On the Partition Funtion of Dyons in $N=4$ String Theory

based on

hep-th/0602254 with D. Jatkar & A. Sen

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Introduction & Statement of Results.

Consider String Compactification to 4d with $N=4$ susy

prototype e.g Heterotic on T^6

$\exists$ BPS states which preserve $1/2$ susy

e.g Purely Electric / purely Magnetic

$\rightarrow$ A string carrying momentum & winding on one circles of T^6

fermionic sector is in ground state.

It is easy to count the degeneracy of such states

Level matching

$$\frac{P_L^2}{2} - \frac{P_R^2}{2} = -\frac{q_e^2}{2} = -nw = -\left(\sum_{\ell, I} \ell N_\ell^I - 1\right)$$

$$d(nw) = \int_0^1 d\tau \, \frac{e^{-2\pi i \tau \frac{q_e^2}{2}}}{[\eta(\tau)]^{24}}$$

- By $SL(2,\mathbb{Z})$ degeracy of purely magnetic states

$$d\left(\frac{q_m^2}{2}\right) = \int_0^1 du \, \frac{e^{-2\pi i \frac{q_m^2}{2} u}}{[\eta(u)]^{24}}$$

- Similarly There are $\frac{1}{4}$ BPS states carrying both electric & magnetic charges (Dyons)

e.g of a configuration.

Take 2 circles of the T^6

	Electric	Magnetic
S^1	$\begin{pmatrix} -n \\ w \end{pmatrix}$	$\begin{array}{r} \text{KK monopole} \\ \text{NS 5-brane} \end{array} \begin{pmatrix} 0 \\ W \end{pmatrix}$
$\tilde{S}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \tilde{N} \\ \tilde{W} \end{pmatrix}$.

$$\frac{Q_e^2}{2} = nw \qquad \frac{Q_m^2}{2} = \tilde{N}\tilde{W} \qquad Q_e \cdot Q_m = wW$$

- The state considered is $\frac{1}{4}$ BPS unless Electric & Magnetic Charges are parallel $\quad Q_e^2 Q_m^2 = (Q_e \cdot Q_m)^2$

→ Goal is to count the degeneracy of $\frac{1}{4}$ BPS dyons in $N=4$ compactifications

For Heterotic on T^6

There is the following duality symmetry.

$$\underbrace{SO(22,6;\mathbb{Z})}_{\text{T-duality}} \times \underbrace{SL(2,\mathbb{Z})}_{\text{S-duality}}$$

Given Charges $\vec{Q}_e$, $\vec{Q}_m$: $SO(22,6;\mathbb{Z})$ vector

T-duality Invariants are

$$\frac{Q_e^2}{2}, \quad \frac{Q_m^2}{2}, \quad Q_e \cdot Q_m.$$

As T-duality is a symmetry the degeracy should be a function of

$$d\left(\frac{Q_e^2}{2}, \frac{Q_m^2}{2}, Q_e \cdot Q_m\right)$$

→ It also should be Invariant under $SL(2,\mathbb{Z})$ S-duality.

e.g $d\left(\frac{Q_e^2}{2}, \frac{Q_m^2}{2}, Q_e \cdot Q_m\right)$

$$= d\left(\frac{Q_m^2}{2}, \frac{Q_e^2}{2}, -Q_e \cdot Q_m\right)$$

→ We will Construct the following partion function which will count the degeracy.

$$\sum_{\frac{Q_e^2}{2}, \frac{Q_m^2}{2}, Q_e \cdot Q_m} d\left(\frac{Q_e^2}{2}, \frac{Q_m^2}{2}, Q_e \cdot Q_m\right) e^{2\pi i\left[\frac{Q_e^2}{2}u + \frac{Q_m^2}{2}\tau, Q_e \cdot Q_m v\right]} = \frac{1}{\Phi(\tau, u, v)}.$$

Results for CHL orbifolds

- CHL orbifolds/ reduce rank of gauge group from $U(1)^{28} \to U(1)^{28-r}$, but keep $N=4$ SUSY.
  of Heterotic

- Simplest e.g.

