R^2 Corrections for Extremal Black Holes

Based on

M. A., hep-th/0703099,

Also

A. Castro, J.L. Davis, P. Kraus, F. Larsen, hep-th/0703087

Closely related papers

A. Castro, J.L. Davis, P. Kraus, F. Larsen, hep-th/0702072, arXiv:0705.1847 [hep-th]

• One of the goal in theoretical physics is to understand the microscopic origin and the meaning of the black hole entropy.

• From macroscopic point of view the black hole entropy is given in the leading order by Bekenstein-Hawking formula

$$S_{BH} = \frac{A}{4G}$$

"One of the surprising feature of this formula is its universality"

• It applies to all kinds of black holes with all charges, shapes and rotations, even to black string

• Following Boltzmann we would like to think about this entropy as something which counts the number of microsates of a system

In general it is difficult to understand this question, though there are several examples in the context of string theory where we can understand the entropy.

The question is how to find the system?

The procedure

1. Identify a system and its degrees of freedom

2. Identify the corresponding ensemble

3. Compute the statistical entropy $S_{St.}$

Now the question is

$$\text{If } S_{BH} = S_{St} \text{ ?}$$

It is important to note that a priori it is not obvious which definition of the statistical entropy should be compared with the macroscopic entropy

This equality has two sides: S_{BH} and S_{St}

S_{BH}: One needs also to develop a way to compute the entropy when higher order corrections to the leading order action are also taken into account.

It is known that with these corrections we have deviation from area law and in fact the entropy can be given by Wald formula for entropy (R.M. Wald, gr-qc/9307038)

In recent years our understanding of corrections to the black hole entropy has increased considerably. In a gravitational theory using the Wald entropy formula one can find the contribution of higher order corrections to the tree level Bekenstein-Hawking area law formula and then compare it with microscopic description of entropy coming from microstate counting in string theory.

How to compute the entropy using Wald formula?

Following Wald the entropy can be interpreted as the Noether charge. Under some assumptions the Wald formula for entropy is given by (R.M. Wald, gr-qc/9307038)

$$S = 2\pi \int_{Hor} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}$$

$\epsilon_{\mu\nu}$ is binormal to the horizon normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$

In general it is also difficult to compute the entropy using the Wald formula. Not only we need the higher order corrections, but we need the corrected geometry too

For extremal black it is a way to compute the black hole entropy : Sen entropy function formalism (A.Sen, hep-th/0506177)

In four dimensions an extremal back hole is defined as a black hole whose near horizon geometry is $AdS_2 \times M_2$

Entropy function formalism

• Assume that the Lagrangian density can be expressed in terms of gauge invariant field strengths.

• Using the near horizon symmetry we consider the following ansatz for near horizon geometry

$$
\begin{aligned}
ds^2 &= v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \\
X_I &= u_I = \text{constant} \\
F^a_{tr} &= e^a, \qquad F^\alpha_{\theta\phi} = \frac{p^\alpha}{2\pi}\sin\theta
\end{aligned}
$$

• Let's denote by $f(\vec{u},\vec{v},\vec{e},\vec{p})$ the Lagrangian density evaluated for the above geometry

$$
f(\vec{u},\vec{v},\vec{e},\vec{p}) = \int d\theta\, d\phi \sqrt{-\det g}\,\mathcal{L}
$$

• Define the entropy function as $\mathcal{E} = 2\pi(\vec{e} \cdot \vec{q} - f)$

• The parameters are fixed by extremaizing the entropy function with respect to them

$$\frac{\partial \mathcal{E}}{\partial v_i} = \frac{\partial \mathcal{E}}{\partial u_I} = \frac{\partial \mathcal{E}}{\partial e^a} = 0$$

• The entropy is given by

$$S = \mathcal{E}|_{extr.}$$

Explicit example

Statistical entropy

- Consider Heterotic string theory compactified on $T^5 \times S^1$.

