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HOLOGRAPHY, DUALITY AND HIGHER-SPINS

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SUMMARY

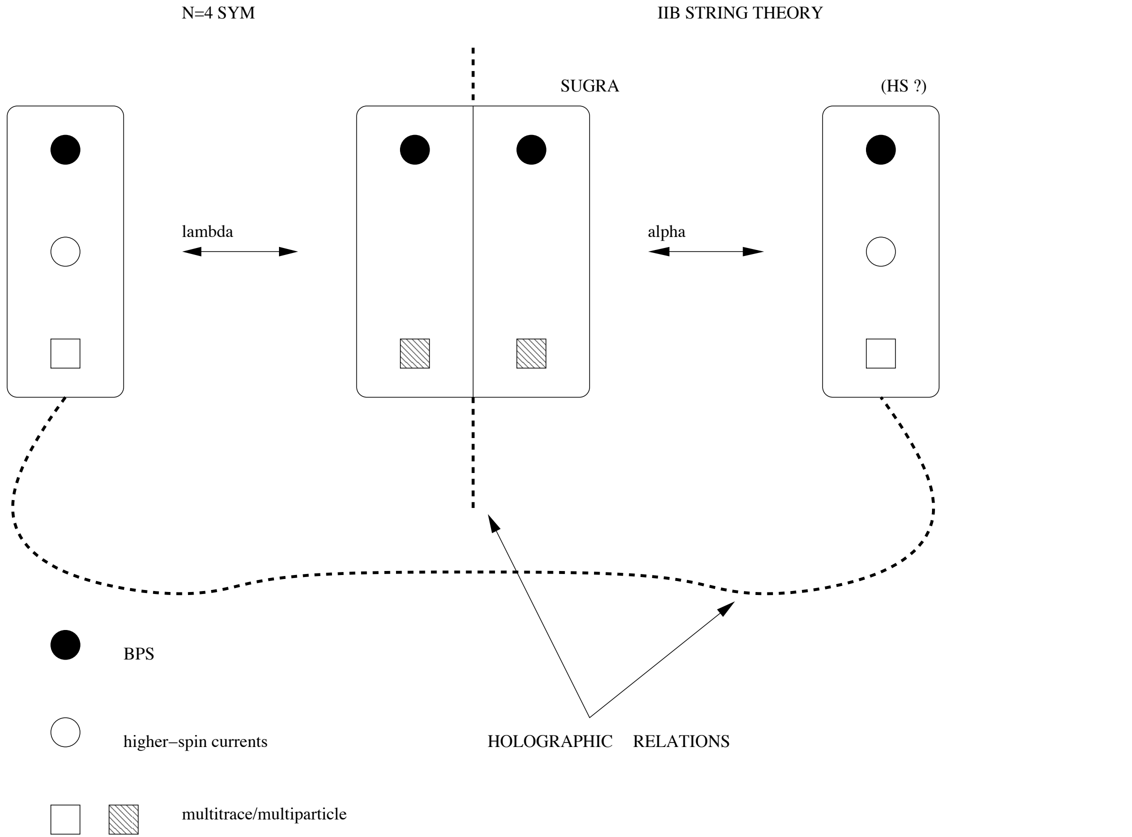
1. Relevance of higher-spins to holography: Type-H holography and the three-dimensional $O(N)$ vector model.

2. "Double-Trace" deformations in Conformal Field Theories and the first signs of higher-spin dualities.

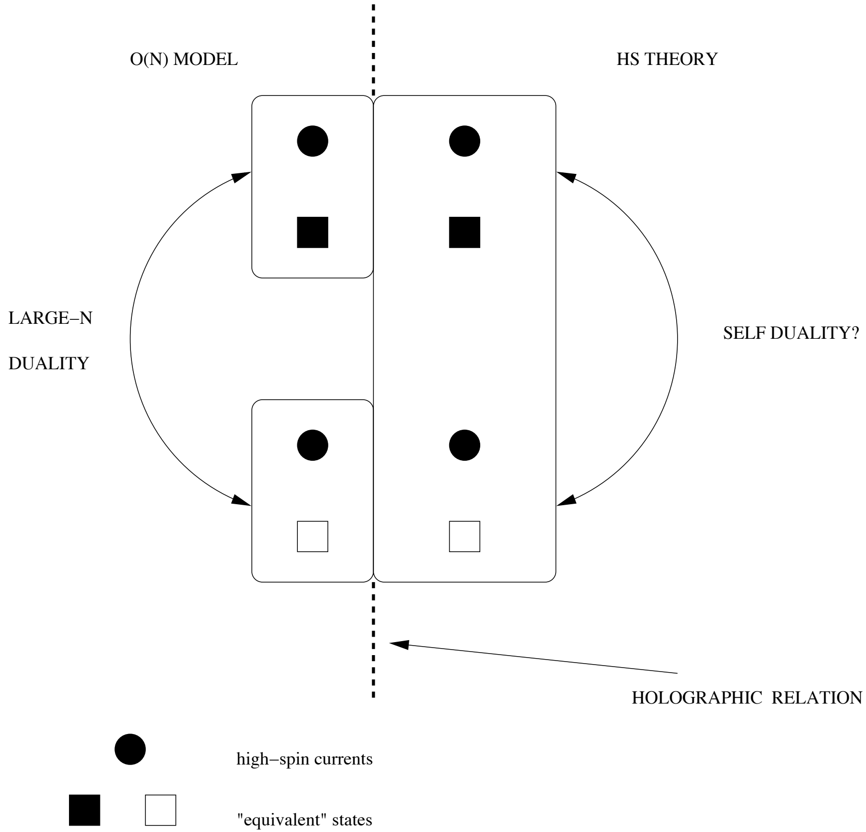
3. Explicitly: duality canonical transformations in a $U(1)$ gauge theory on AdS_4 induce boundary S-transformations.