$N=2$: case : exchange the 2 E_8 with a $\frac{1}{2}$ shift on a circle S^1,

Duality group $SO(22-r, 6; \mathbb{Z}) \times \Gamma_1(N)$.

$$\Gamma_1(N) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad ad-bc=1.$$

$a, d = 1 \bmod N$, $c = 0 \bmod N$.

The $r = 2\left[12 - \frac{24}{N+1}\right]$

$N=1$
$r=0$

$N=2$.
$r=8$

$N=1$: unorbifolded case.

Let

$$d(Q_e, Q_m) = g\left(\frac{Q_m^2}{2}, \frac{Q_e^2}{2}, Q_e \cdot Q_m\right)$$

$$\frac{1}{\Phi_k(u,T,v)} = \sum_{\substack{n,m,p \\ m \geq -1 \\ n > -\frac{1}{N}}} e^{2\pi i (mu + nT + pv)} g(m,n,p)$$

$$k = \frac{24}{N+1} - 2.$$

$N=1$
$k = 10$
$N=2$ $k=6$.

$\Phi_k(u,T,v)$ is a modular form of weight k under a subgroup of $Sp(2,\mathbb{Z})$. Let $\Omega = \begin{pmatrix} T & v \\ v & u \end{pmatrix}_{2\times 2}$

$$\Phi_k\left((A\Omega + B)(C\Omega + D)^{-1}\right) = \left[\det(C\Omega + D)\right]^k \Phi_k(\Omega)$$

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{4\times 4} \in Sp(2,\mathbb{Z})$ ie $AB^T = BA^T$

$CD^T = DC^T$, $AD^T - BC^T = 1$

Explicitly

$$\Phi_k(u,T,v) = \frac{-1}{(i\sqrt{N})^{k+2}} \exp 2\pi i\left[\frac{T}{N} + u + v\right]$$

$$\times \prod_{r=0}^{N-1} \prod_{\substack{\ell, b \in \mathbb{Z} \\ k' \in \mathbb{Z} + \frac{r}{N}}} \left[1 - e^{2\pi i (k'T + \ell u + b v)}\right]^{\sum_{s=0}^{N-1} e^{-\frac{2\pi i \ell s}{N}} C^{(r,s)}(4k'\ell - b^2)}$$

$$(k', \ell, b) > 0 \rightarrow \begin{array}{lll} k' > 0 & \ell \geq 0 & b \in \mathbb{Z} \\ k' = 0 & \ell > 0 & b \in \mathbb{Z} \\ k' = 0 & \ell = 0 & b < \mathbb{Z} \end{array}$$

$C^{(r,s)}$: coefficients of the Twisted Elliptic genus of K3.

$$F^{(r,s)}(\tau,z) = \frac{1}{N} \mathrm{Tr}^{K3}_{RR\,\tilde{g}^r}\left[(-1)^{F_{K3}+\bar{F}_{K3}} (\tilde{g}^s)\, e^{2\pi i z F_{K3}}\, q^{L_0} \bar{q}^{\bar{L}_0}\right]$$

$0 \leq r, s \leq N-1$

$\tilde{g}$ is $\mathbb{Z}_N$ action on K3 used to construct the CHL model. on Type II A side

- It can be obtained by performing the integral

$$d(Q_e, Q_m) = \int du\, d\tau\, dv \; \frac{e^{-2i\pi\left[u\frac{Q_e^2}{2} + \tau\frac{Q_m^2}{2} + v Q_e Q_m\right]}}{\Phi_k(u,\tau,v)}$$

$\mathrm{Im}\,u = M_1$ $\quad 0 \le \mathrm{Re}\,u \le N$

$\mathrm{Im}\,\tau = M_2$ $\quad 0 \le \mathrm{Re}\,v \le 1$

$\mathrm{Im}\,v = M_3$ $\quad 0 \le \mathrm{Re}\,\tau \le 1$

- Performing a saddle point evaluation including the "1-loop term" we are led to the following minimization problem. to determine the entropy $S = \ln d(Q_e, Q_m)$ for large charges.