- Consider a fundamental string wrapping w times along S^1 and carrying momentum n

- One has $N_L - N_R = 1 + nw$

- For SUSY ($n > 0$) one may set $N_R = 0$ and therefore the partition function is given by $Z(q) = \eta(q)^{-24}$

- The degeneracy is given by $Z(q) = q^{-1} \sum d_{N-1} q^N$

- For large N one has

$$d_N \sim N^{-27/4} \exp(4\pi\sqrt{N})$$

- The naive statistical entropy is given by

$$S_{St} = \ln d_N = 4\pi\sqrt{nw} - \frac{27}{2}\ln(\sqrt{nw})$$

Macroscopic entropy

- At leading order the action is give by

$$S = \frac{1}{32\pi}\int d^4x\sqrt{-\det g}\, e^{-2\phi}\Big[R + 4\partial_\mu\phi\partial^\mu\phi - X^{-2}\partial_\mu X\partial^\mu X - X^2(F^1)^2 - X^{-2}(F^2)^2\Big]$$

- Using the near horizon geometry ansatz the function f at leading reads (A.Sen, hep-th/0508042)

$$f = \frac{v_1 v_2 u_1}{8}\left[-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2u_2^2 e_1^2}{v_1^2} + \frac{2e_2^2}{u_2^2 v_1^2}\right]$$

- At leading order we get $S_{BH} = 0$

- At R^2 level we have

$$\Delta\mathcal{L} = A\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$

with

$$A = -\frac{3}{16\pi^2}\ln(2e^{-2\phi}\eta(e^{-2\phi})^4) \approx \frac{1}{16\pi}e^{-2\phi} + \cdots$$

- From first term we find $\Delta f = -2u_1$ and therefore we get

$$S_{BH} = 4\pi\sqrt{nw}$$

- Taking other terms we get

$$S_{BH} = 4\pi\sqrt{nw} - 12\ln\sqrt{nw}$$

which disagrees with S_{St}

One needs to be careful about the definition of the entropy. If we had used different ensemble, we would have gotten another answer.

Consider a grand canonical ensemble and a chemical potential μ conjugate to nw . So the partition function is

$$e^{\mathcal{F}(\mu)} = \sum_N d_{N-1} e^{-\mu(N-1)}$$

The entropy is $S_{st} = \mathcal{F}(\mu) + \mu\, nw$ which is going to be the same as S_{BH} (A.Sen. hep-th/0411255)

This method works for both supersymmetric and non-supersymmetric.

- In non-supersymmetric case using using the equations for parameters one gets

$$\mathcal{E} = V_{eff}(\phi)$$

To get stable solution

$$\frac{\partial \mathcal{E}}{\partial \phi} = 0, \qquad \frac{\partial^2 \mathcal{E}}{\partial \phi^2} > 0$$

See for example:

K. Goldestein, N. Iizuka, R.P. Jena and S.P. Trivedi, hep-th/0507097

A. Sen, hep-th/0506177

M.A. and H. Ebrahim, hep-th/0601016

• One may also consider $\mathcal{N}=2$ in for dimensions which could be obtained from type IIA string theory compactified on a Calabi-Yau. The theory can be described in terms of special geometry where prepotential plays an essential role.

Higher order corrections are given by one-loop correction to the prepotential

In this case for the SUSY solution one gets

$$f(e^I, p^I) = -2\text{Im}\left(\frac{4}{\omega}F(X^I, \omega)\right)$$

with

$$X^I = \frac{\omega}{8}(e^I + ip^I)$$

See

B. Sahoo and A. Sen, hep-th/0603149

M.A. and H. Ebrahim, hep-th/0605279

G.L. Cardoso, B. de Wit and S. Mahapatra, hep-th/0612225

Five dimensioal Black hole

Consider five dimensional $\mathcal{N}=2$ supergravity. This can be obtained from M-theory compactified on Calabi-Yau threefold

- The five dimensional theory contains the gravity multiple coupled to $h_{(1,1)}-1$ vector multiplets

- The $(h_{(1,1)}-1)$ dimensional space of the scalars can be regarded as a hypersurface of a $h_{(1,1)}$ dimensional manifold. The hypersurface is defined by

$$\frac{1}{6}C_{IJK}X^I X^J X^K = X^I X_I = 1, \quad I,J,K = 1,\cdots,h_{(1,1)}$$

where

- C_{IJK}: intersection number of C.Y.

- X^I: the volumes of 2-cycles

- X_I: the volumes of 4-cycles

- The bosonic part of the Lagrangian is

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{4}G_{IJ}F^I_{\mu\nu}F^{\mu\nu\ J} - \frac{1}{2}\mathcal{G}_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \frac{g^{-1}}{48}c^{\mu\nu\rho\sigma\lambda}C_{IJK}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda$$

where $(X = X(\phi))$

$$G_{IJ} = -\frac{1}{2}\partial_I\partial_J \log\nu|_{\nu=1}\ , \qquad \mathcal{G}_{ij} = G_{IJ}\ \partial_i X^I \partial_j X^J|_{\nu=1}$$

There are several known supersymmetric solutions for this model including

Black hole, Black string, Black ring, Rotating black hole

The near horizon geometry could be either $AdS_2 \times S^3$ or $AdS_3 \times S^2$ or some deformation of them.