4. Outlook (and preliminary results on duality transformations in linearized higher-spin theories).

Type-S Holographic Correspondence: $\mathcal{N}$=4 SYM/IIB String Theory



Type-H Holographic Correspondence: O(N) Vector Model/HS on AdS_4



1. Type-H Holography

The set of quasi-primary operators in a CFT form an algebra under the OPE [Mack (74)]. For two such operators $A(x)$ and $B(x)$ we may expand as

$$A(x)B(0) = \sum_{\{Q\}} C(x,\partial)Q(0)$$

The coefficients $C(x,\partial)$ are fully determined in terms of the spin, scaling dimension and 3-pt function couplings of the quasi-primary operators $\{Q\}$ involved in the OPE [Hoffmann, Petkou, Rühl (00)].

Quasi-primary operators are the "building-blocks" of a CFT and they give information about its dynamics.

Example: consider a scalar quasi-primary operator $\Phi(x)$ with dimension Δ.

$$\begin{aligned}\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle &= \sum_{\{Q\}} C(x_{12},\partial_2)C(x_{34},\partial_4)\langle Q(x_2)Q(x_4)\rangle \\ &= \frac{1}{(x_{12}^2x_{34}^2)^{\Delta}}\sum_{\{\Delta_s,s\}} g_{\Delta_s,s}\, H_{\Delta_s,s}(v,Y) \\ &= \frac{1}{(x_{12}^2x_{34}^2)^{\Delta}}\sum_{\{\Delta_s,s\}} g_{\Delta_s,s}\, v^{\frac{\Delta_s-s}{2}}Y^s\Big[1+O(v,Y)\Big]\ ,\end{aligned}$$

where

$$v=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\ ,\qquad u=\frac{x_{12}^2x_{34}^2}{x_{14}^2x_{23}^2}\ ,\qquad Y=1-\frac{v}{u},\qquad x_{ij}^2=|x_i-x_j|^2\ .$$

$g_{\Delta_s,s}$ are the "3-pt function couplings".

$H_{\Delta_s,s}(u,Y)$ are explicitly known functions that give the contribution of an operator with spin s and dimension Δ_s to the OPE.

The terms

$$v^{\frac{\Delta_s - s}{2}} Y^s$$

correspond to the leading short-distance behavior of the 4-pt function for $x_{12}^2, x_{34}^2 \to 0$.

In a free CFT the set $\{Q\}$ includes, among others, an infinite set of quasi-primary conserved tensor operators with spin s and *canonical* dimension

$$\Delta_s = d - 2 + s$$

These are the **higher-spin conserved currents** or **conformal higher-spins**. Their *leading* contributions in the OPE are terms of the form

$$v^{\frac{d-2}{2}} Y^s \, .$$

[*Such terms in an OPE also come from descendant operators.*]

Holographic description of 4-pt functions:

The *classical* action on AdS gives non-trivial boundary correlators via *tree-level* bulk-to-boundary graphs [Witten (98)] [e.g. $\alpha' = 0 \Rightarrow g^2N = \infty$ *limit of the AdS/CFT Correspondence.*]

It appears that in order to get boundary 4-pt functions that have higher-spin current contributions - e.g. free field theory ones - it is *necessary* to consider bulk massless (gauge) fields.

A concrete example of the above is provided by explicit calculations in $\mathcal{N} = 4$ SYM_4 via the AdS/CFT correspondence [Arutyunov, Frolov, Petkou (00)].