For $N=1$ (un orbifolded) $C(4n-b^2)$ coefficient of Elliptic genus of K3

$$F^{(r,s)}(\tau,z) = \sum_{b\in \mathbb{Z}', n} C^{(r,s)}(4n-b^2)\, q^n e^{2\pi i z b}.$$

$C^{(r,s)}$ can be explicitly computed.

$\Rightarrow$ The degeneracy $d(Q_e, Q_m)$ are Integers.
(given by the formula)

$\rightarrow$ They obey S — duality.
$\Gamma(N)$.

$\rightarrow$ For large charges The dyons become black holes.

It is of Interest to obtain the Asymptotic formula for the degeracy.

Consider

$$f(\tau_1,\tau_2) = \frac{\pi}{2\tau_2}\left|Q_e + \tau Q_m\right|^2 - \ln f^k(\tau) - \ln f^k(-\bar{\tau}) - (k+2)\ln(2\tau_2)$$

A function of τ_1, τ_2.

$$\underline{f^k(\tau) = \eta^{k+2}(\tau)\,\eta^{k+2}(N\tau)}$$

↳ Dedekind η-fn

Entropy is obtained by. 1st

$$\left.\frac{\partial f}{\partial \tau_1}\right|_{\tau_1^*, \tau_2^*} = 0, \quad \left.\frac{\partial f}{\partial \tau_2}\right|_{\tau_1^* \tau_2''} = 0.$$

$$S = f(\tau_1^*, \tau_2^*)$$

$$\approx \pi\sqrt{Q_e^2 Q_m^2 - (Q_e Q_m)^2} - (k+2)\ln\left[\frac{2\sqrt{Q_e^2 Q_m^2 - (Q_e Q_m)^2}}{Q_m^2}\right].$$

$$\tau_1 \approx -\frac{(Q_e \cdot Q_m)}{Q_m^2}, \quad \tau_2 = \frac{\sqrt{Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2}}{Q_m^2}$$

Can this be reproduced from the gravity description ?

- For these black holes "Einstein Gravity" the 2-derivative action gives

$$S = \pi \sqrt{Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2}$$

- Consider the 4-derivative Gauss - Bonnet Term in the effective action

$$\phi(\tau,\bar{\tau}) \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right]$$

$$\tau = a + iS$$
$$= \tau_1 + i\tau_2$$

τ_1: axion, τ_2: Dilaton

$$\phi(\tau,\bar{\tau}) = \frac{-1}{64\pi^2} \left[(k+2) \ln \tau_2 + \ln f^k(\tau) + \ln f^k(-\bar{\tau}) \right]$$

- Note that we are including <u>all string loop corrections</u> for ~~at~~ this term.

- $\varphi(\tau,\bar\tau)$ is evaluated by a 1-loop computation in the dual Type II description

- We now evaluate the Entropy using the <u>Entropy function formalism</u>

- We obtain the following Minimization problem

Let

$$f(\tau_1,\tau_2) = \frac{\pi}{2\tau_2}\left|Q_e + \tau Q_m\right|^2 - \ln f^k(\tau) - \ln f^k(-\bar\tau) - (k+2)\ln(2\tau_2)$$

$$f^k(\tau) = \eta^{k+2}(\tau)\,\eta^{k+2}(N\tau)$$

Entropy is obtained by

$$\left.\frac{\partial f}{\partial \tau_1}\right|_{\tau^*} = 0 \qquad \left.\frac{\partial f}{\partial \tau_2}\right|_{\tau^*} = 0$$

$$S = f(\tau_1^*, \tau_2^*)$$

→ This is the <u>same</u> minimization problem from direct counting.

<u>Comments</u>

- There is some evidence that one has taken into account all the correction at the linear order in α' (Sahoo & Sen)

. Taking into account all 4-derivative tree level corrections for susy black holes and evaluating the entropy agree with.

→ Including only the Gauss-Bonnet Term

→ Including all terms related to the susy completion of the R^2 term

→ Arguments based on the assumption of AdS_3 near horizon geometry.