- For a black hole with near horizon geometry $AdS_2 \times S^3$ we use the entropy function formalism

- For the case with $AdS_3 \times S^2$ as near horizon geometry it is useful to use the c-extremization method (P. Kraus and F. Larsen, hep-th/0506176)

c-extremization

1. Consider a black string solution with near horizon geometry $AdS_3 \times S^2$

2. Consider an ansatz

$$ds = l_A^2 ds^2_{ADS_3} + l_S^2 ds^2_{S^2}, \qquad X^I = cont. \qquad F^I_{\theta\phi} = \frac{p^I}{2} \sin\theta$$

3. Define the c-function as

$$c = -6 l_A^3 l_S^2 \mathcal{L}$$

4. Extremize the function with respect to the parameters

5. For a theory with equal left and right moving central charge one gets

$$c = c|_{extr.}$$

An example

Using the two-derivative action the parameters of the ansatz are obtained by extremizing this function with respect to them

$$\frac{\partial c}{\partial l_A} = 0, \qquad \frac{\partial c}{\partial l_S} = 0, \qquad \frac{\partial c}{\partial X^I} = 0.$$

The above equations can be solved leading to

$$l_A = 2l_S = (\frac{1}{6}C_{IJK}p^I p^J p^K)^{1/3}, \; X^I = \frac{p^I}{(\frac{1}{6}C_{IJK}p^I p^J p^K)^{1/3}}$$

The central charge for the black string at two-derivative level is

$$c = C_{IJK}p^I p^J p^K$$

As far as the five dimensional black holes are concerned, We note that although the microscopic origin of $\mathcal{N} = 8, 4$ five dimensional rotating black holes has been understood for a decade , the origin of the entropy for five dimensional $\mathcal{N} = 2$ black hole has not been fully understood yet.

We would like study higher derivative corrections to $\mathcal{N} = 2$ five dimensional black holes.

To do this we will work with the full 5D supersymmetry invariant four-derivative action, corresponding to the supersymmetric completion of the four-derivative Chern-Simons term which has recently been obtained (K. Hanaki, K. Ohashi and Y. Tachikawa, hep-th/0611329)

Although in the leading order we use very special geometry, sometime it is useful to work with a more general context, namely the superconformal approach. This approach, in particular, is useful when we want to write the explicit form of the action. In this approach we start with a five dimensional theory which is invariant under a larger group that is superconformal group and therefore we construct a conformal supergravity. Then by imposing a gauge fixing condition, one breaks the conformal supergravity to standard supergravity model.

The representation of superconformal group includes Weyl, vector and hyper multiples.

1. Weyl multiplet

$$e_\mu^a,\ \psi_\mu,\ V_\mu,\ v^{ab},\ \chi,\ D$$

2. Vector multiplet

$$A_\mu^I,\ X^I,\ \Omega^I, Y^I, \quad I = 1, \cdots, n_v$$

The hypermultiplet contains scalar fields $\mathcal{A}_\alpha^i$ where $i = 1, 2$ is $SU(2)$ doublet index and $\alpha = 1, \cdots, 2r$ refers to $USp(2r)$ group.

Although we won't couple the theory to matters, we shall consider the hyper multiplet to gauge fix the dilataional symmetry reducing the action to the standard $\mathcal{N} = 2$ supergravity action.

Since the Weyl and vector multiplets are irr. rep. the variations of the fields under SUSY transformations are independent of the action. This is the point of retaining the auxiliary fields

The supersymmetry variations of the fermions in Weyl, vector and hyper multiplets are

$$\begin{aligned}
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i + \frac{1}{2}v^{ab}\gamma_{\mu ab}\varepsilon^i - \gamma_\mu\eta^i, \\
\delta\chi^i &= D\varepsilon^i - 2\gamma^c\gamma^{ab}\varepsilon^i\mathcal{D}_a v_{bc} - 2\gamma^a\varepsilon^i\epsilon_{abcde}v^{bc}v^{de} + 4\gamma\cdot v\eta^i, \\
\delta\Omega^{Ii} &= -\frac{1}{4}\gamma\cdot F^I\varepsilon^i - \frac{1}{2}\gamma^a\partial_a X^I - X^I\eta^i, \\
\delta\zeta^\alpha &= \gamma^a\partial_a\mathcal{A}_j^\alpha - \gamma\cdot v\varepsilon\mathcal{A}_j^\alpha + 3\mathcal{A}_j^\alpha\eta^i
\end{aligned}$$