Consider the 4-pt function of the lowest dimension chiral primary operators (CPOs) of $\mathcal{N}=4$, [scalar operators with $\Delta=2$ in the [0,2,0] of the SU(4) R-symmetry group]. In the free field theory limit we have

$$\frac{\delta^{I_1 I_2}\delta^{I_3 I_4}}{400}\langle Q^{I_1}(x_1)...Q^{I_4}(x_4)\rangle_{free} =$$
$$= \frac{1}{(x_{12}^2 x_{34}^2)^2}\Big[1+\frac{1}{20}v^2+\frac{1}{20}v^2(1-Y)^{-2}$$
$$+\frac{4}{N^2}(\frac{1}{6}[v+v(1-Y)^{-1}]+\frac{1}{60}v^2(1-Y)^{-1})\Big]$$
$$= \frac{1}{(x_{12}^2 x_{34}^2)^2}\Big[...+\frac{4}{6N^2}\sum_{l=2}^{\infty} vY^l+...\Big]$$

In the last line of we see the contribution of higher-spin conserved currents in the *connected* part of the correlator.

The leading contribution comes from the energy-momentum tensor which appears *always* in the OPE. Its contribution is

$$
\begin{aligned}
H_{e.m.}(v,Y) &= vF_1(Y)[1+O(v,Y)] \\
F_1(Y) &= \frac{4Y^2-8Y}{Y^3}+\frac{4(-6+6Y-Y^2)}{Y^3}\ln(1-Y) \\
&\longrightarrow Y^2+O(Y^3)
\end{aligned}
$$

One can identify the contributions from all the higher-spin currents and even calculate their "couplings" after subtracting the descendants.

The perturbative corrections to the connected part are of the form

$$
\frac{1}{N^2}[connected]=\frac{1}{N^2}[connected]_{free}+\frac{1}{N^2}g_{YM}^2N\,F(v,Y)\ ,
$$

$$
F(v,Y)\sim\sum_l v\,Y^l\ln v+...
$$

The above terms can be attributed to an infinite set of ”nearly conserved” higher-spin currents i.e. quasi-primary operators whose scaling dimensions have being shifted from their canonical values as

$$\Delta_{HS} \longrightarrow \Delta_{HS} + \gamma = 2 + s + (g^2_{YM} N)\eta_s + \cdots$$

We may view the above effect as a small deformation of the energy spectrum of the boundary CFT.

[On AdS, this deformation should correspond to a *monster Higgs effect* [Bianchi (04)]].

One can calculate the same 4-pt function using IIB supergravity [Arutyunov, Frolov (99)]. The result is highly non trivial, but has the simple short-distance expansion

$$\frac{1}{N^2}[connected]_{IIB} = \frac{1}{N^2}\frac{1}{(x_{12}^2 x_{34}^2)^2}\Big[vF_1(Y) + O(v^2, Y)\Big]$$

We notice that the expansion of such a non-trivial function reveals the presence of *only* the energy momentum tensor and the absence of *all* higher-spin currents.

This shows *how far away supergravity is from a holographic description of perturbative CFTs*. This shows also the *necessity to consider HS gauge theories if we wish to describe holographically perturbative CFTs*.

QUESTION: Why bother for a holographic description of a well-understood, non-gravitational system such as perturbative QFT?

ANSWER: Perhaps that well-understood system will teach us something about a (quantum?) gravitational system on an (A)dS space.

If that is the case, we should better aim for a description of a *four-dimensional* gravitational system - it is slightly easier then to argue that we do Physics.

Therefore, we will study *three-dimensional* CFTs. A concrete proposal for the holographic correspondence between a 3-d CFT and a 4-d HS gauge theory has been made [Klebanov, Polyakov (02)]: the $O(N)$-singlet part of the critical three-dimensional $O(N)$ vector model is the holographic dual of the simplest HS gauge theory on AdS_4, (hs_4) a theory that contains bosonic symmetric traceless even-rank tensors.

The elementary fields of the (Euclidean) three-dimensional $O(N)$ vector model are the scalars

$$\Phi^a(x)\,, \quad a = 1, 2, .., N$$

constrained by

$$\Phi^a(x)\Phi^a(x) = \frac{1}{g}$$

The model approaches a free field theory for $g \to 0$.

To calculate the partition function in the presence of sources $J^a(x)$ it is convenient to introduce the Lagrange multiplier field $G(x)$ as

$$Z[J^a] = \int (\mathcal{D}\Phi^a)(\mathcal{D}G) e^{-\frac{1}{2}\int \Phi^a(-\partial^2)\Phi^a + \frac{i}{2}\int G(\Phi^a\Phi^a - \frac{1}{g}) + \int J^a \Phi_a}$$

We then consider the effective coupling $\widehat{g} = gN$ which for large-N may be adjusted to remain $O(1)$ as $g \to 0$.