→ Suggests at least to tree level we ~~have~~ includeding all 4-derivative terms. $\equiv$ Including only Gauss-Bonnet Term

· Generalization to Type II $N=4$ orbifolds

→ We have obtained the degeneracy formula for dyons for Type II $N=4$ orbifolds.

e.g Type II A on $T^4 \times S^1 \times \tilde{S}^1 / \mathbb{Z}_2$

$\mathbb{Z}_2'$ is $(-1)^{F_L}$ projection together with ½ shift on S^1

→ The degeneracy formula has the required T & S symmetries

→ The asymptotic formula for the degeneracy for large charges agree with that obtained by including the Gauss-Bonnet Term

- This fact is suprising since There is no Tree level 'R^2' term in these theories and the counting of purely Electric States (Small black holes) does not agree with that obtained by including the 'R^2' term.

We will now outline the counting of the dyons for the simplest e.g Heterotic on T^6.

Strategy for Counting Dyons

- Consider 'D1-D5' system on Type IIB on $K3 \times S^1$ (5-dimension)

- Use 4D-5D connection to obtain a configuration in 4-D

- Perform a set of dualities to obtain the dyon configuration in Heterotic string on T^6

- Count Dyons by counting the degeneracy of the CFT description of the D1-D5 system

- Consider II B string theory on $K3 \times S^1$

$$\underset{}{0\ 1\ 2\ 3\ 4}\ \ \underset{S^1}{\underset{\uparrow}{5}}\ \ \underbrace{6\ 7\ 8\ 9}_{K3}$$

Q_5: D5 - branes wrapped on $K3 \times S^1$

Q_1 : D1 - branes wrapped on S^1

$-n$: units of momentum along S^1

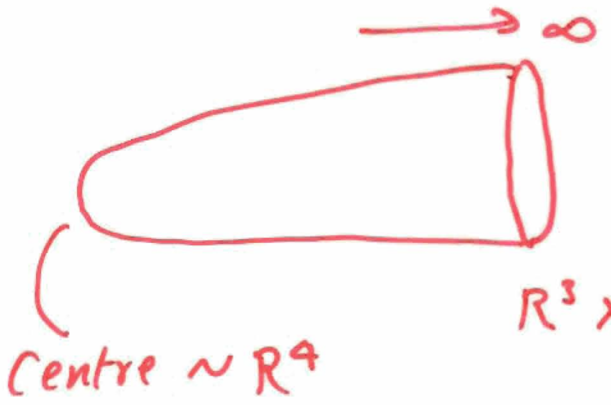
. $J_1 + J_2 = J$ units of angular momentum in the (1-2) & (3-4) plane.

J sits in $U(1)_L$ of $SU_L(2) \times SU(2)_R$ of (1 2 3 4).

Net D1- brane charge $Q_1 - Q_5$

- Place the system at the centre of a Taub-Nut space.

$$\underbrace{0\ 1\ 2\ 3\ 4}_{TN}\quad \underset{S^1}{5}\quad \underbrace{6\ 7\ 8\ 9}_{K3}$$



isometry of TN
||
coincides with angular direction φ rotation of BH

View the system at ∞

$\to$ 4D system $\tilde{S}^1$ compact direction

Charges in 4D

$\underline{Q5}$: D5-branes on $K3 \times S^1$

$\underline{Q_1 - Q_5}$: D1-branes on S^1

$\underline{-n}$: units of momentum on S^1

$\underline{J}$: units of momentum on $\tilde{S}^1$

$\underline{1}$: unit of Kaluza-Klein monopole on $\tilde{S}^1$

Perform the following dualities.

S-duality in Type II B.
$\rightarrow$ T-duality on $\tilde{S}^1$ to II A
$\rightarrow$ string-string duality to Heterotic on T^6

Final Configuration

	Electric	Magnetic
S^1	$\begin{pmatrix} -n \\ 1 \end{pmatrix}$ — momentum, → winding	NS 5-branes on $T^4 \times \tilde{S}^1$ $\begin{pmatrix} 0 \\ J \end{pmatrix}$ → KK monopoles
$\tilde{S}^1$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	KK monopoles, NS 5-branes on $T^4 \times S^1$ $\begin{pmatrix} Q_5 \\ Q_1 - Q_5 \end{pmatrix}$.