In this notation at leading order the bosonic part of the action is

$$I = \frac{1}{16\pi G_5}\int d^5x\, \mathcal{L}_0$$

with

$$\begin{aligned}
\mathcal{L}_0 &= \partial_a \mathcal{A}^i_\alpha \partial^a \mathcal{A}^\alpha_i + (2\nu + \mathcal{A}^2)\frac{D}{4} + (2\nu - 3\mathcal{A}^2)\frac{R}{8} \\
&+ (6\nu - \mathcal{A}^2)\frac{v^2}{2} + 2\nu_I F^I_{ab} v^{ab} \\
&+ \frac{1}{4}\nu_{IJ}(F^I_{ab}F^{J\,ab} + 2\partial_a X^I \partial^a X^J) \\
&+ \frac{g^{-1}}{24} C_{IJK}\epsilon^{abcde} A^I_a F^J_{bc} F^K_{de}
\end{aligned}$$

where $\mathcal{A}^2 = \mathcal{A}^i_{\alpha\, ab}\mathcal{A}^{\alpha\, ab}_i$, $v^2 = v_{ab}v^{ab}$ and

$$\nu = \frac{1}{6}C_{IJK}X^I X^J X^K,\ \nu_I = \frac{1}{2}C_{IJK}X^J X^K,\ \nu_{IJ} = C_{IJK}X^K$$

To fix the gauge it is convenient to set $\mathcal{A}^2 = -2$. Then integrating out the auxiliary fields by making use of their equations of motion one finds

$$\begin{aligned}\mathcal{L}_0 &= R - \frac{1}{2} G_{IJ} F^I_{ab} F^{Jab} - \mathcal{G}_{ij} \partial_a \phi^i \partial^a \phi^j \\ &+ \frac{g^{-1}}{24} \epsilon^{abcde} C_{IJK} F^I_{ab} F^J_{cd} A^K_e\end{aligned}$$

The supersymmetrized higher order Lagrangian (K. Hanaki, K. Ohashi and Y. Tachikawa, hep-th/0611329)

$$
\begin{aligned}
\mathcal{L}_1 &= \frac{c_{2I}}{24}\Bigg[\frac{1}{16}g^{-1}\epsilon_{abcde}A^{Ia}C^{bcfg}C^{de}{}_{fg} + \frac{1}{8}X^I C^{abcd}C_{abcd} \\
&+ \frac{1}{12}X^I D^2 + \frac{1}{6}F^{Iab}v_{ab}D - \frac{1}{3}X^I C_{abcd}v^{ab}v^{cd} \\
&- \frac{1}{2}F^{Iab}C_{abcd}v^{cd} + \frac{8}{3}X^I v_{ab}\widehat{\mathcal{D}}^b\widehat{\mathcal{D}}_c v^{ac} + \frac{4}{3}X^I\widehat{\mathcal{D}}^a v^{bc}\widehat{\mathcal{D}}_a v_{bc} \\
&+ \frac{4}{3}X^I\widehat{\mathcal{D}}^a v^{bc}\widehat{\mathcal{D}}_b v_{ca} - \frac{2}{3}e^{-1}X^I\epsilon_{abcde}v^{ab}v^{cd}\widehat{\mathcal{D}}_f v^{ef} \\
&+ \frac{2}{3}e^{-1}F^{Iab}\epsilon_{abcde}v^{cd}\widehat{\mathcal{D}}_f v^{ef} + e^{-1}F^{Iab}\epsilon_{abcde}v^c{}_f\widehat{\mathcal{D}}^d v^{ef} \\
&- \frac{4}{3}F^{Iab}v_{ac}v^{cd}v_{db} - \frac{1}{3}F^{Iab}v_{ab}v^2 + 4X^I v_{ab}v^{bc}v_{cd}v^{da} \\
&- X^I(v_{ab}v^{ab})^2\Bigg],
\end{aligned}
$$

where C_{abcd} is the Weyl tensor defined as

$$
C^{ab}{}_{cd} = R^{ab}{}_{cd} + \frac{1}{6}R\delta^{[a}{}_{[c}\delta^{b]}{}_{d]} - \frac{4}{3}\delta^{[a}{}_{[c}R^{b]}{}_{d]} \ .
$$