Integrating out the $\Phi^a s$ and setting $G(x) = G_0 + \lambda(x)/\sqrt{N}$ we obtain a [renormalizable] $1/N$ expansion. Then, we set $G_0 = 0$ to obtain the critical (i.e. CFT) theory whose generating functional is

$$\frac{Z[J^a]}{Z[0]} = \int (\mathcal{D}\lambda) e^{-\frac{N}{2}[Tr(\ln(1 - \frac{i}{\sqrt{N}}\frac{\lambda}{-\partial^2})) + \frac{i}{\sqrt{N}}\frac{\lambda}{\widehat{g}}]} \times e^{\frac{1}{2}\int J^a \frac{1}{-\partial^2}(1 - \frac{i}{\sqrt{N}}\frac{\lambda}{-\partial^2})^{-1} J^a}$$

and the critical coupling is determined by the gap equation

$$\frac{1}{\widehat{g}_*} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{p^2}$$

The basic propagator of the $\Phi^a(x)s$ is

$$\Delta(x) = \frac{1}{4\pi}\frac{1}{(x^2)^{1/2}}$$

The composite field λ acquires an effective propagator K^{-1} with

$$K^{-1} = \left[\frac{\Delta^2}{2}\right]^{-1} = -\frac{16}{\pi^2}\frac{1}{x^4}$$

Now we can calculate all n-pt functions of Φ^a. For example, the 2-pt function is given by

$$\langle \Phi^a(x)\Phi^b(0)\rangle = \delta^{ab}\frac{1}{(x^2)^{1/2}}[1-\eta_1 \ln x^2 + ...] \ , \qquad \eta_1 = \frac{1}{N}\frac{4}{3\pi^2}$$

Notice that the elementary fields have acquired an anomalous dimension η_1 - contrast with $\mathcal{N}=4$ SYM.

A holographic description of the $O(N)$ vector model should reproduce this result from a bulk calculation, however, such a calculation is still elusive. The difficulty is that Φ^a would correspond to the singleton $D(1/2,0)$ of $SO(3,2)$ and we do not know the bulk action for that.

On the other hand, bulk fields would give the correlation functions of composite boundary operators. The generating functional for one such operator may be obtained if we consider a external source A for the fluctuations of the auxiliary field λ as

$$\frac{Z[J^a]}{Z_0} \rightarrow Z[A]$$
$$= \int (\mathcal{D}\lambda) e^{-\frac{1}{2}\int \lambda K \lambda - \frac{i}{3!\sqrt{N}}\int K_3 \lambda\lambda\lambda - \frac{1}{8N}\int K_4 \lambda\lambda\lambda\lambda + \cdots + \int A\lambda}$$

K_3, K_4,.. are connected correlators.

This is a generating functional $e^{\widehat{W}[A]}$ for a conformal scalar operator λ with a dimension

$$\Delta = 2 - \frac{32}{3\pi^2}\frac{1}{N} + O(\frac{1}{N^2})$$

[*To be precise, since*

$$\int (\mathcal{D}\lambda) e^{-\frac{1}{2}\int \lambda K \lambda + \int A\lambda} = e^{\frac{1}{2}\int A \Pi \lambda}$$

and Π *gives a non-positive 2-pt function, we should consider*

$$W[A] = \widehat{W}[iA]$$

]

The proposal of Klebanov-Polyakov is then

$$e^{W[A]} \equiv \int_{AdS_4} (\mathcal{D}\Phi) e^{-I_{HS}(\Phi)}$$

$$I_{HS}(\Phi) = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{g} \left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2} m^2 \Phi^2 + ... \right]$$

$m^2 = -2$ that corresponds to a conformally coupled scalar.

The problem in hand naturally asks for a "bottom-up" approach: use the full knowledge of the boundary effective action in order to calculate the bulk path integral.

In principle we should be able to have control of the *fully quantized* bulk theory: bulk quantum corrections would correspond to the $1/N$ corrections of a renormalizable boundary theory.

For the time being, however, one can be content if knowledge of the boundary generating functional for composite operators can help the calculation of the elusive non-linear classical bulk action for a HS gauge theory.

The "lifting program" [Petkou (02)]:

A possible form of the bulk HS action is

$$I_{HS}(\Phi) = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{g}\Big[\frac{1}{2}(\partial\Phi)^2 - \Phi^2 + \frac{g_3}{3!}\Phi^3 + \frac{g_4}{4!}\Phi^4 + \sum_{s=2}^{\infty}\mathcal{G}_s h^{\{\mu_1\ldots\mu_s\}}\Phi\,\partial_{\{\mu_1}\ldots\partial_{\mu_s\}}\Phi + \cdots\Big]$$

$\mathcal{G}_s$ denote the couplings of Φ to the higher-spin gauge fields $h^{\{\mu_1,..,\mu_s\}}$.

From the above, by the standard holographic procedure, we get *unambiguously* the 2- and 3-pt functions of the boundary field λ.

The 4-pt function depends on g_4, g_3^2 and on *all* the higher-spin couplings $\mathcal{G}_s$. It is technically not impossible to calculate the bulk tree-level exchange of higher-spin currents [Manvelyan, Rühl (04-05)]. Bulk gauge invariance should be sufficient to determine *all* the $\mathcal{G}_s$ in terms of only one of them.

