WORLD-VOLUME THEORIES

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Abstract

In these lectures we discuss properties of superstring and D-brane world-volume actions, with emphasis on the local fermionic symmetry known as kappa symmetry.

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I. Bosonic actions

Relativistic point particle

Since the classical motion of a massive point particle is along geodesics, the action is proportional to the invariant length of the world-line

$$S_0 = -m \int ds, \tag{1}$$

where $\hbar = c = 1$. The line element is given by

$$ds^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu .$$

Here $g_{\mu\nu}(x)$ with $\mu, \nu = 0, \ldots, D-1$ describes the background geometry, with signature $(- + \cdots +)$.

The minus sign has been introduced so that ds is real for a time-like trajectory. The action, which is reparametrization invariant, can be rewritten in the form

$$S_0 = -m \int \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} d\tau, \qquad (2)$$

This action contains a square root, and it cannot be used to describe a massless particle. These problems can be circumvented by introducing an auxiliary field $e(\tau)$

$$\widetilde{S}_0 = \frac{1}{2}\int d\tau \left(e^{-1}\dot{x}^2 - m^2 e\right), \tag{3}$$

where $\dot{x}^2 = g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu$. Solving for $e(\tau)$ and substituting back into $\widetilde{S}_0$ gives S_0. $\widetilde{S}_0$ has a smooth $m \to 0$ limit.

p-brane actions

The action (2) can be generalized to a p-brane sweeping out a $(p+1)$-dimensional world volume in D dimensions

$$S_p = -T_p \int d\mu_p. \tag{4}$$

T_p is the p-brane tension. The dimension of T_p is $(\text{length})^{-p-1}$.

$d\mu_p$ is the $(p+1)$-dimensional volume element

$$d\mu_p = \sqrt{-\det G_{\alpha\beta}}\, d^{p+1}\sigma,$$

where the induced metric is

$$G_{\alpha\beta} = g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu.$$

This action is valid in a probe-brane approximation in which back reaction is neglected.

The reparametrization (or diffeomorphism) invariance of the p-brane action allows $p+1$ gauge choices. Static gauge identifies world-volume coordinates with spacetime coordinates

$$X^\mu = (\sigma^\alpha, X^i),$$

where X^i describes the $D-p-1$ directions transverse to the p-brane. In a Minkowski spacetime background the p-brane action becomes

$$S = -T_p \int d^{p+1}\sigma \sqrt{-\det(\eta_{\alpha\beta} + \partial_\alpha X^i \partial_\beta X^i)}.$$

T_p is the energy density of the undisturbed brane.

Strings

Now consider a string in Minkowski spacetime. Denoting $\sigma^0 = \tau$ and $\sigma^1 = \sigma$, one obtains the Nambu–Goto action

$$S_{\mathrm{NG}} = -T \int d\sigma d\tau \sqrt{\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2},$$

where

$$\dot{X}^{\mu} = \frac{\partial X^{\mu}}{\partial \tau} \qquad \text{and} \qquad X^{\mu\prime} = \frac{\partial X^{\mu}}{\partial \sigma},$$

and $A \cdot B = \eta_{\mu\nu} A^{\mu} B^{\nu}$. Classical string motion extremizes the world-sheet area.

A classically equivalent action, usually called the Polyakov action, is expressed in terms of an auxiliary world-sheet metric $h_{\alpha\beta}(\sigma, \tau)$.

The Polyakov action is

$$S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X.$$

The $h_{\alpha\beta}$ equation of motion implies the vanishing of the energy–momentum tensor $T_{\alpha\beta}$

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X = 0.$$

Taking the square-root of minus the determinant of both sides of the equation

$$\frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X = \partial_\alpha X\cdot\partial_\beta X$$

gives

$$\frac{1}{2}\sqrt{-h}\,h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X = \sqrt{-\det(\partial_\alpha X\cdot\partial_\beta X)}.$$

Substitution gives back S_{NG}.

In addition to global Poincaré invariance and local diffeomorphism invariance, S_{P} has local Weyl invariance

$$h_{\alpha\beta}\to e^{\phi(\sigma,\tau)}h_{\alpha\beta} \qquad \text{and} \qquad X^\mu\to X^\mu.$$

This explains why the energy–momentum tensor is traceless.

II. Supersymmetric actions

A natural supersymmetric generalization is based on maps into superspace, so that the basic fields are $X^{\mu}(\sigma^{\alpha})$ and $\Theta^{a}(\sigma^{\alpha})$.