Black string

Consider five dimensional extremal black string solution whose near horizon geometry is $AdS_3 \times S^2$ given by the following ansatz

$$ds = l_A^2 ds^2_{ADS_3} + l_S^2 ds^2_{S^2}, \qquad X^I = cont. \qquad F^I_{\theta\phi} = \frac{p^I}{2}\sin\theta$$

The c-function is

$$c = -6\, l_A^3 l_S^2\, \mathcal{L}$$

In general it is difficult to solve the equations of motion explicitly. Nevertheless one may use the supersymmetry transformations to fix some of the parameters. The remaining parameters can then be found by equation of motion of the auxiliary field D.

From the supersymmetry transformations for our ansatz one finds

$$D = \frac{12}{l_S^2}, \qquad p^I = -\frac{8}{3} V X^I, \qquad V = -\frac{3}{8} l_A, \qquad l_A = 2 l_S$$

On the other hand the equation of motion of auxiliary field D is

$$\frac{1}{6}C_{IJK}X^I X^J X^K + \frac{c_{2I}}{72}\left(DX^I + \frac{Vp^I}{l_S^4}\right) = 1$$

which can be recast to the following form

$$\frac{1}{6}C_{IJK}X^I X^J X^K + \frac{1}{12 l_A^2}c_{2I}X^I = 1$$

Setting $c_{2I} = 0$, it reduces to $\nu = 1$ where one can define very special geometry underlines the theory at leading order.

It is then natural to define the dual coordinates X_I as

$$X_I = \frac{1}{6}C_{IJK}X^J X^K + \frac{1}{12 l_A^2}c_{2I}$$

such that $X_I X^I = 1$.

From the supersymmetry conditions one has $X^I = \frac{p^I}{l_A}$. Plugging this into the above expressions we get

$$l_A^3 = \frac{1}{6}C_{IJK}p^I p^J p^K + \frac{1}{12}c_{2I}p^I$$

$$X^I \;=\; \frac{p^I}{(\frac{1}{6}C_{IJK}p^Ip^Ip^K + \frac{1}{12}c_{2I}p^I)^{1/3}},$$

$$X_I \;=\; \frac{\frac{1}{6}C_{IJK}p^Ip^K + \frac{1}{12}c_{2I}}{(\frac{1}{6}C_{IJK}p^Ip^Ip^K + \frac{1}{12}c_{2I}p^I)^{2/3}}$$

The corrected central charge is

$$c = C_{IJK}p^Ip^Jp^K + \frac{3}{4}c_2 \cdot p$$

We have observed that adding higher derivative terms will change the feature of the very special geometry in such a way that the leading order constraint $\nu = 1$ is not satisfied any more.

We would like to generalize the notion of very special geometry when higher order corrections are also taken into account. It is possible to do that due to new progresses which have recently been made in computing the higher order correction using the fully supersymmetrized higher derivative terms. We note, however, that the notion of *generalized* very special geometry depends on the explicit solution we are considering.

Since the leading order constraint $\nu = 1$ defining the very special geometry is, indeed, the equation of motion of the auxiliary field D, we will also define the constraint for the generalized very special geometry by making use of the equation of motion of field D in the presence of higher derivative terms.

Using the black string ansatz and taking into account the supersymmetry constraints the corresponding equation can be recast to the following form

$$\frac{1}{6}C_{IJK}X^I X^J X^K + \frac{1}{12 l_A^2} c_{2I} X^I = 1.$$

There is an auxiliary two-form field $v_{\mu\nu}$ which could be treated as an additional gauge field in the theory with charge p^0. Accordingly, we could introduce new scalar field X^0 such that in the near horizon geometry one may set $X^0 = \frac{p^0}{l_A}$.

$$\frac{1}{6}C_{IJK}X^I X^J X^K + \frac{1}{12} c_{2I} (X^0)^2 X^I |_{p^0=1} = 1.$$

One may also define $F^0_{\theta\phi} = \frac{p^0}{2}\sin\theta$ such that $v_{\theta\phi} = -\frac{3}{4 l_A} F^0_{\theta\phi}$.

1. Obviously working with this notation all expressions reduce to those we had for $p^0 = 1$, though it is not a solution for $p^0 \neq 1$. Nevertheless we will work with $p^0 \neq 1$ with the understanding that the solution is obtained by setting $p^0 = 1$.