Schematically we have

$$\langle \lambda\lambda\lambda\lambda \rangle \quad \sim \quad g_4 \bigotimes + \quad g_3^2 \Big[\bigotimes + \text{crossed} \Big] + \sum_s \mathcal{G}_s \Big[\cdots \Big]$$

The above must be compared with the corresponding result obtained from the $W[A]$. This would allow us to fix the scalings of various coefficients as (we set the AdS radius to 1)

$$\frac{1}{2\kappa_4^2} \sim N \ , \quad g_3 \, , g_4 \sim O(1) \ , \quad \mathcal{G}_s^2 \sim O(\frac{1}{N}) \ \text{ iff } \ \langle h_s \, h_s \rangle \sim O(1)$$

The first result obtained this way was [Petkou (02)]

$$g_3 = 0$$

This has been confirmed by a direct calculation using the Vasiliev equations on AdS_4 [Sezgin, Sundell (03)]. Further results have been reported in [Manvelyan, Rühl (03)-(05)].

To Do

- Reproduce, from bulk loops, the *known* boundary anomalous dimension of the operator λ and of all the higher-spin currents [for $s > 2$]

$$\eta_s = \frac{1}{N}\frac{16}{3\pi^2}\frac{s-2}{(2s-1)} \rightarrow_{s\rightarrow\infty} 2\eta_1[1 - \frac{3}{2s} + ...]$$

  [*Manifestation of and old argument by Parisi*.] Understand the bulk Higgs mechanism responsible for the above anomalous dimensions.

- Is there "integrability" of the spectrum of anomalous dimensions? Is there a "semiclassical" AdS string (membrane?) behind?

- Temperature: How is the bulk Hawking temperature related to the known critical temperature for the $O(N) \rightarrow O(N-1)$ phase transition?

2. "Double-Trace" Deformations and the First Sign of Duality

In the $O(N)$ Vector Model we have started with an elementary field Φ^a with dimension $\Delta = 1/2 + O(1/N)$ and obtained a composite operator λ with $\Delta_\lambda = 2 + O(1/N)$. It follows that we are dealing with an interacting CFT even for $N \to \infty$, since the free CFT would have had a composite operator like

$$\Phi^2 \sim \frac{1}{\sqrt{N}} \Phi^a \Phi^a$$

with $\Delta_{\Phi^2} = 1$.

Where is the free theory?

Consider the Legendre transform of $W[A]$ as

$$\begin{aligned} W[A] + \int AQ &= \Gamma[Q] \\ \Gamma[Q] &= \Gamma_0[Q] + \frac{1}{N}\Gamma_1[Q] + ... \\ \Gamma_0[Q] &= \frac{1}{2}\int Q\, K^{-1}\, Q + \frac{1}{3!\sqrt{N}}\int K_3^{-1} QQQ + \cdots \end{aligned}$$

K_3^{-1} ... are the amputated correlators.

$\Gamma_0[Q]$ is the generating functional for the correlation functions of the free field $\frac{1}{\sqrt{N}}\Phi^a\Phi^a$ with dimension $\Delta = 1$.

Therefore, via the Legendre transform we have the holographic description of a free field theory!

The theory described by $\Gamma[Q]$ has imaginary couplings and anomalous dimensions below the unitarity bounds [Petkou (96)]. Nevertheless, it seems that the theories described by $W[A]$ and $\Gamma[Q]$ are holographic duals of a *unique* HS bulk theory.

For $N \to \infty$ the spectra of the two above theories are *almost* the same. The $W[A]$ theory also contains higher-spin conserved currents for $N \to \infty$ [recall the values of the anomalous dimensions].

The *only* difference between the two theories at leading-N is the interchange of the two scalar operators λ and Φ^2, that correspond to the following *Weyl equivalent* UIRs of SO(3,2)

$$D(1,0) \longleftrightarrow D(2,0)$$

The crucial observation is that the above two theories are related to each other by an *underlying dynamics that appears to be generic in non-trivial models of three-dimensional CFTs i.e the* $O(N)$ *vector model, the Gross-Neveu model, the Thirring model*.

This dynamics is commonly reffered to as *double-trace deformations*, takes its name from real double-trace deformations of conformal gauge theories - i.e. $\mathcal{N} = 4$ SYM.

We conjecture that this particular underlying dynamics is related to some kind of a duality.