D0-brane action

Let us begin with a spacetime supersymmetric world-line action for a point particle of mass m. The D0-brane, a nonperturbative excitation in the type IIA theory, is a special case of more general Dp-branes, which will be discussed later.

Our goal here is to find a $D = 10$, $\mathcal{N} = 2$ supersymmetric generalization of

$$S = -m \int \sqrt{-\dot{X}_\mu \dot{X}^\mu} d\tau \tag{5}$$

by introducing a Majorana spinor coordinate $\Theta^a(\tau)$ with $a = 1, 2, \ldots, 32$. This encodes two MW spinors of opposite chirality defined by

$$\Theta^1 = \frac{1}{2}(1 + \Gamma_{11})\Theta \qquad \text{and} \qquad \Theta^2 = \frac{1}{2}(1 - \Gamma_{11})\Theta,$$

where

$$\Gamma_{11} = \Gamma_0 \Gamma_1 \ldots \Gamma_9.$$

Supersymmetry can be represented in terms of infinitesimal supersymmetry transformations of superspace

$$\delta\Theta^a = \varepsilon^a,$$

$$\delta X^\mu = \bar{\varepsilon}\Gamma^\mu\Theta.$$

These supersymmetry transformations will be realized as global symmetries of the action. We define the supersymmetric combination

$$\Pi_0^\mu = \dot{X}^\mu - \bar{\Theta}\Gamma^\mu\dot{\Theta}.$$

The corresponding formula for a Dp-brane is

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - \bar{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A, \quad \alpha = 0, 1, \ldots, p \quad A = 1, 2.$$

which explains the subscript 0.

Making the replacement $\dot{X}^\mu \rightarrow \Pi^\mu_0$ in the action (5) gives

$$S_1 = -m \int \sqrt{-\Pi_0^2}\, d\tau, \tag{6}$$

where $\Pi_0^2 = \Pi_0 \cdot \Pi_0$. This action is invariant under global super-Poincaré transformations and local diffeomorphisms of the world-line.

The action S_1, by itself, is not the theory that we want. This can be seen by deriving the equations of motion associated with X^μ and Θ^A. The canonical conjugate momentum to X^μ is

$$P_\mu = \frac{\delta S_1}{\delta \dot{X}^\mu} = \frac{m}{\sqrt{-\Pi_0^2}} \left(\dot{X}_\mu - \bar{\Theta} \Gamma_\mu \dot{\Theta} \right) . \tag{7}$$

The X^μ equations of motion imply $\dot{P}_\mu = 0$. Not all the components of the momentum are independent. Squaring both sides of eq. (7) gives the mass-shell condition

$$P^2 = -m^2.$$

On the other hand, the equation of motion for Θ is

$$P \cdot \Gamma \dot{\Theta} = 0. \tag{8}$$

Multiplying this with $P \cdot \Gamma$ gives $m^2 \dot{\Theta} = 0$, so for $m \neq 0$ one obtains $\dot{\Theta} = 0$.

The factor $P \cdot \Gamma$ is singular in the massless case, corresponding to saturation of a BPS bound and enhanced supersymmetry. By adding another contribution to the action we can ensure saturation of a BPS bound and enhanced supersymmetry in the massive case as well.

Suppose that there is a second contribution to the action that changes eq. (8) to form suggested by KK reduction of massless particle in 11d

$$(P \cdot \Gamma - m\Gamma_{11})\dot{\Theta} = 0.$$

This equation only forces half the components of Θ to be constant, because half of the eigenvalues of $P \cdot \Gamma - m\Gamma_{11}$ are zero. As evidence of this consider its square

$$\begin{aligned}(P \cdot \Gamma - m\Gamma_{11})^2 &= (P \cdot \Gamma)^2 - m\{P \cdot \Gamma, \Gamma_{11}\} + (m\Gamma_{11})^2 \\ &= P^2 + m^2 = 0.\end{aligned}$$

Thus the number of independent equations is only half the number of components of Θ. There are local fermionic symmetries such that half the components of Θ are actually gauge degrees of freedom.

The missing contribution to the action that gives this additional term in the Θ equation of motion is

$$S_2 = -m \int \bar{\Theta} \Gamma_{11} \dot{\Theta} \, d\tau. \tag{9}$$

If this sign describes a D0-brane, then the opposite sign would describe an anti-D0-brane.