2. It is worth noting that in general the auxiliary field $v_{\mu\nu}$ cannot be treated as a gauge field, though it can be seen that for the models we are going to study it may be considered as a gauge field.

Let us define new indexes

$$A, B, \cdots = 0, I, J \cdots$$

so that

$$\frac{1}{6} C_{ABC} X^A X^B X^C = 1$$

where $C_{ABC} = C_{IJK}$ for $A, B, C = I, J, K$ and $C_{00I} = \frac{c_{2I}}{6}$ and the other components are zero.

It is natural to define X_A and the metric C_{AB} as follows

$$X_A = \frac{1}{6} C_{ABC} X^B X^C, \;\; C_{AB} = \frac{1}{6} C_{ABC} X^C$$

More explicitly one has

$$X_I = \frac{1}{6} C_{IJK} X^J X^K + \frac{c_{2I}}{36} (X^0)^2, \;\; X_0 = \frac{c_{2I} X^I}{18} X^0$$

and

$$C_{IJ} = \frac{1}{6} C_{IJK} X^K, \; C_{I0} = \frac{c_{2I}}{36} X^0, \; C_{00} = \frac{c_{2I} X^I}{36}$$

It is easy to verify that

$$X_A X^A = 1, \;\; X_A = C_{AB} X^B, \;\; C_{AB} X^A X^B = 1$$

Following the standard notion of very special geometry, the magnetic central charge may be defined as

$$Z_m = X_A p^A$$

The near horizon parameters are fixed by extremizing it

$$\partial_i Z_m = \partial_i X_A p^A = 0$$

So that $X^A = p^A / Z_m$

$$\begin{aligned} Z_m &= \left(\frac{1}{6} C_{ABC} p^A p^B p^C \right)^{1/3} \\ &= \left(\frac{1}{6} C_{IJK} p^I p^J p^K + \frac{(p^0)^2}{12} c_{2I} p^I \right)^{1/3} \end{aligned}$$

Example

At R^2 level the solution can be written as follows

$$
\begin{aligned}
ds^2 &= \frac{l_A^2}{4}(dx^2 - 2r\,dx\,dr + \frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta\, d\phi^2), \\
A^I_\theta &= -\frac{p^I}{2}\cos\theta\, d\phi, \qquad X^I = \frac{p^I}{l_A}, \\
A^0_\theta &= -\frac{1}{2}\cos\theta\, d\phi, \quad \text{with} \quad v = -\frac{3}{4} l_A F^0_{\theta\phi}
\end{aligned}
$$

where $l_A = \left(\frac{1}{6}C_{IJK}p^I p^J p^K + \frac{1}{12}c_{2I}p^I\right)^{1/3}$.

In writing the metric we have used the fact that AdS_3 can be written as S^1 fibered over AdS_2.

In the notation of generalized very special geometry one get

$$
\begin{aligned}
ds^2 &= \frac{Z_m^2}{4}(dx^2 - 2r dx dr + \frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta\, d\phi^2), \\
A^A_\theta &= -\frac{p^A}{2}\cos\theta\, d\phi, \quad Z_m^3 = \frac{1}{6}C_{ABC}p^A p^B p^C, \quad X^A = \frac{p^A}{Z_m}
\end{aligned}
$$

Consider the total space of $U(1)$ bundle over the above metric to define a six dimensional manifold with the metric

$$
\begin{aligned}
ds_6^2 &= \frac{Z_m^2}{4}(dx^2 - 2r dx dt + \frac{dr^2}{r^2} + \sigma_1^2 + \sigma_2^2) \\
&+ (2X_A X_B - C_{AB})(dy^A + A^A)(dy^B + A^B)
\end{aligned}
$$

Here σ_i are right invariant one-forms such that

$$
\sigma_1^2 + \sigma_2^2 = d\theta + \sin^2\theta\, d\phi^2
$$

.

Define new coordinates z, ψ as

$$y^A = z^A + (\sin B - 1)X^A X_B z^B - \frac{1}{2}p^A\psi, \qquad x = \frac{2\cos B}{Z_m}X_A z^A$$

where $\sin B$ is a constant which its physical meaning will become clear later.

We define the coordinates such that the new coordinates have the following identification

$$\psi \sim \psi + 4\pi m, \qquad z^A \sim z^A + 2\pi n^A,$$

where m and n^A are integers.

