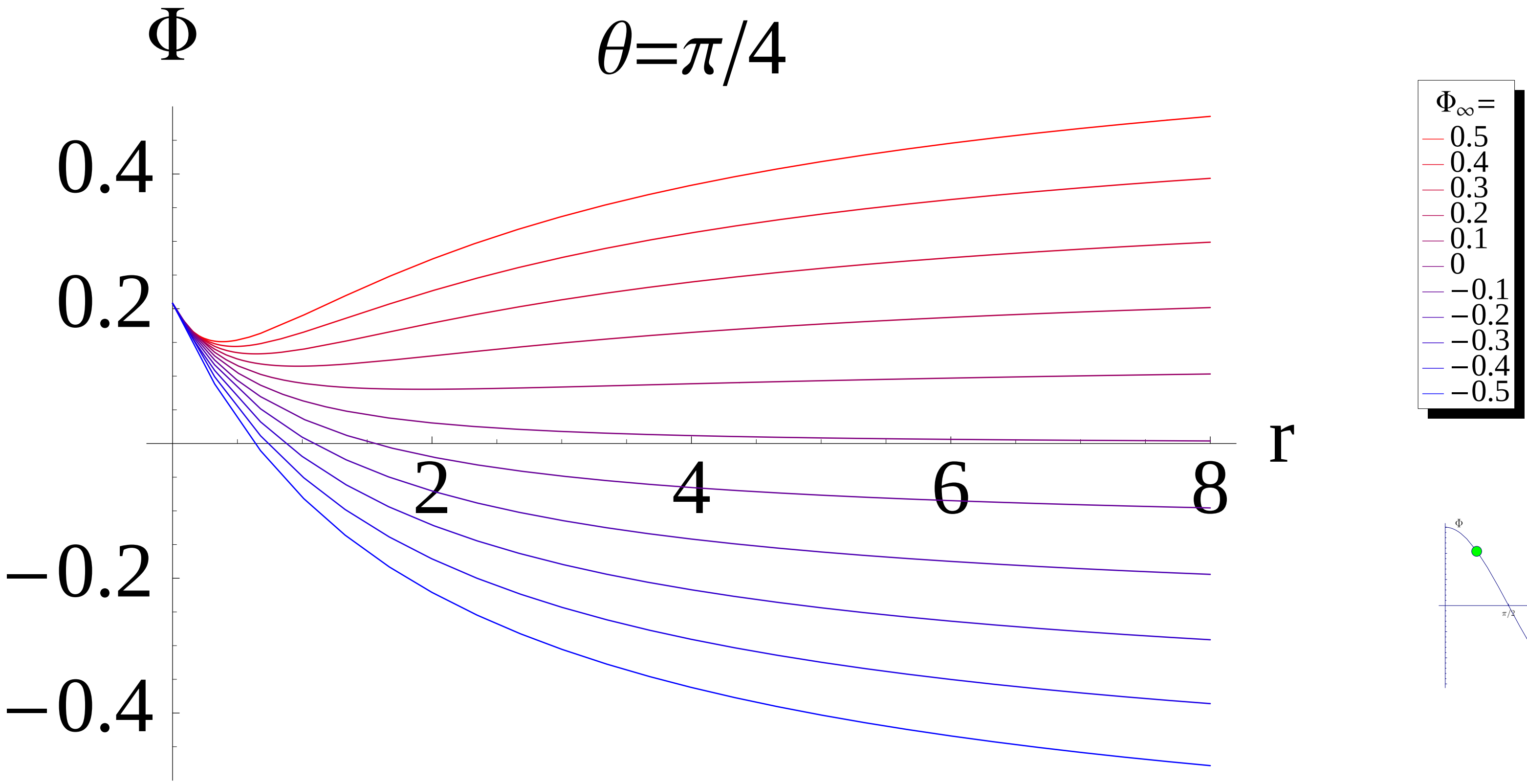
2. Ergo-free branch

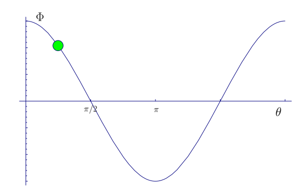
   ☞ no Ergo-sphere
   ☞ $S = 2\pi\sqrt{-J^2 - D}$
   ☞ $\mathcal{E}$ has no flat directions