To summarize, the complete action for a D0-brane of mass m is

$$S = -m \int \sqrt{-\Pi_0 \cdot \Pi_0} \, d\tau - m \int \bar{\Theta} \Gamma_{11} \dot{\Theta} \, d\tau.$$

κ symmetry

Under a κ transformation

$$\delta X^\mu = \bar{\Theta}\Gamma^\mu \delta\Theta = -\delta\bar{\Theta}\Gamma^\mu \Theta.$$

The variation $\delta\Theta$ will be determined later. It follows that

$$\delta\Pi_0^\mu = -2\delta\bar{\Theta}\Gamma^\mu \dot{\Theta}.$$

The variation of the action S_1 under a κ transformation is

$$\delta S_1 = m \int \frac{\Pi_0 \cdot \delta\Pi_0}{\sqrt{-\Pi_0^2}}\, d\tau.$$

Substituting for $\delta\Pi_0^\mu$ gives

$$\delta S_1 = -2m \int \frac{\Pi_0^\mu \delta\bar{\Theta}\Gamma_\mu \dot{\Theta}}{\sqrt{-\Pi_0^2}}\, d\tau = -2m \int \delta\bar{\Theta}\gamma\Gamma_{11}\dot{\Theta}\, d\tau,$$

where

$$\gamma = \frac{\Gamma \cdot \Pi_0}{\sqrt{-\Pi_0^2}}\Gamma_{11}.$$

Since $\gamma^2 = 1$, γ can be used to construct projection operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma).$$

The second contribution to the action, S_2, has the variation

$$\delta S_2 = -2m \int \delta\bar{\Theta}\Gamma_{11}\dot{\Theta}\, d\tau.$$

Thus

$$\delta(S_1{+}S_2) = -2m \int \delta\bar{\Theta}(1{+}\gamma)\Gamma_{11}\dot{\Theta}\, d\tau = -4m \int \delta\bar{\Theta}P_+\Gamma_{11}\dot{\Theta}\, d\tau.$$

For a transformation $\delta\bar{\Theta}$ that takes the form

$$\delta\bar{\Theta} = \bar{\kappa}P_-,$$

with $\kappa(\tau)$ an arbitrary Majorana spinor, the action is invariant. So this is a local symmetry of the action.

To summarize, the D0-brane action S is invariant under the local κ transformations

$$\delta\bar{\Theta} = \bar{\kappa}P_{-} \qquad \text{and} \qquad \delta X^{\mu} = -\bar{\kappa}P_{-}\Gamma^{\mu}\Theta.$$

κ symmetry implies that half of the components of Θ are decoupled and can be gauged away. Without this symmetry there would be too many fermionic degrees of freedom.

Supersymmetric string actions

The type IIA and type IIB superstring theories both have $\mathcal{N} = 2$ and hence two MW fermionic coordinates Θ^1 and Θ^2.

$$\Gamma_{11}\Theta^A = (-1)^{A+1}\Theta^A \qquad \text{type IIA}$$

$$\Gamma_{11}\Theta^A = \quad \Theta^A \qquad \text{type IIB.}$$

The obvious guess is that the supersymmetric string action (for $\alpha' = 1/2$ or $T = 1/\pi$) takes the form

$$S_1 = -\frac{1}{\pi}\int d^2\sigma\sqrt{-G},$$

with $G = \det G_{\alpha\beta}$ and $G_{\alpha\beta} = \Pi_\alpha \cdot \Pi_\beta$, where

$$\Pi_\alpha^\mu = \partial_\alpha X^\mu - \bar{\Theta}^A\Gamma^\mu\partial_\alpha\Theta^A.$$

As in the case of the D-particle, a second term S_2 has to be added in order to produce local κ symmetry. Global super-Poincaré symmetry and local reparametrization symmetry must be preserved by S_2.

As before, we require that under κ transformations

$$\delta X^\mu = \bar{\Theta}^A \Gamma^\mu \delta \Theta^A = -\delta \bar{\Theta}^A \Gamma^\mu \Theta^A,$$

which implies

$$\delta \Pi_\alpha^\mu = -2\delta \bar{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A. \tag{10}$$

Using (10) one obtains

$$\delta S_1 = \frac{1}{\pi} \int d^2\sigma \sqrt{-G} G^{\alpha\beta} \Pi_\alpha^\mu \delta \bar{\Theta}^A \Gamma_\mu \partial_\beta \Theta^A.$$

Construction of S_2

Whereas S_1 has the structure of a supersymmetrized volume, S_2 is naturally described as the integral of a two-form

$$S_2 = \int \Omega_2 = \frac{1}{2} \int d^2\sigma \epsilon^{\alpha\beta} \Omega_{\alpha\beta},$$

where Ω_2 does not depend on the world-sheet metric. More generally, for a p-brane it would be an integral of a $(p+1)$-form. Such a geometric structure has manifest diffeomorphism symmetry.