1 unit of KK monopole (TN) $\longrightarrow$ 1 unit of fundamental string in Heterotic

Invariants.

$$\frac{Q_e^2}{2} = n \quad : \quad \frac{Q_m^2}{2} = (Q_1 - Q_5)\cdot Q_5 \; ; \quad Q_e \cdot Q_m = J$$

To evaluate the degeneracy of the Dyons we evaluate the degeneracy in 5D of the D1- D5 system.

The Degeracy arises from 3- sources

$$H^{I} \otimes H^{II} \otimes H^{III}$$

H^{I}: Intrinsic modes of Taub Nut

H^{II}: CM motion of D1-D5 system in Taub-Nut

H^{III}: Degrees of freedom of D1- D5 system

- We have assumed systems are decoupled

→ This ~~is bare~~ holds in all models studied

→ The Mass of Taub-Nut is large

→ Final Answer has 'good' modular properties.

We will count the degeneracy of each of the Hilbert spaces

· H^I : Intrinsic modes of the Taub-Nut

$$\underset{}{0\ 1\ 2\ 3\ \underset{\tilde{S}^1}{4}}\quad \underset{S^1}{5}\ \underbrace{6\ 7\ 8\ 9}_{K3}$$

. Taub-Nut can carry momentum $-\ell$ on S^1

→ we have seen the Taub-Nut is dual to (Chain as before) to a Heterotic string on S^1 with momentum $-\ell$

$$Z_{TN} = \sum d_{TN}(\ell)\, e^{-2\pi i \ell u}$$

$$= 16\, e^{-2\pi i u} \prod_{n=1}^{\infty} \left(1 - e^{+2\pi i n u}\right)^{-24}$$

$$= \frac{16}{[\eta(u)]^{24}}$$

due to fermionic zero-modes

Note : • We are Tensoring basic degrees of the super multiplet.
X No: of super multiplets (Bosons - fermions).

• Right movers are in ground state

This counting can be done directly in Type II by collective coordinates of T.N space.

• T.N has the world volume action identical to the Heterotic string. (K3 → small).

Totally 24 left movers } Bosons
8 right movers
fermions

Counting is identical

H^{II}: Degeneracy due to centre of mass motion of D1-D5 system in Taub-Nut.

Eeffective Theory: low energy dynamics of centre of Mass described by 1+1 theory of 4 Bosons & 4 Majorana fermion with TN Target space

World sheet coordinate σ along S^1

Looking for states which carry $L_0 = \ell$ & j_0 on $\tilde{S}$ in $U_L(1)$

Then $\sum_{\ell, j} d_{cm}(\ell_0, j_0)$

Fermions: Right movers in ground state.

- One can show left moving fermions do not carry $U_L(1)$ and are free.

$$Z_{\text{free fermions}} = \mathrm{Tr}_{\text{free fermion}}\left[(-1)^{F+\bar{F}}\, e^{2\pi i(uL_0+2\pi i j_0)}\right]$$

$$= \underset{\substack{\| \\ 2^2}}{4} \prod_{n=1}^{\infty}\left(1 - e^{2\pi i n u}\right)^4$$

Bosons: Zero modes.

The No: of zero modes (bosonic) has bee computed by POPE

→ No: of boundstate $= j_0$ for $j_0 > 0$.

→ for Radius of Taub-Nut space large.

From here

$$Z_{\text{zero modes}} = \sum_{j=1}^{\infty} j \, e^{2\pi i v j} = \frac{e^{2\pi i V}}{(1-e^{2\pi i v})^2}$$

$$= \frac{e^{-2\pi i v}}{(1-e^{-2\pi i v})^2}$$

Now $Z_{\text{Bosonic oscillators}}$.

$$= \mathrm{Tr}_{\text{oscillators}}\left[(-1)^{F+\tilde{F}} e^{2\pi i u L_0} e^{2\pi i j V}\right]$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-e^{2\pi i u n + 2\pi i v})^2 (1-e^{2\pi i n v - 2\pi i v})^2}$$

→ Evaluating the partition fn near $r=0$ ⟶ flat space.