Ergo-free branch

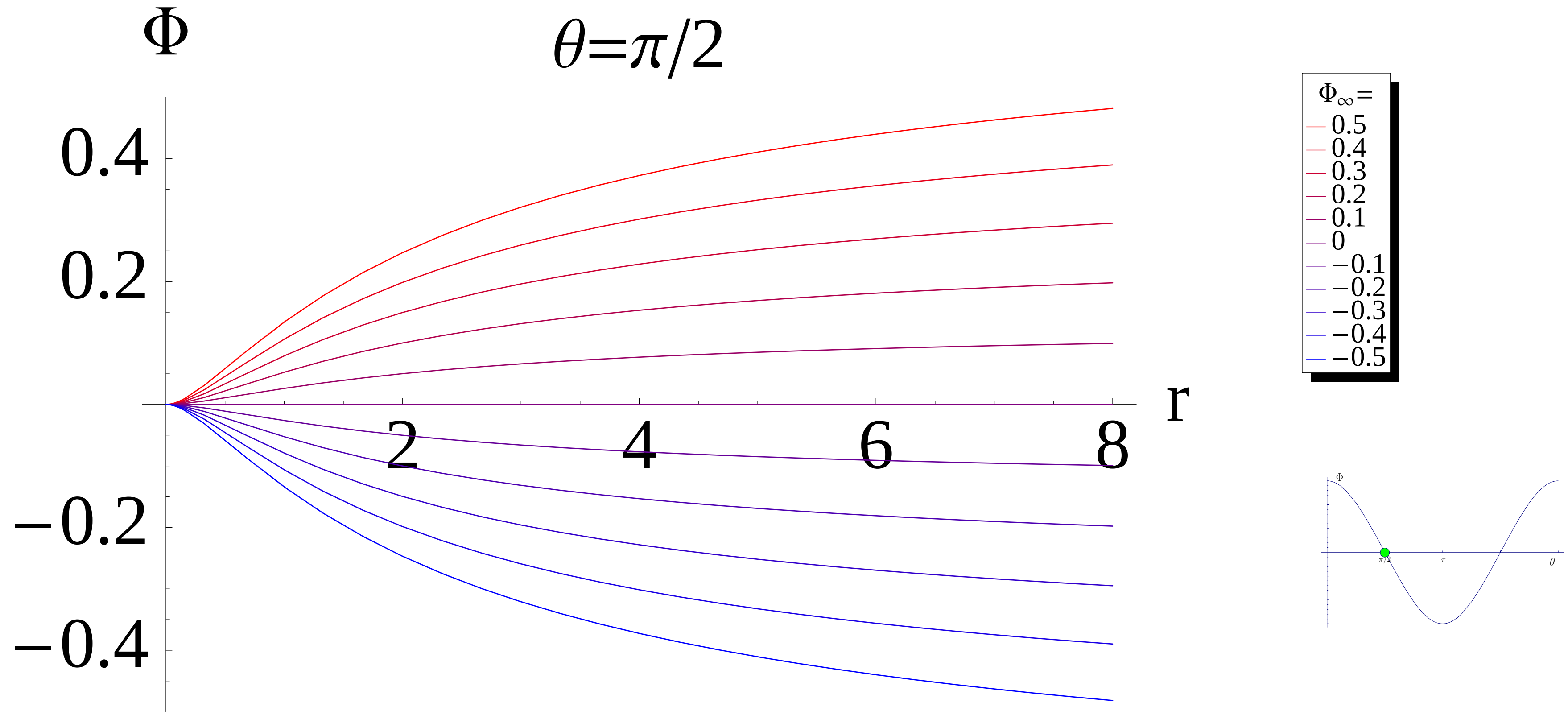


Ergo-free branch

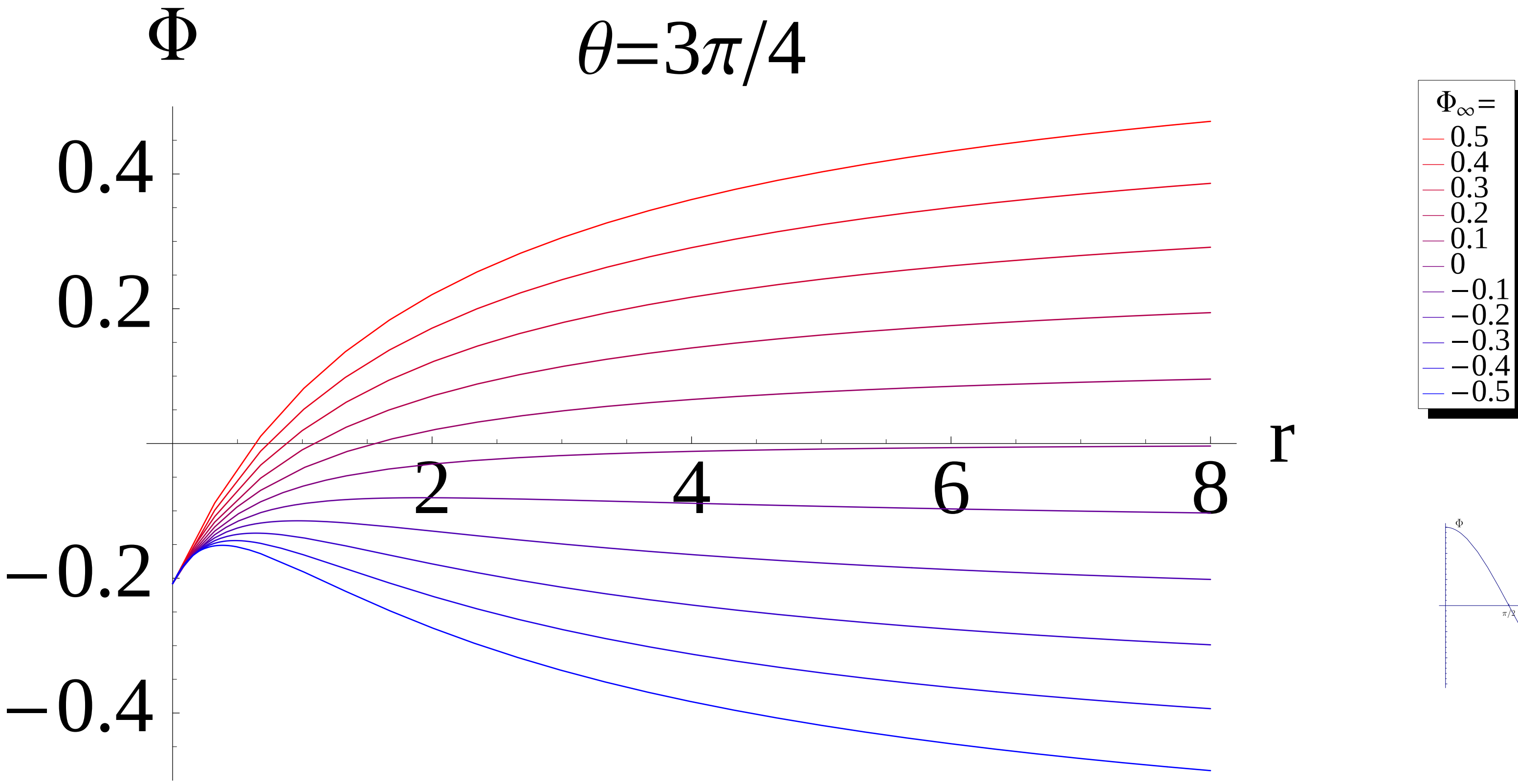


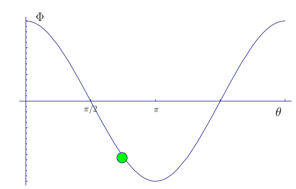


Ergo-free branch

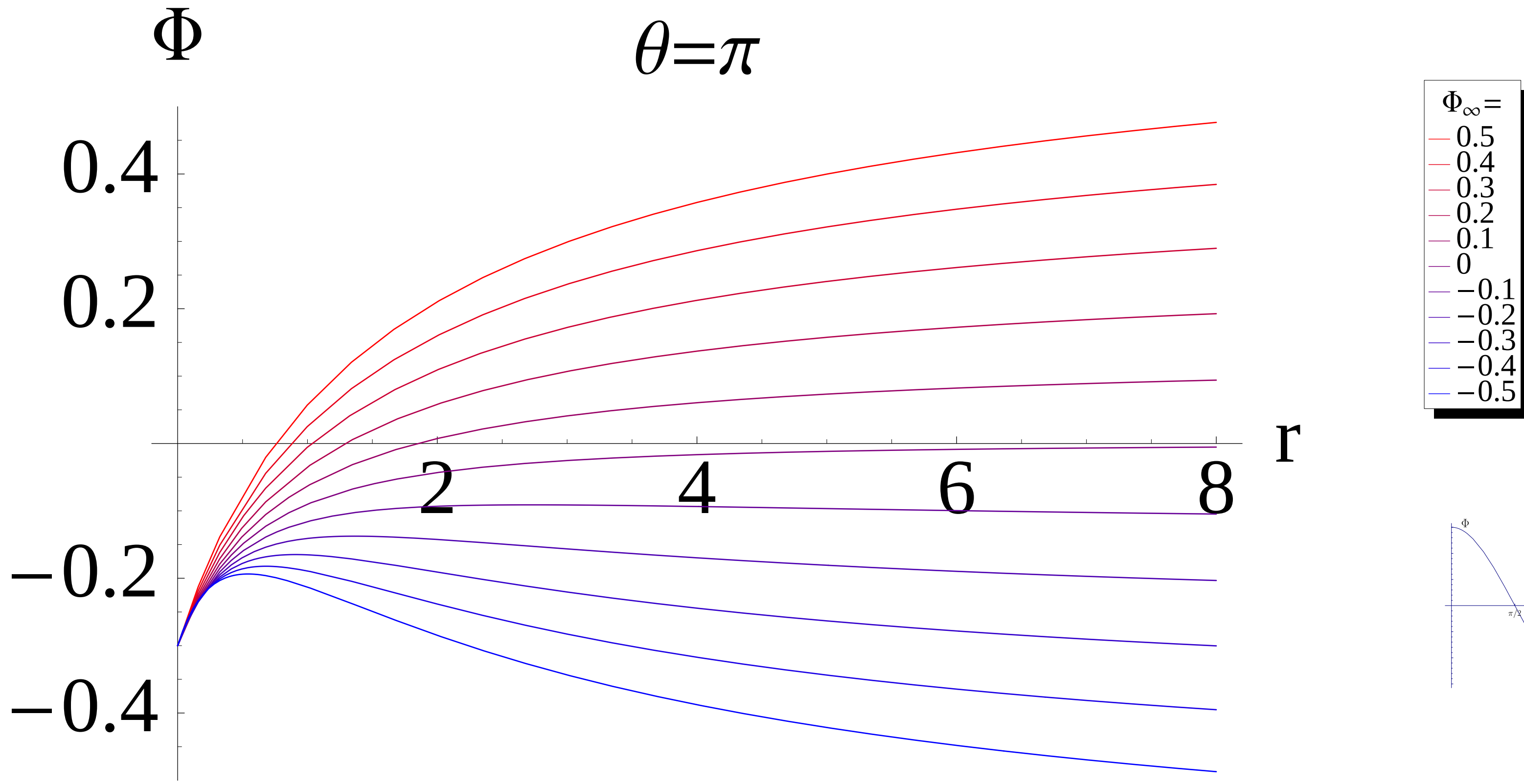


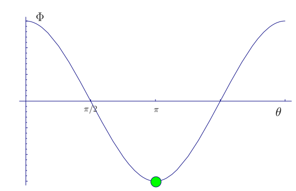
Ergo-free branch



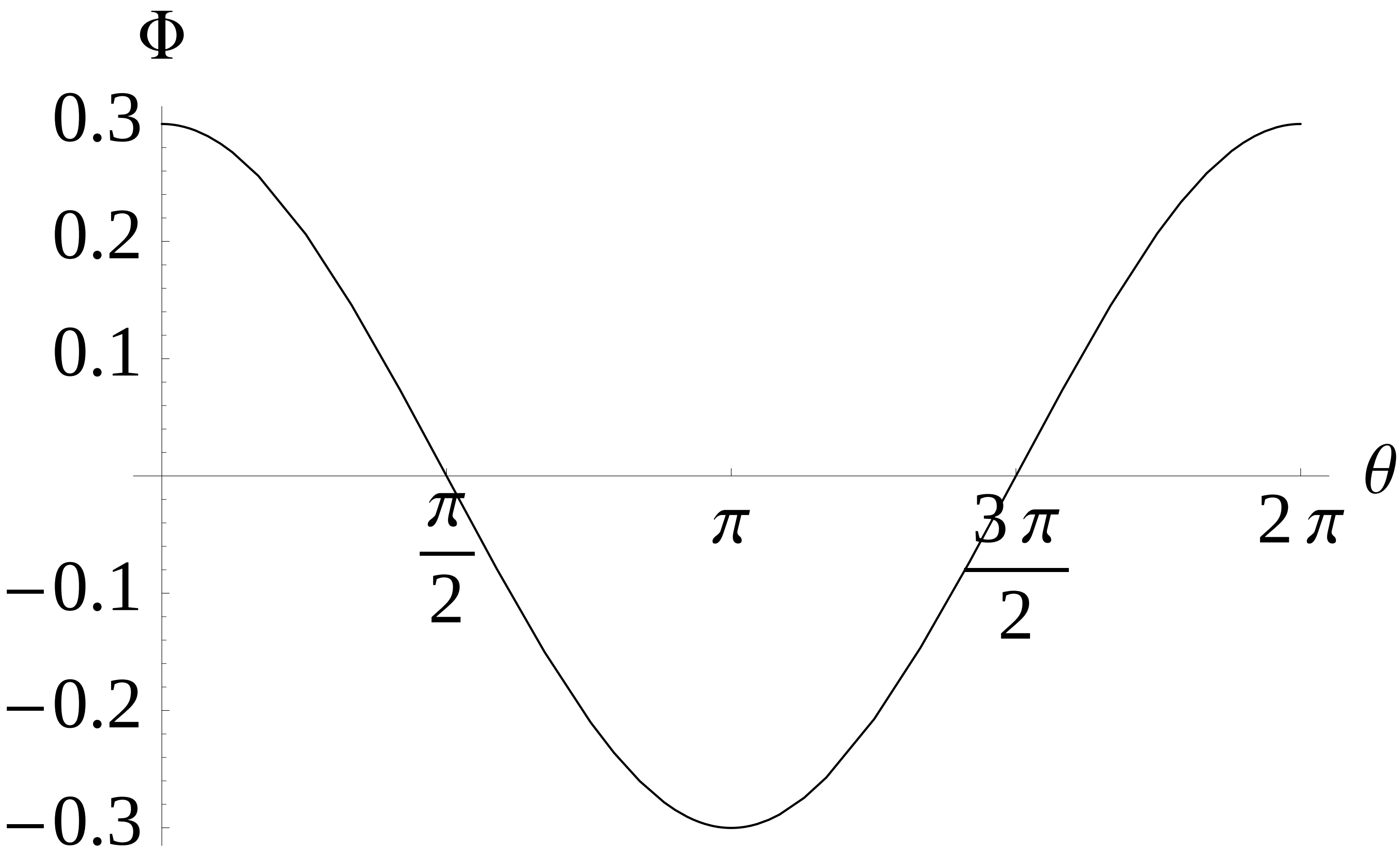