The way to make the symmetries of the problem manifest is to formally introduce an additional dimension and consider the three-form

$$\Omega_3 = d\Omega_2.$$

Similarly, introduce a 3d region D whose boundary is the string world-sheet M ($M = \partial D$). Then by Stokes' theorem

$$\int_D \Omega_3 = \int_M \Omega_2.$$

The form Ω_3 is like a characteristic class in that it is closed and invariant under the symmetries in question. The form Ω_2 is the corresponding Chern–Simons term. In general its variation under the corresponding symmetry transformations is a total derivative, which is all we need.

A useful identity satisfied by a MW spinor Θ in 10d is

$$\Gamma^{\mu} d\Theta \, d\bar{\Theta} \Gamma_{\mu} d\Theta = 0. \tag{11}$$

Wedge products are implicit. This formula, which is crucial to the existence of supersymmetric Yang–Mills theory in 10d, is proved by considering Fierz rearrangements of the spinors.

There are three supersymmetric one-forms:

$$d\Theta^1, \quad d\Theta^2, \quad \Pi^{\mu} = dX^{\mu} - \bar{\Theta}^A \Gamma^{\mu} d\Theta^A.$$

So Ω_3 should be a Lorentz-invariant three-form constructed out of these. Up to a constant c to be determined later, the appropriate choice is

$$\Omega_3 = c(d\bar{\Theta}^1 \Gamma_\mu d\Theta^1 - d\bar{\Theta}^2 \Gamma_\mu d\Theta^2)\Pi^\mu. \tag{12}$$

To verify that Ω_3 is closed, substitute

$$d\Pi^\mu = -(d\bar{\Theta}^1 \Gamma^\mu d\Theta^1 + d\Theta^2 \Gamma^\mu d\Theta^2)$$

into

$$d\Omega_3 = c(d\bar{\Theta}^1 \Gamma_\mu d\Theta^1 - d\bar{\Theta}^2 \Gamma_\mu d\Theta^2) d\Pi^\mu.$$

The minus sign ensures the cancellation of the cross terms with two powers of $d\Theta^1$ and two powers of $d\Theta^2$. Terms quartic in $d\Theta^1$ or $d\Theta^2$ vanish due to eq. (11).

Let us now compute the κ symmetry variation of Ω_3:

$$\delta\Omega_3 = 2c(d\delta\bar{\Theta}^1\Gamma_\mu d\Theta^1 - d\delta\bar{\Theta}^2\Gamma_\mu d\Theta^2)\Pi^\mu$$

$$-2c(d\bar{\Theta}^1\Gamma_\mu d\Theta^1 - d\bar{\Theta}^2\Gamma_\mu d\Theta^2)\delta\bar{\Theta}^A\Gamma^\mu d\Theta^A.$$

Using eq. (11) again, one obtains

$$\delta\Omega_3 = d\Big[2c(\delta\bar{\Theta}^1\Gamma_\mu d\Theta^1 - \delta\bar{\Theta}^2\Gamma_\mu d\Theta^2)\Pi^\mu\Big],$$

and thus

$$\delta\Omega_2 = 2c(\delta\bar{\Theta}^1\Gamma_\mu d\Theta^1 - \delta\bar{\Theta}^2\Gamma_\mu d\Theta^2)\Pi^\mu.$$

To be explicit, setting $c = 1/2\pi$ gives

$$\delta S_2 = \frac{1}{\pi}\int d^2\sigma\varepsilon^{\alpha\beta}(\delta\bar{\Theta}^1\Gamma_\mu\partial_\alpha\Theta^1 - \delta\bar{\Theta}^2\Gamma_\mu\partial_\alpha\Theta^2)\Pi^\mu_\beta. \quad (13)$$

Then the variation of the entire action under κ transformations takes the form

$$\delta S = \frac{2}{\pi}\int d^2\sigma\varepsilon^{\alpha\beta}(\delta\bar{\Theta}^1 P_+\Gamma_\mu\partial_\alpha\Theta^1 - \delta\bar{\Theta}^2 P_-\Gamma_\mu\partial_\alpha\Theta^2)\Pi^\mu_\beta.$$

The orthogonal projection operators $P_\pm$ are defined by

$$P_\pm = \frac{1}{2}(1 \pm \gamma)$$

with

$$\gamma = -\frac{\varepsilon^{\alpha\beta}\Pi_\alpha^\mu \Pi_\beta^\nu \Gamma_{\mu\nu}}{2\sqrt{-G}},$$

which again satisfies $\gamma^2 = 1$.