Using the new coordinate ψ one can define the right invariant one-forms by

$$\sigma_1 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi,$$
$$\sigma_2 = \cos\psi d\theta + \sin\psi \sin\theta d\phi,$$
$$\sigma_3 = d\psi + \cos\theta d\phi$$

In terms of the new coordinates the six dimensional metric reads

$$ds^2 = \frac{Z^2}{4}[-(\cos B\, rdt + \sin B\sigma_3)^2 + \frac{dr^2}{r^2} + \sigma_1^2 + \sigma_2^2 + \sigma_3^2] + (2X_A X_B - C_{AB})(dz^A + \tilde{A}^A)(dz^B + \tilde{A}^B),$$

where $\tilde{A}^A = -\frac{p^A}{2}(\cos B\, rdt + \sin B\sigma_3)$.

The obtained six dimensional manifold can be treated as the total space of a $U(1)$ bundle over BMPV black hole at R^2 level. Therefore we can reduce to five dimensions to get BMPV black hole where higher derivative corrections are also taken into account.

The resulting five dimensional black hole solution is

$$ds^2 = \frac{l_A^2}{4}[-(\cos B\, rdt + \sin B\sigma_3)^2 + \frac{dr^2}{r^2} + \sigma_1^2 + \sigma_2^2 + \sigma_3^2],$$

$$\tilde{A}^I = -\frac{p^I}{2}(\cos B\, rdt + \sin B\sigma_3), \qquad X^I = \frac{p^I}{l_A}$$

$$\tilde{A}^0 = -\frac{1}{2}(\cos B\, rdt + \sin B\sigma_3), \qquad v = -\frac{3}{4} l_A \tilde{F}^0$$

From this solution we can identify $\sin B$ as the angular momentum, J, of the BMPV solution through the relation $\sin B = \frac{J}{Z^3}$.

Black Hole

Consider a five dimensional extremal black hole whose near geometry is $AdS_2 \times S^3$.

Let us start with the following ansatz for near horizon geometry

$$ds = l_A^2 ds^2_{ADS_2} + l_S^2 ds^2_{S^3}, \quad X^I = cont. \quad F^I_{rt} = e^I, \quad v_{rt} = V$$

The entropy function is given by

$$\mathcal{E} = 2\pi(e^I q_I - f_0 + f_1)$$

$$
\begin{aligned}
f_0 &= \frac{1}{2} l_A^2 l_S^3 \Big[\frac{\nu - 1}{2} D + \frac{\nu + 3}{2} \left(\frac{3}{l_S^2} - \frac{1}{l_A^2} \right) - \frac{2(3\nu + 1)}{l_A^4} V^2 \\
&- \frac{4\nu_I e^I}{l_A^4} - \frac{\nu_{IJ} e^I e^J}{2 l_A^4} \Big] \\
f_1 &= \frac{c_{2I}}{48} l_A^2 l_S^3 \left[\frac{X^I}{4} \left(\frac{1}{l_S^2} - \frac{1}{l_A^2} \right)^2 + \frac{4V^4}{l_A^8} X^I + \frac{4V^3}{3 l_A^8} e^I \right. \\
&- \frac{DV}{3 l_A^4} e^I + \frac{D^2}{12} X^I - \frac{2V^2 X^I}{3 l_A^4} \left(\frac{3}{l_S^2} + \frac{5}{l_A^2} \right) \\
&- \left. \frac{V e^I}{l_A^4} \left(\frac{1}{l_A^2} - \frac{1}{l_S^2} \right) \right]
\end{aligned}
$$

At leading order where only f_0 contributes, one has

$$
\begin{aligned}
f_0 &= \frac{1}{2} l_A^2 l_S^3 \left[\frac{6}{l_S^2} - \frac{2}{l_A^2} + \frac{1}{2 l_A^4} (\nu_I \nu_J - \nu_{IJ}) e^I e^J \right] \\
&= \frac{1}{2} l_A^2 l_S^3 \left(\frac{6}{l_S^2} - \frac{2}{l_A^2} + \frac{G_{IJ} e^I e^J}{l_A^4} \right)
\end{aligned}
$$

It is easy to extremize the entropy function with respect to the parameters to find l_A, l_S, X^I and e^I.