Ergo-free branch

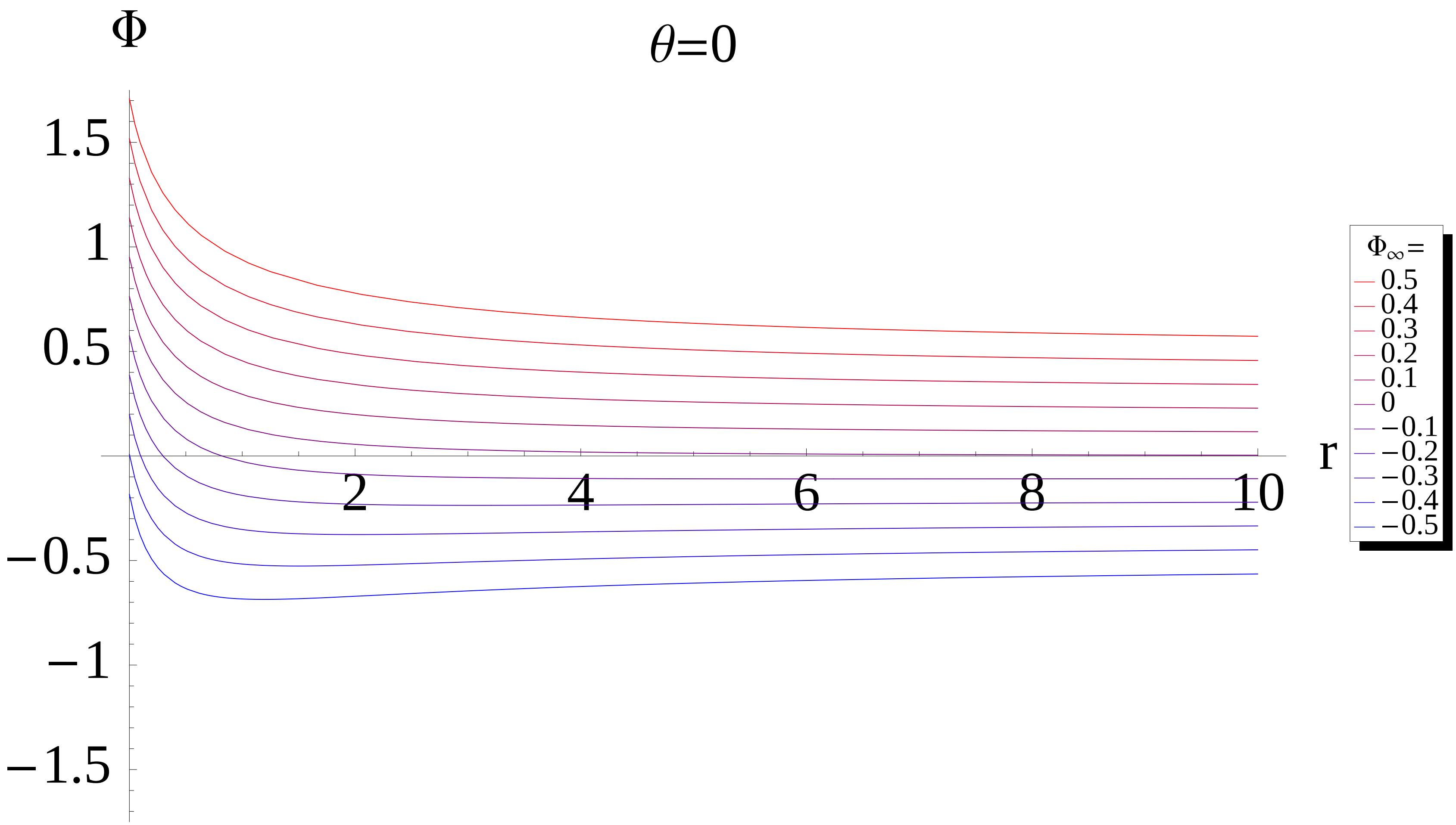




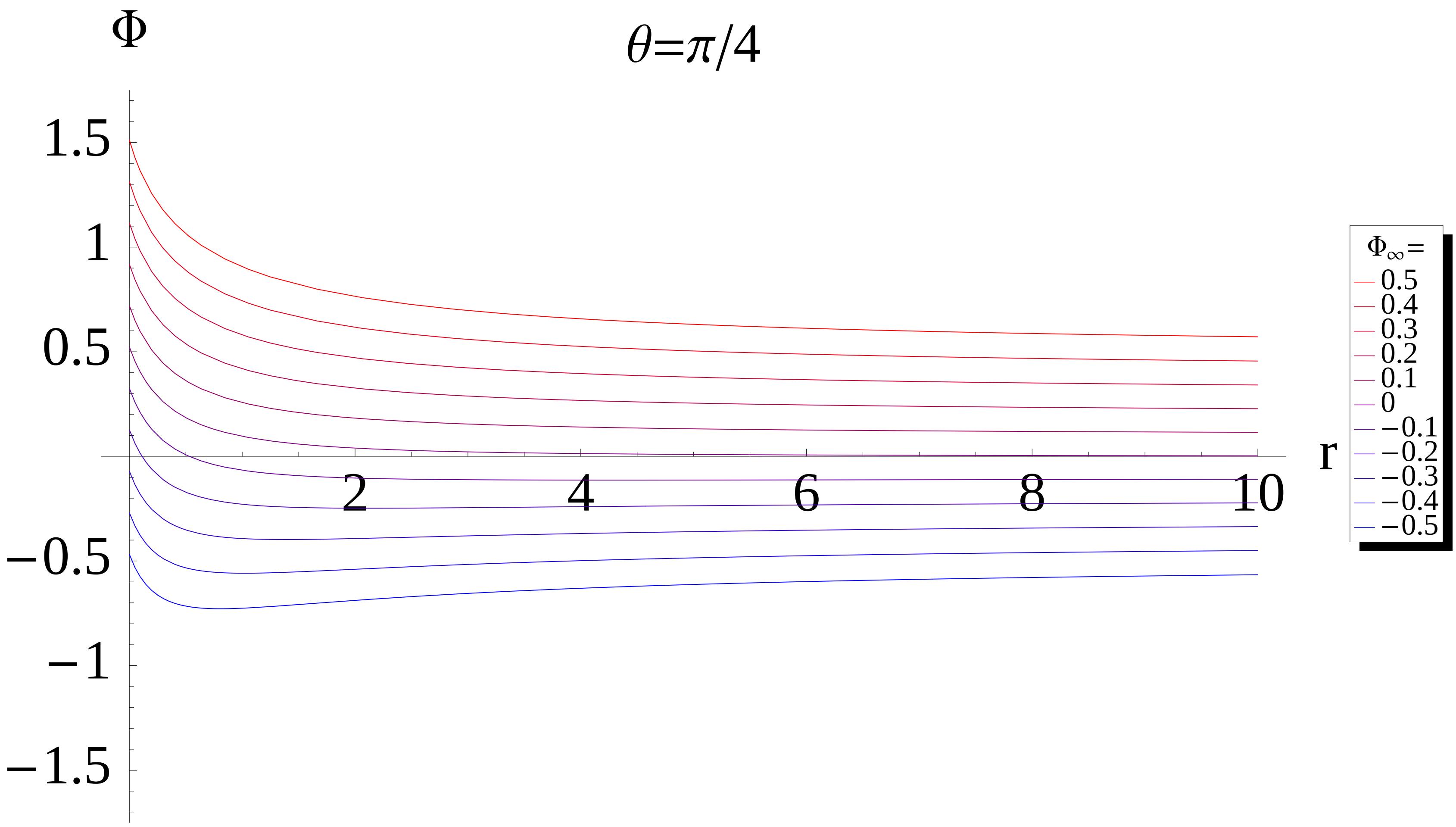
Scalar Field at Horizon



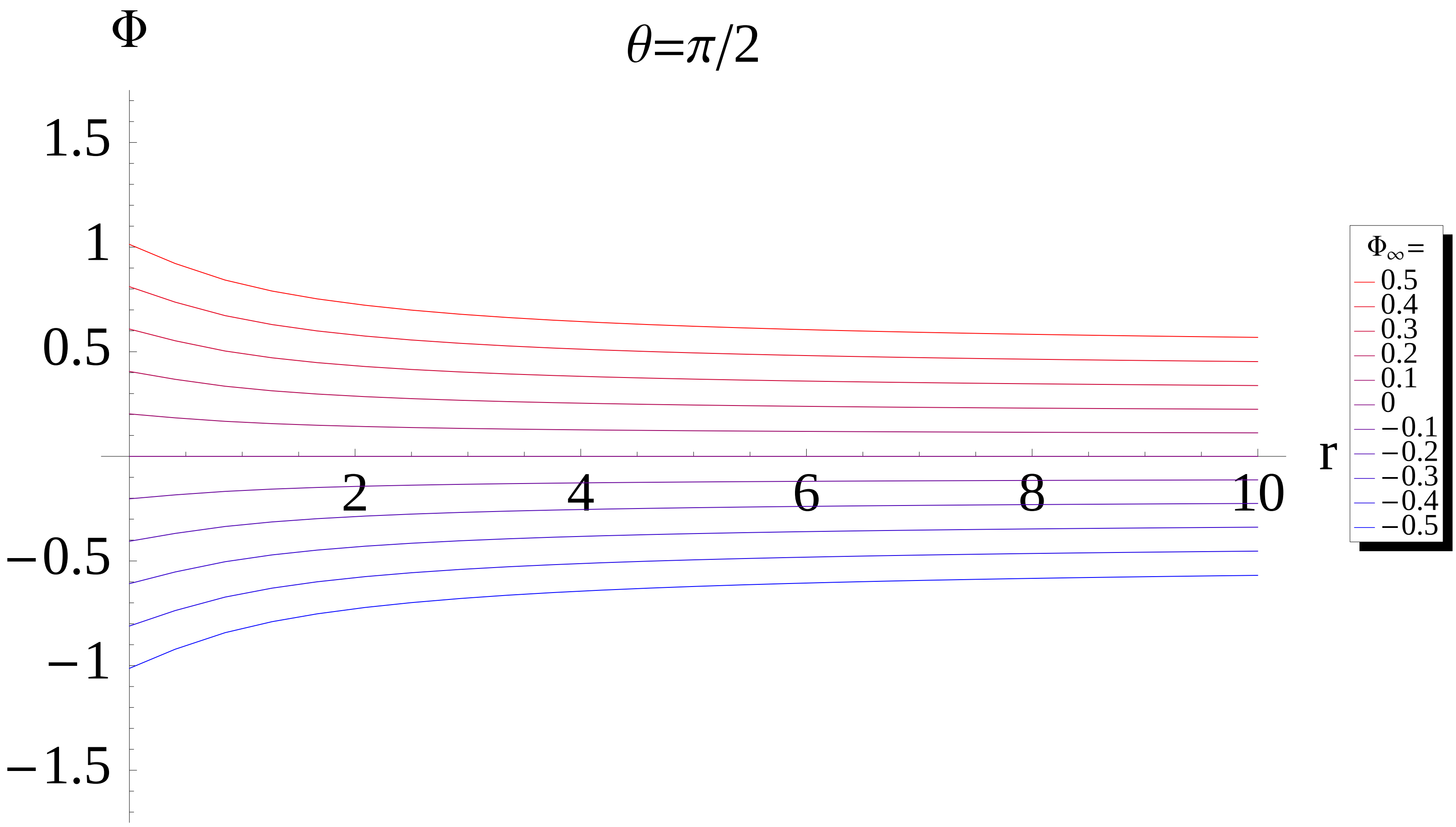
Ergo-branch



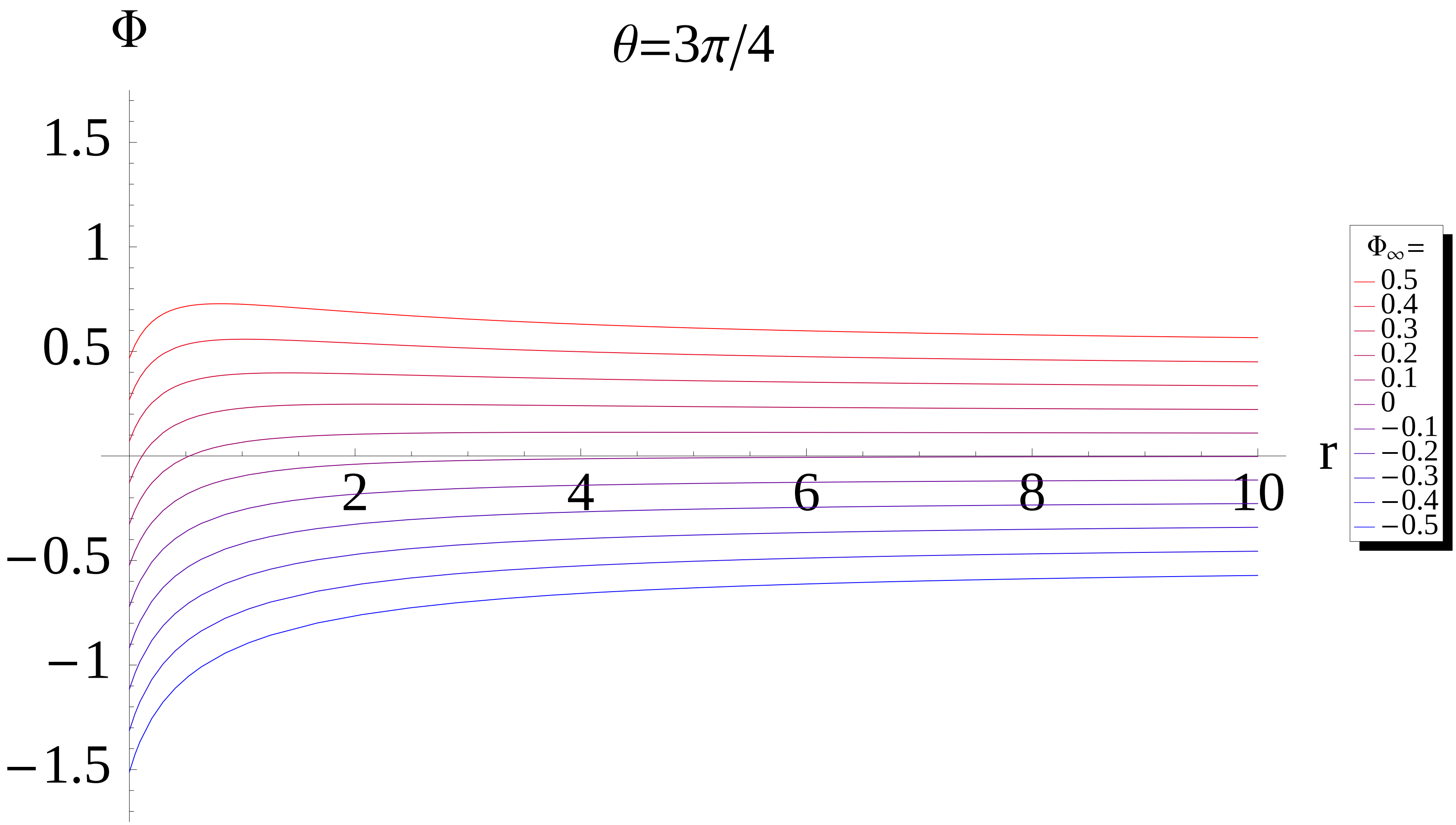
Ergo-branch



Ergo-branch

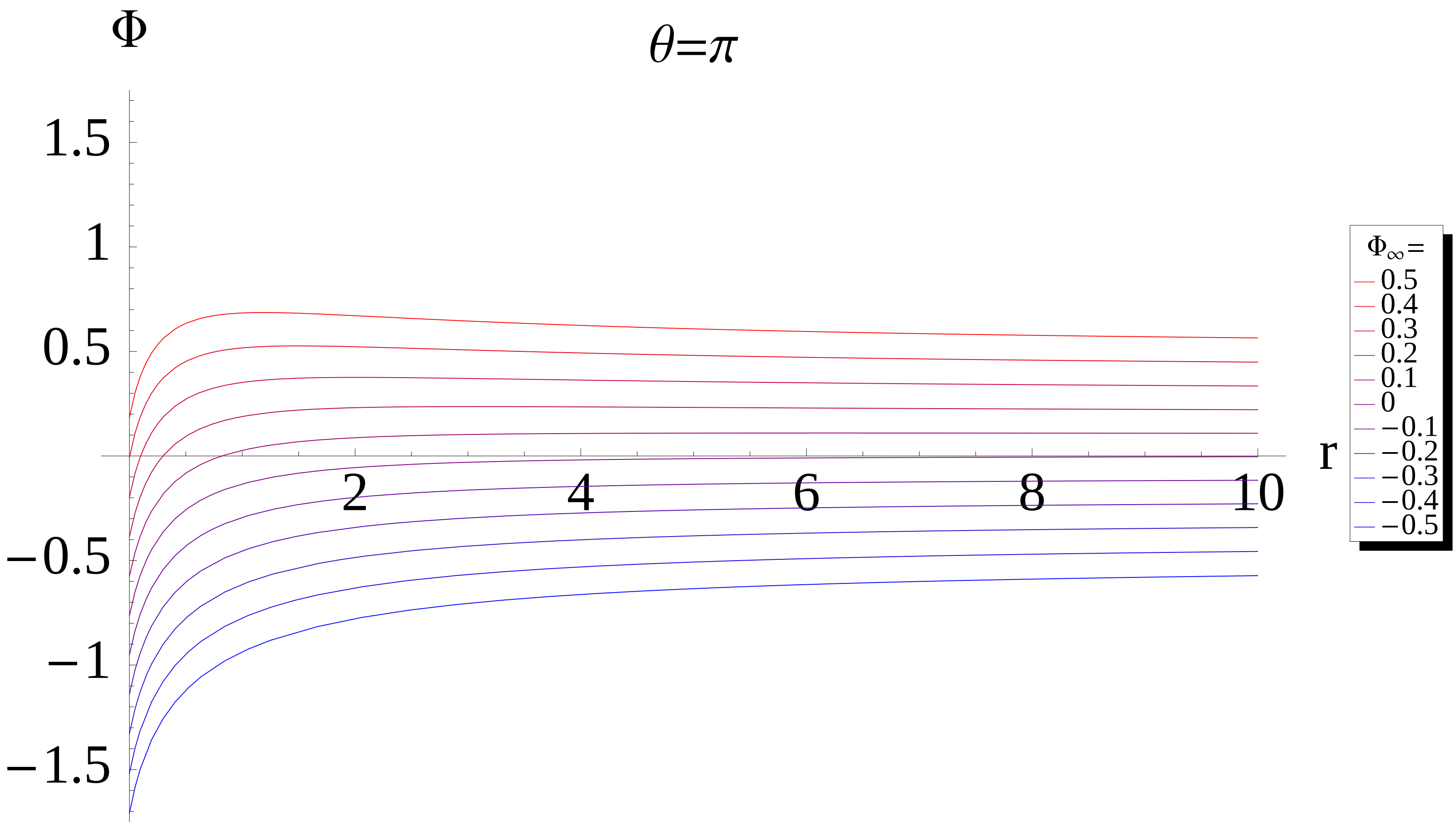


Ergo-branch