It now follows that the action is invariant under the transformations

$$\delta\bar{\Theta}^1 = \bar{\kappa}^1 P_- \quad \text{and} \quad \delta\bar{\Theta}^2 = \bar{\kappa}^2 P_+$$

for arbitrary MW spinors κ^1 and κ^2 of appropriate chirality.

To make S_2 more explicit, we can solve $\Omega_3 = d\Omega_2$ for Ω_2. The solution is

$$\Omega_2 = c(\bar{\Theta}^1\Gamma_\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2)dX^\mu - c\bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\Theta}^2\Gamma^\mu d\Theta^2,$$

where $c = 1/2\pi$. Note that changing the sign of c corresponds to interchanging Θ^1 and Θ^2.

III. Kappa symmetric D-brane actions

The D-brane world-volume theories that follow contain

$$X^{\mu}(\sigma), \quad \Theta^{1a}(\sigma), \quad \Theta^{2a}(\sigma), \quad A_{\alpha}(\sigma)$$

σ^{α} parametrizes the Dp-brane world-volume ($\alpha = 0, 1, \ldots, p$). The new ingredient is an Abelian world-volume gauge field.

The physical content is the massless open-string spectrum, which is maximally supersymmetric Maxwell theory in $p+1$ dimensions — 8 physical fermions and 8 physical bosons.

The fields Θ^{Aa} have 32 real components. κ symmetry gives a factor of two reduction and the Dirac equation removes half again. So there are 8 physical fermions.

The bosonic degrees of freedom come from X^μ and A_α. Accounting for $p+1$ diffeomorphism symmetries

$$10-(p+1)=9-p$$

components of X^μ are physical. These describe transverse excitations of the Dp-brane. The gauge field A_α has $p+1$ components, but two of them are nondynamical, so A contributes $p-1$ physical degrees of freedom.

Altogether, there are

$$(9-p)+(p-1)=8$$

physical bosonic degrees of freedom, as required by supersymmetry.

Construction of S_1

Born and Infeld proposed the theory

$$S_{\rm BI} \sim \int \sqrt{-\det(\eta_{\alpha\beta} + kF_{\alpha\beta})}\, d^4\sigma,$$

where k is a constant. This structure combines nicely with the usual Nambu–Goto structure for a p-brane to give the DBI action

$$S_1 = -T_{{\rm D}p} \int d^{p+1}\sigma \sqrt{-\det(G_{\alpha\beta} + k\mathcal{F}_{\alpha\beta})},$$

where $T_{{\rm D}p}$ is the tension (or energy density), and $k = 2\pi\alpha'$. As before

$$G_{\alpha\beta} = \eta_{\mu\nu}\Pi^\mu_\alpha \Pi^\nu_\beta,$$

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - \bar{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A.$$

We also define

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + b_{\alpha\beta},$$

where $F = dA$ is the Maxwell field strength, and the two-form b is required in order that $\mathcal{F}$ is supersymmetric. The result is

$$b = (\bar{\Theta}^1 \Gamma_\mu d\Theta^1 - \bar{\Theta}^2 \Gamma_\mu d\Theta^2)(dX^\mu - \frac{1}{2}\bar{\Theta}^A \Gamma^\mu d\Theta^A).$$

Construction of S_2

A Chern–Simons term S_2 still needs to be added for κ symmetry. It is the integral of a $(p+1)$-form

$$S_2 = \int \Omega_{p+1}.$$

As in the case of the superstring, it is easier to construct the $(p+2)$-form $d\Omega_{p+1}$, which is manifestly invariant under supersymmetry. The answer again takes the form

$$d\Omega_{p+1} = d\bar{\Theta}^A \mathcal{T}_p^{AB}\, d\Theta^B,$$

where $\mathcal{T}_p^{AB}$ is a 2×2 matrix of p-form valued Dirac matrices.