In particular for STU model where $C_{123} = 1$ and the other components are zero one gets

$$l_S = 2l_A = (q_1 q_2 q_3)^{1/6}, \ X^I = \frac{(q_1 q_2 q_3)^{1/3}}{q_I}, \ e^I = \frac{1}{2}\frac{(q_1 q_2 q_3)^{1/2}}{q_I}$$

and the corresponding black hole entropy is

$$S = 2\pi\sqrt{q_1 q_2 q_3}$$

To study the higher order corrections we use the SUSY transformation to simplify the equations. For our ansatz the supersymmetry conditions lead to

$$D = -\frac{3}{l_A^2}, \qquad e^I = -\frac{4}{3}VX^I, \qquad V = -\frac{3}{4}l_A, \qquad l_S = 2l_A$$

Setting $X^I = \frac{e^I}{l_A}$ and defining $E = \frac{1}{6}C_{IJK}e^I e^J e^K$ one has

$$\nu = \frac{1}{l_A^3}E, \qquad \nu_I e^I = \frac{3}{l_A^2}E, \qquad \nu_{IJ}e^I e^J = \frac{6}{l_A}E$$

In this notation the equation of motion for auxiliary field D reads

$$E - l_A^3 + \frac{l_A^3}{12}c_{2I}\left(\frac{DX^I}{6} - \frac{Ve^I}{3l_A^4}\right) = 0$$

so that $l_A = \frac{1}{2}(8E - \frac{c_{2I}e^I}{6})^{1/3}$.

By making use of the expressions for the parameters D, V, X^I and l_S given above, the entropy function gets the following simple form

$$\begin{aligned} \mathcal{E} &= 2\pi(q_I e^I - 4E + \frac{1}{8}c_{2I}e^I) \\ &= 2\pi\left[(q_I + \frac{1}{8}c_{2I})e^I - \frac{2}{3}C_{IJK}e^I e^J e^K\right] \end{aligned}$$

Extremizing the entropy function with respect to e^I we get

$$2C_{IJK}e^J e^K = q_I + \frac{1}{8}c_{2I}$$

The entropy is also given by

$$S = \frac{4\pi}{3} q_I^+ e^I$$

where $q_I^+ = q_I + \frac{1}{8}c_{2I}$

In particular for STU model we get $e^I = (q_1^+ q_2^+ q_3^+)^{1/2}/2q_I^+$ so that

$$\begin{aligned} 2l_A &= (q_1^+ q_2^+ q_3^+)^{1/6}(1 - \frac{c_{2I}}{12}\frac{1}{q_I^+})^{1/3} \\ X^I &= \frac{(q_1^+ q_2^+ q_3^+)^{1/3}}{q_I^+}(1 - \frac{c_{2I}}{12}\frac{1}{q_I^+})^{-1/3} \end{aligned}$$

The entropy is

$$S = 2\pi\sqrt{q_1^+ q_2^+ q_3^+}$$

The entropy of the black hole at R^2 level has the same form as the tree level except that the charges q_I's are replaced by shifted charges q_I^+'s

This result can be used to see how the higher order corrections stretch horizon. For example if we start by a classical solution with $q_1 = 0$ we get vanishing horizon (small black hole) and therefore the entropy is zero.

Adding the R^2 terms we get a smooth solution with a non-zero entropy give by

$$S = \pi\sqrt{\frac{c_{21}}{2} q_2^+ q_3^+}$$

It is also instructive to compare the results with the case where the higher order corrections are given in terms of the Gauss-Bonnet action. In our notation the Gauss-Bonnet term is given by

$$\mathcal{L}_{GB} = \frac{c_{2I} X^I}{2^8 \cdot 3\pi^2} \left(R^{abcd} R_{abcd} - 4 R^{ab} R_{ab} + R^2 \right)$$

It can be shown that with the specific coefficient we have chosen for the Gauss-Bonnet action, we get the same results as those in supersymmetrized action.

To extend the solution for a general model we will have to solve a set of equations

$$C_{IJK}x^J x^K = a_I, \qquad I = 1, \cdots, N$$

for given C_{IJK} and a_I.

An interesting observation we have made in this paper is that the solution of the above equation can be used for both tree level case and the case when the R^2 corrections are added. The only difference is that in the presence of R^2 terms one just needs to replace a_I by another constant which is related to it by a constant shift.

The corresponding microscopic description of this black hole has not fullu understood

(See however C. Vafa, hp-th/9711067, and M.X. Huang, A. Klemm, M. Marino and A. Tavanfar, arXive:0704.2440[hep-th])

One may compare this with other method like D4/D5 connection

Using the correct definition of charges in 4D and 5D we get the same result