Evidence for our conjecture: consider an operator $Q(x)$ with a dimension $\Delta = 1$ i.e. an operator in free field theory. $Q^2(x)$ is a relevant deformation and we can consider the deformed 2-pt function as

$$\langle Q(x_1)Q(x_2)\, e^{\frac{f}{2}\int Q^2(x)}\rangle = \langle Q(x_1)Q(x_2)\rangle_f$$
$$= \langle Q(x_1)Q(x_2)\rangle_0 + \frac{f}{2}\int d^3x \langle Q(x_1)Q(x_2)Q^2(x)\rangle_0 + ...$$

We now make a large-N factorization assumption such that

$$\frac{1}{2}\langle Q(x_1)Q(x_2)Q^2(x)\rangle_0 \simeq \langle Q(x_1)Q(x)\rangle_0 \langle Q(x_2)Q(x)\rangle_0 + O\left(\frac{1}{N}\right)$$

Then the series can be summed and we obtain

$$\begin{aligned}\langle Q(x_1)Q(x_2)\rangle_f &= \langle Q(x_1)Q(x_2)\rangle_0 \\ &\quad + f\int d^3x \langle Q(x_1)Q(x)\rangle_0 \langle Q(x_2)Q(x)\rangle_f + O\left(\frac{1}{N}\right)\end{aligned}$$

In momentum space this becomes

$$Q_f(p) = \frac{Q_0(p)}{1 - fQ_0(p)}\,, \quad Q_0(p) \simeq \frac{1}{p}\,.$$

In the infrared, i.e. for small momenta $|p| \ll f$, we find

$$f^2 Q_f(p) = -\frac{f}{1 - \frac{1}{fQ_0(p)}} \simeq -f - Q_0^{-1}(p) + O(\frac{1}{f})\ldots\,.$$

Dropping the non-conformal constant f term on the r.h.s. we obtain the 2-pt function of an operator with dimension $\Delta_f = 2$.

We see that the UV dimension $\Delta_0 = 1$ has changed to the IR dimension $\Delta_f = 2$ and that this change is induced by the double-trace deformation.

The above dynamics must be seen in AdS_4. The on-shell bulk action of a conformally coupled scalar, using the standard Poincaré coordinates, is

$$I_\varepsilon = -\frac{1}{2}\frac{1}{\varepsilon^2}\int d^3x\, \Phi(\bar{x};\varepsilon)\partial_r\Phi(\bar{x};r)\Big|_{r=\varepsilon\ll 1}$$

To evaluate it we need to solve the Dirichlet problem

$$(\nabla^2+2)\Phi(\bar{x};r)=0\ , \quad \Phi(\bar{x};r=\infty)=0\ , \quad \Phi(\bar{x};\varepsilon)=\Phi_\epsilon(\bar{x})$$

$$\Phi(\bar{x};r)=\int\frac{d^3p}{(2\pi)^3}e^{i\bar{x}\bar{p}}\Phi_\epsilon(\bar{p})\,\frac{r}{\varepsilon}\,e^{-|p|(r-\varepsilon)}$$

We proceed via the Dirichlet-to-Neumann map that relates the boundary value of a field in a certain manifold $\mathcal{M}$ to its normal derivative at the boundary

$$\Phi(x)\Big|_{x\in\partial\mathcal{M}} = f(\bar{x}) \; ; \quad \hat{\Lambda} f = n^\mu \partial_\mu \Phi(x)\Big|_{x\in\partial\mathcal{M}}$$

where n^μ is the normal to the boundary vector. Knowledge of the map $\hat{\Lambda}$ allows (in most cases) the reconstruction of the bulk metric. For the conformally coupled scalar we have the remarkably simple expression

$$\partial_r \Phi(\bar{p}; r)\Big|_{r=\varepsilon} = (\frac{1}{\varepsilon} - |p|)\Phi(\bar{p}; \varepsilon) \; .$$

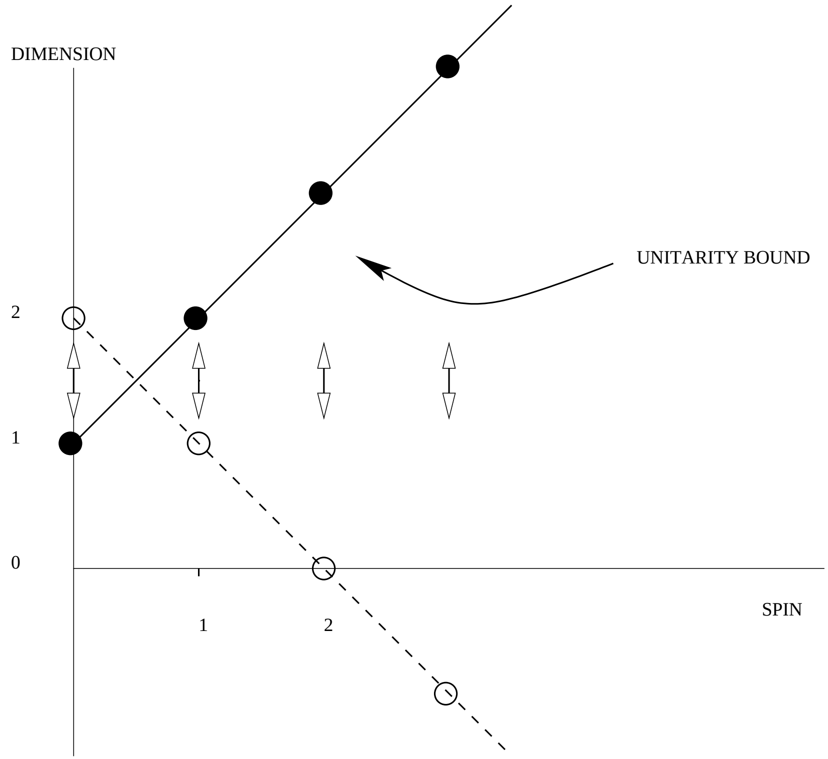
The terms in parenthesis on the r.h.s. may be viewed as a *generalized Dirichlet-to-Neumann map* since we have taken the boundary to be at $r = \epsilon$. If we set $f = 1/\epsilon$, we find