In the case of the D0-brane

$$\Omega_2 = -md\bar{\Theta}\Gamma_{11}d\Theta = m(d\bar{\Theta}^1 d\Theta^2 - d\bar{\Theta}^2 d\Theta^1),$$

which implies that

$$\mathcal{T}_0 = m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the mass is

$$m = T_{\mathrm{D0}} = \frac{1}{g_{\mathrm{s}}\sqrt{\alpha'}}.$$

It is simpler to give the results for all p together rather than to enumerate them one by one.

$$\mathcal{T}^{AB} = \sum_{p=0}^{\infty} \mathcal{T}_p^{AB}$$

In the IIA case the sum is over even values of p, and in the IIB case the sum is over odd values of p.

Given $\mathcal{T}$, one simply extracts the p-form part to obtain $\mathcal{T}_p$. The expression for $\mathcal{T}$ turns out to be

$$\mathcal{T}^{AB} = m\, e^{2\pi\alpha'\mathcal{F}} f^{AB}(\psi),$$

where ψ is a matrix-valued one-form given by

$$\psi = \frac{1}{\sqrt{2\pi\alpha'}}\,\Gamma_\mu\,\Pi^\mu_\alpha\,d\sigma^\alpha.$$

In the type IIA case

$$f(\psi) = \begin{pmatrix} 0 & \cos\,\psi \\ -\cosh\,\psi & 0 \end{pmatrix}$$

and in the type IIB case

$$f(\psi) = \begin{pmatrix} 0 & \sin\,\psi \\ \sinh\,\psi & 0 \end{pmatrix}.$$

This structure ensures that the matrix is symmetric or anti-symmetric for the appropriate powers of ψ, as required when

$\mathcal{T}$ is sandwiched between MW spinors.

Let us now choose static gauge and relabel the remaining $9-p$ coordinates as

$$X^i = k\Phi^i.$$

Then the bosonic part of the DBI action becomes

$$-T_{\mathrm{D}p}\int d^{p+1}\sigma\sqrt{-\det(\eta_{\alpha\beta}+k^2\partial_\alpha\Phi^i\partial_\beta\Phi^i+kF_{\alpha\beta})}.$$

Now let us include the fermions and make a gauge choice for the κ symmetry. A particularly nice choice is

$$\Theta^2 = 0.$$

This completely kills the Chern–Simons term, because the matrices $f^{(A)}$ and $f^{(B)}$ are entirely off-diagonal.

Making this gauge choice in the case $p = 9$ and renaming the remaining MW spinor $\Theta^1 = \lambda$ gives the action

$$-T_{\mathrm{D9}} \int d^{10}\sigma \sqrt{-\det\left(\eta_{\alpha\beta} + Z_{\alpha\beta}\right)}, \tag{14}$$

where

$$Z_{\alpha\beta} = kF_{\alpha\beta} - 2k^2\bar{\lambda}\Gamma_\alpha\partial_\beta\lambda + k^4\bar{\lambda}\Gamma^\gamma\partial_\alpha\lambda\bar{\lambda}\Gamma_\gamma\partial_\beta\lambda.$$

It is truly remarkable that this nonlinear extension of 10d super-Maxwell theory has exact unbroken supersymmetry.

The static gauge Dp-brane actions with $p < 9$ can be obtained by dimensional reduction of the gauge-fixed D9-brane action in eq. (14). The supersymmetry transformations are complicated, because induced κ transformations must be added to the original ε transformations of the fields.

IV. Bosonic D-brane actions with background fields

The D-brane actions obtained so far are for a Minkowski spacetime background. Actions that describe D-branes in more general backgrounds are also of interest.

Abelian case

The background fields in the NS-NS sector are the spacetime metric $g_{\mu\nu}$, the two-form $B_{\mu\nu}$ and the dilaton ϕ. These can be pulled back to the world volume by

$$P[g+B]_{\alpha\beta} = (g_{\mu\nu} + B_{\mu\nu})\partial_\alpha X^\mu \partial_\beta X^\nu.$$

Henceforth, this will be denoted $g_{\alpha\beta} + B_{\alpha\beta}$.

The DBI term in static gauge takes the form

$$-T_{\mathrm{D}p}\int d^{p+1}\sigma e^{-\phi}\sqrt{-\det\left(g_{\alpha\beta}+B_{\alpha\beta}+k^2\partial_\alpha\Phi^i\partial_\beta\Phi^i+kF_{\alpha\beta}\right)}.$$

Since the string coupling constant g_{s} is already included in the tension, the dilaton here is shifted by a constant so that it approaches zero at infinity.