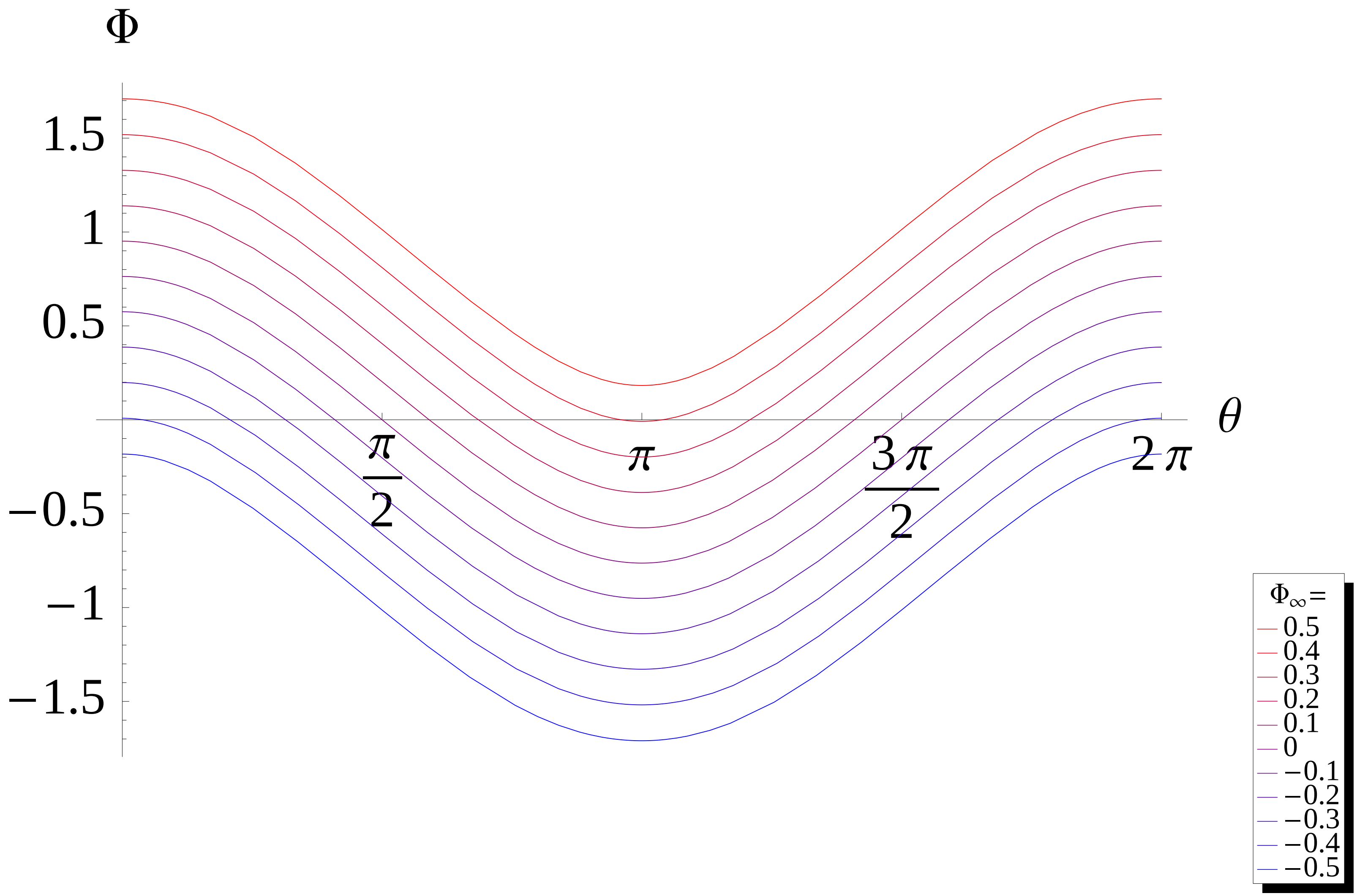


θ=3π/4
Φ
1.5
1
0.5
−0.5
−1
−1.5
2
4
6
8
10
r
Φ_∞=
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5

Ergo-branch



Scalar Field at Horizon



General Entropy function in five dimensions: $AdS_2 \times U(1)^2$

We now relax our symmetry assumptions to $SO(2,1) \times U(1)^2$:

$$
\begin{aligned}
ds^2 &= w^{-1}(\theta)\Omega^2(\theta)e^{2\Psi(\theta)}\left(-r^2dt^2 + \frac{dr^2}{r^2} + \beta^2 d\theta^2\right) \\
&\quad + w^{-1}(\theta)e^{-2\Psi(\theta)}(d\phi + e_\phi r dt)^2 \\
&\quad + w^2(\theta)(d\psi + e_0 r dt + b_0(\theta)d\phi)^2 \\
A^I &= e^I r dt + b^I(\theta)(d\phi + e_\phi r dt) + a^I(\theta)(d\psi + e_0 r dt + b_0(\theta)d\phi) \\
\phi^S &= u^S(\theta).
\end{aligned}
$$

Using dimensional reduction we can now use the four dimensional entropy function to get the five dimensional one.

Summary

☞ Discussed black ring and black hole attractors in a unified way using Sen's entropy function.

☞ Constructed the entropy function for black holes with $AdS_2 \times S^2$ horizons which can be lifted to $(AdS_2 \times S^2) \otimes U(1)$ black things.

☞ Generalised this to $SO(2,1) \times U(1) \rightarrow SO(2,1) \times U(1)^2$

☞ Attractor behaviour seems only to need the presence of an AdS_2

Puzzles and Future directions

☞ Consider higher derivative corrections

❑ (Alishahiha ; Cai,Pang; Castro,Davis,Kraus,Larsen)

☞ Consider non-extremal generalisations

☞ Further investigate the role of flat directions.

☞ What role does the ergo-sphere play ?

☞ Why are these bosonic symmetries sufficient for the attractor mechanism?

Thank you