$$\frac{1}{\epsilon^2}\left[\hat{\Lambda}_\epsilon(p)\right]^{-1} \sim f^2 Q_f(p) = f^2 \frac{Q_0(p)}{1 - f Q_0(p)} \; .$$

Hence, the inversion of the generalized Dirichlet-to-Neumann map for a conformally coupled scalar corresponds to the resummation induced by a double-trace deformation on the free boundary 2-pt function. Notice that the limit $\epsilon \rightarrow 0$ drives the boundary theory in the IR.

In other words, we have found that a dynamical process [double-trace deformation] in the boundary corresponds to a "duality" in the bulk [$D(1,0) \leftrightarrow D(2,0)$].

The UIRs $D(1,0)$ and $D(2,0)$ may be viewed as "spin=0 conserved currents". Indeed, the higher-spin bosonic currents of hs$_4$ correspond to the UIRs $D(s+1,s)$ and saturate the unitarity bound $\Delta \geq s+1$.



The filled circles correspond to the higher-spin bosonic currents of SO(3,2), $D(s+1,s)$ and the empty circles to their "shadows" $D(2-s,s)$.

Looking at the plot, one could conclude that the above "duality" is a special feature of the scalar UIRs. [*Note: A similar "duality" between scalar operators with dimensions 3/2 and 5/2 is proposed to underline the Seiberg duality in four-dimensions [Klebanov, Maldacena (04)]*].

It does not seem to work for the higher-spin UIRs since the shadows have dimensions below the unitarity bound, and for $s > 2$ below zero.

On the other hand, one may ask whether the group theoretic "shadow invariance" of the higher-spin current spectrum can be detectable at all. Or else, whether there exist some particular property of 3-d CFTs that corresponds to the "shadow symmetry" of higher-spin currents on AdS_4. It is natural to imagine that this dynamical property is a "double-trace" deformation.

Motivated by the spin=0 case, consider boundary deformations of the form

$$\frac{f_s}{2}\int d^3x\, h^{(s)}h_{(s)}\,,$$

where $h^{(s)}$ denote symmetric traceless and conserved currents. Such deformations are of course *irrelevant* for all $s \geq 1$, nevertheless in many cases they lead to well-defined UV fixed points.

The crucial example is the 3-d Thirring models in the large-N limit [e.g. Hands (94), Anselmi (00)]. In this model one can show the existence of an non-trivial UV fixed point. The irrelevant operator that drives the free (IR) theory to that fixed point(UV) is of the "double-trace" form above. [A similar phenomenon occurs in the 3-d Gross-Neveu model].

Nevertheless, we will still be left with the problem of unitarity...

3. $U(1)$ Field on AdS$_4$: Bulk Duality and Boundary Transformations

The remedy of the above problem is suggested by old studies of three- dimensional gauge theories. In particular, it is well-known that to a three-dimensional gauge potential $A_i(p)$ corresponds [in momentum space], a conserved current

$$J_i(x) \propto i\epsilon_{ijk}\, p_j\, A_k(p)$$

Similarly, to a three-dimensional spin-2 gauge potential $g_{ij}(p)$ [symmetric, conserved, traceless tensor], corresponds a conserved current

$$T_{ij}(p) \propto \Pi^{(1.5)}_{ijkl}(p)\, g_{kl}(p)$$

[see below for the definitions].

A similar construction associates to each gauge field belonging to the irrep $D(2-s,s)$ a physical current in the irrep $D(s+1,s)$.

Therefore, we need actually two steps to understand the effect of the double-trace deformations: Starting with a theory that has a field in the $D(s+1,s)$ irrep:

i) The deformation will produce an operator that transforms under $D(2-s,s)$. This may be viewed as a gauge field and hence there is no necessity to belong to a unitary irrep.

ii) To this operator we associate a "dual" conserved current $D(s+1,s)$ and hence we are led to a "dual" CFT.

Moreover, these conserved currents have opposite parity from the gauge fields, so we expect that parity plays a role in our discussion.