The R-R background fields only contribute to the Chern–Simons term. Let us denote an n-form R-R field by C_n and the corresponding field strength by $F_{n+1} = dC_n$.

The complete list of R-R fields in type II superstring theories involves only $n = 0, 1, 2, 3, 4$. However, it is convenient to introduce redundant fields C_n for $n = 5, 6, 7, 8$. This leads to more elegant formulas. The idea is to generalize the self-duality of the 5-form field strength by requiring that

$$\star F_{n+1} = F_{9-n}.$$

This can be generalized to allow for interactions by adding additional terms in the definitions of the field strengths $F_{n+1} = dC_n + \ldots$.

The C_n fields can also be pulled back to the D-brane world volume, after which they are represented by the same symbols. Then the Chern–Simons term must contain a contribution $\mu_p \int C_{p+1}$, where μ_p denotes the Dp-brane charge.

In the presence of a background B field or world-volume gauge fields, the D-brane also couples to R-R potentials of lower rank. In terms of the total R-R potential

$$C = \sum_{n=0}^{8} C_n$$

the result is

$$S_{\rm CS} = \mu_p \int \left(C \, e^{B + 2\pi\alpha' F} \right)_{p+1} .$$

Since B and F are 2-forms, only odd-rank R-R fields contribute for even p (the IIA case) and only even-rank R-R fields contribute for odd p (the IIB case). The B and F fields appear in the same combination as in the DBI term.

A Dp-brane in a suitable background can also carry induced D$(p-2n)$-brane charge for $n = 0, 1, \ldots$. This charge can be smeared over the D-brane world-volume, or it can be concentrated on a lower-dimensional hypersurface (*e.g.*, a brane within a brane).

In the presence of spacetime curvature the Chern–Simons term contains an additional factor involving differential forms constructed from the curvature tensor. This factor reduces to 1 in a flat spacetime, which is the case described here.

Non-Abelian case

When N Dp-branes coincide, the world-volume theory is a

$U(N)$ gauge theory. Almost all studies of non-Abelian D-brane actions use the static gauge from the outset, since otherwise it is unclear how to implement diffeomorphism and κ symmetry.

In the static gauge the world-volume fields are just those of a maximally supersymmetric YM supermultiplet: gauge fields, scalars and spinors, all in the adjoint representation of $U(N)$. The leading nontrivial term in a weak-field expansion is exactly super YM theory. This approximation is sufficient for many purposes including Matrix theory, based on D0-branes, and AdS/CFT duality, based on D3-branes.

One should include higher powers of fields to give formulas that correctly describe non-Abelian D-brane physics for strong fields. The goal is to capture the physics in the regime of approximation in which the background fields and the world-volume gauge fields are allowed to be arbitrarily large, but their variation is small over distances of order the string scale.

The requirement of slow variation is meant to justify dropping terms involving derivatives of fields. The tricky issue in the non-Abelian case is that one should use covariant derivatives to maintain gauge invariance, but there are relations of the form

$$[D_\alpha, D_\beta] \sim F_{\alpha\beta}.$$

This makes it somewhat ambiguous whether a term is derivative or not. Nonetheless, some success has been achieved, which will now be sketched.

In addition to the background fields g, B, ϕ and C, the desired actions contain adjoint gauge fields A and $9-p$ adjoint scalars Φ^i, both of which are represented as hermitian $N \times N$ matrices. The notation is

$$A_\alpha = \sum_n A_\alpha^{(n)} T_n \quad \text{and} \quad \Phi^i = \sum_n \Phi^{i(n)} T_n,$$

where T_n are N^2 hermitian $N \times N$ matrices satisfying

$$\mathrm{Tr}(T_m T_n) = N \delta_{mn}.$$

Let us start with the non-Abelian D9-brane action, which is relatively simple, because there are no scalar fields. In this case the proposed DBI term S_1 is

$$-T_{\rm D9}\int d^{10}\sigma e^{-\phi}\,{\rm Str}\left(\sqrt{-\det\left(g_{\alpha\beta}+B_{\alpha\beta}+2\pi\alpha' F_{\alpha\beta}\right)}\right).$$

The determinant refers to the 10×10 matrix labelled by the Lorentz indices. However, the expression inside the determinant is also an $N\times N$ matrix. The square root of the determinant is computed for each of the N^2 matrix elements, and then the trace of the resulting $N\times N$ matrix is evaluated. Str is the symmetrized trace.