Putting it Together

$$Z_{CM} = Z_{\text{free fermion}} \, Z_{\substack{\text{bosonic} \\ \text{zero modes}}} \, Z_{\substack{\text{bosonic} \\ \text{oscillator}}}$$

$$= 4\, e^{-2\pi i V} \frac{1}{(1-e^{2\pi i v})^2} \prod_{n=1}^{\infty} \frac{(1-e^{-2\pi i n u})^4}{(1-e^{2\pi i n u + 2\pi i V})^2 (1-e^{2\pi i n u - 2\pi i V})^2}$$

H^{III} The No. of states of D1-D5 system.

Low Energy description of D1/D5 system

is CFT with

$$S^P(K3) = \frac{(K3)^P}{S_P} \qquad \text{Vafa}$$

$$P = Q_5(Q_1 - Q_5) + 1 > 0$$

One needs to evaluate.

$$\mathrm{Tr}_{S^P(K3)}\left[(-1)^{F+\tilde{F}} e^{2\pi i j F} q^{L_0} \bar{q}^{L_0}\right]$$

$$= \sum_{n,j} h_p(n,j)\, e^{2\pi i (nu + vj)}$$

It is infact easier to evaluate

$$\sum_{P,n,j} h_p(n,j)\, e^{+2\pi i (nu + vj + T(P-1))}$$

$$= e^{-2\pi i T} \prod_{\substack{P>0 \\ l\geq 0,\, j\in\mathbb{Z}}} \left(1 - e^{+2\pi i (lu + PT + jv)}\right)^{-C(4Pl - j^2)}$$

$C(4Pl - j^2) \rightarrow$ coefficients of elliptic genus of K3.

We Almost have Φ_{10} except $p=0$

Lets put the partition fn together

$$Z = Z_{TN} \cdot Z_{CM} \cdot Z_{D1/D5}$$

$$= 16 \times 4 \; e^{-2\pi i(T+U+V)} \times \prod \left(1 - e^{+2\pi i(mU+pT+bV)}\right)^{-c(4mp-b^2)}$$

$16 \times 4 = 64 = 2^6$

$\frac{1}{4}$ BPS state multiplicity

$p>0 \quad \ell>0 \quad b \in \mathbb{Z}$

$p=0 \quad \ell>0 \quad b \in \mathbb{Z}$

$p>0 \quad \ell=0 \quad b \in \mathbb{Z}$

$p=0 \quad \ell=0 \quad b<0$

To obtain this form $Z_{TN} \cdot Z_{CM}$ arrange themselves nicely since

$c(0) = 20$

$c(1) = 2$

$c(-1) = 2$.

Now we identify the quantum number by the duality chain

$$P = Q_5(Q_1 - Q_5) + 1 = \frac{Q_m^2}{2} + 1$$

$$\frac{Q_e^2}{2} = n \qquad Q_e \cdot Q_m = j$$

$$\therefore \frac{1}{\Phi_{10}(u,\tau,v)} = \sum h\left(\frac{Q_m^2}{2}+1, \frac{Q_e^2}{2}, Q_e \cdot Q_m\right) e^{2\pi i\left(\frac{Q_e^2}{2}u + \frac{Q_m^2}{2}\tau + Q_e \cdot Q_m\right)}$$

$$\therefore d\left(\frac{Q_m^2}{2}, \frac{Q_e^2}{2}, Q_e \cdot Q_m\right).$$

$$= \int du\, d\tau\, dv \frac{e^{-i\pi\left[Q_e^2 u + Q_m^2 \tau + 2v Q_e Q_m\right]}}{\Phi_{10}(u,\tau,v)}.$$

- The generalization to CHL orbifolds & Type II orbifolds work similarly
- In all cases one can obtain the partition function of Dyons as

$$Z_{TN} \cdot Z_{CM g D1/D5} \times Z_{D1/D5}$$

The Twisted elliptic genus coefficient for K3 in each case is different

but : $Z_{TN} \cdot Z_{CM g D1/D5}$

appropriately complete $Z_{D1/D5}$ to a good modular form on $Sp(2, \mathbb{Z})$

We have studied a class of black holes which admit a microscopic description.

The entropy agrees with the gravity description not just in the leading approximation but to the next leading order.