The proposed non-Abelian D9-brane Chern–Simons term is

$$S_2 = \mu_9 \int \mathrm{Str}\left(C\, e^{B+2\pi\alpha' F}\right)_{10}.$$

Starting from this ansatz for the $p = 9$ case, Myers was able to deduce a unique formula for all the $p < 9$ cases by implementing consistency with T-duality.

Myers effect

The formula that Myers obtained has a very complicated Φ dependence. We will settle here for pointing out an interesting feature of the result: In the Abelian case a Dp-brane can couple to the R-R potentials $C_{p-1}, C_{p-3}, \ldots$ in addition to the usual C_{p+1}. In the non-Abelian case the Dp-brane can also couple to the higher-rank R-R potentials $C_{p+3}, C_{p+5}, \ldots$.

The coupling of non-Abelian D-branes to higher-rank R-R potentials has interesting consequences. The simplest example concerns N coincident D0-branes in the presence of constant 4-form flux $F_4 = dC_3$. The flux is chosen to be electric, with nonzero components

$$F_{0ijk} = f\epsilon_{ijk},$$

where f is a constant. All other background fields are set to zero, and the background geometry is assumed to be 10d Minkowski spacetime.

The relevant terms that need to be considered are a kinetic energy term proportional to $\mathrm{Tr}(\dot{\Phi}^i\dot{\Phi}^i)$, which comes from the DBI term, and a potential energy term

$$V(\Phi) \sim -\frac{1}{4}\mathrm{Tr}([\Phi^i, \Phi^j][\Phi^i, \Phi^j]) - \frac{i}{3} f \epsilon_{ijk} \mathrm{Tr}(\Phi^i \Phi^j \Phi^k).$$

The first term in the potential comes from the DBI action. The second term is the coupling to the R-R 4-form electric field, which comes from the non-Abelian CS action.

Now let's look for a static solution for which the potential is extremal, which requires

$$[[\Phi^i, \Phi^j], \Phi^j] + i f \epsilon_{ijk} [\Phi^j, \Phi^k] = 0.$$

A class of solutions of this equation is obtained by letting

$$\Phi^i = f\alpha^i/2,$$

where α^i is an N-dimensional representation of $SU(2)$ satisfying

$$[\alpha^i, \alpha^j] = 2i\epsilon_{ijk}\alpha^k.$$

This gives many possible solutions (besides zero) if N is large—one for each partition of N. However, the one of lowest energy is given by the N-dimensional irreducible representation of $SU(2)$, which satisfies

$$\mathrm{Tr}(\alpha^i\alpha^j) = \frac{1}{3}N(N^2-1)\delta_{ij}.$$

In the Abelian theory $2\pi\alpha'\Phi^i$ is interpreted as a transverse coordinate of the D-brane. In the non-Abelian theory this is an $N \times N$ matrix, so this identification is not so straightforward anymore. In the absence of the four-form electric field, the preferred configurations that minimize the potential have $[\Phi^i, \Phi^j] = 0$. This allows one to define a moduli space on which these matrices are simultaneously diagonal. Then one can interpret the diagonal entries as characterizing the positions of the N D-branes.

In the presence of the four-form flux, the Φ^i no longer commute at the extrema of the potential, and so the classical interpretation of the D-brane positions breaks down. There is an irreducible fuzziness in the description of their positions.

One can define the mean-square value of the ith coordinate (averaged over all N D-branes) to be

$$\langle (X^i)^2 \rangle = \frac{1}{N}(2\pi\alpha')^2 \mathrm{Tr}[(\Phi^i)^2].$$

Summing over the three coordinates gives a fuzzy sphere whose radius R squared is the sum of three such terms. Substituting the ground-state solution gives

$$R^2 = (\pi\alpha' f)^2 (N^2 - 1).$$

For large N the uncertainty δR is proportional to $1/N$, and the radius is approximately $R = \pi\alpha' f N$. This is proportional to the electric field and the number of D0-branes.

If one used a reducible representation of $SU(2)$ instead, one would find a set of concentric fuzzy spheres, one for each irreducible component. However, such solutions are energetically disfavored.

The fuzzy sphere has an alternative interpretation as a spherical D2-brane with N dissolved D0-branes. For large N this can be analyzed using the Abelian D2-brane theory. The total D2-brane charge is zero, but there is a nonzero D2-charge electric dipole moment, which couples to the four-form electric field. The previous results can be reproduced, at least for large N, in this picture.