The explicit example [Witten (03); Leigh, Petkou (03)]:

Consider a boundary CFT with a conserved current J_i having momentum space 2-pt function

$$\langle \mathcal{J}_i \mathcal{J}_i \rangle_0 \equiv (\mathcal{J}_0)_{ij} = \tau_1 \frac{1}{|p|} \Pi_{ij} + \tau_2 \varepsilon_{ijk} p_k \,, \quad \Pi_{ij} \equiv p_i p_j - \delta_{ij} p^2 \,.$$

The term proportional to τ_2 is a parity breaking term, special to three dimensions.

Consider the irrelevant double-trace deformation

$$\frac{f_1}{2} \int \mathcal{J}_i \mathcal{J}_i \,,$$

and calculate

$$(\mathcal{J}_{f_1})_{ij} \equiv \langle \mathcal{J}_i \mathcal{J}_i \, e^{\frac{f_1}{2} \int \mathcal{J}\mathcal{J}} \rangle = (\mathcal{J}_0)_{ij} + \frac{f_1}{2} \int \langle \mathcal{J}_i \mathcal{J}_j \Big| \mathcal{J}_k \mathcal{J}_k \rangle + ...$$

Now assume:

i) large-N expansion

$$\mathcal{J}_i \mathcal{J}_j \sim (\mathcal{J})_{ij} + O(1/N)$$

ii) existence of a UV fixed-point.

The leading-N resummation yields

$$f_1^2 (\mathcal{J}_{f_1})_{ij} = \hat{\tau}_1 \frac{1}{|p|} \Pi_{ij} + \hat{\tau}_2 \varepsilon_{ijk} p_k$$

$$\hat{\tau}_1 \simeq \frac{f_1}{|p|} + \frac{1}{|p|^2} \frac{\tau_1}{\tau_1^2 + \tau_2^2} + \ldots$$

$$\hat{\tau}_2 \simeq -\frac{1}{|p|^2} \frac{\tau_2}{\tau_1^2 + \tau_2^2} + \ldots$$

Dropping the non-conformally invariant term $f_1/|p|$ we get

$$f_1^2 (\mathcal{J}_{f_1})_{ij} = \frac{\tau_1}{\tau_1^2 + \tau_2^2} \frac{1}{|p|^3} \Pi_{ij} - \frac{\tau_2}{\tau_1^2 + \tau_2^2} \frac{1}{p^2} \varepsilon_{ijk} p_k \ .$$

This is the 2-pt function of a conformal operator $\widehat{A}_i(\bar{p})$ transforming in the irrep $D(1,1)$. It lies below the unitary bound $\Delta \geq s+1$ of $SO(3,2)$, therefore it must be a gauge field.

Define then the current $\widehat{\mathcal{J}}_i = i\varepsilon_{ijk} p_j \widehat{A}_k$ that has 2-pt function

$$\langle \widehat{\mathcal{J}}_i \widehat{\mathcal{J}}_j \rangle = \frac{\tau_1}{\tau_1^2+\tau_2^2} \frac{1}{|p|} \Pi_{ij} - \frac{\tau_2}{\tau_1^2+\tau_2^2} \varepsilon_{ijk} p_k \, .$$

It follows that there exists a "dual" theory with current $\widehat{\mathcal{J}}_i$ that has 2-pt function obtained from the initial one by

$$\tau \to -\frac{1}{\tau} \, , \quad \tau = \tau_2 + i\tau_1 \, .$$

We conclude that the "double-trace" deformation has induced an S-transformation on the parameters of the 2-pt function of the conserved current.

The above generalized to all (bosonic) higher-spin currents in three dimensional CFTs. Example: the energy momentum tensor.

$$\langle T_{ij} T_{kl} \rangle = \kappa_1 \frac{1}{|p|} \Pi^{(2)}_{ij,kl} - \kappa_2 \Pi^{(1.5)}_{ij,kl} \, ,$$

$$\Pi^{(2)}_{ij,kl} = \frac{1}{2} \Big[\Pi_{ik} \Pi_{jl} + \Pi_{il} \Pi_{jk} - \Pi_{ij} \Pi_{kl} \Big] \, ,$$

$$\Pi^{(1.5)}_{ij,kl} = \frac{1}{4} \Big[\varepsilon_{ikp} \Pi_{jl} + \varepsilon_{jkp} \Pi_{il} + \varepsilon_{ilp} \Pi_{jk} + \varepsilon_{jlp} \Pi_{ik} \Big] \, .$$

The boundary irrelevant "double-trace" deformation

$$\frac{f_2}{2} \int T_{ij} T_{ij} \, ,$$

leads [under the same assumption regarding a large-N expansion and the existence of UV fixed point], to a "dual" theory with an energy momentum tensor that has 2-pt function obtained from the initial one by

$$\kappa \to -\frac{1}{\kappa} \, , \;\; \kappa = \kappa_2 + i\kappa_1 \, .$$

What do all that mean for the bulk HS gauge theory?

The bulk action on AdS_4

$$I = \frac{1}{8\pi}\int d^4x\sqrt{g}\left[\frac{4\pi}{e^2}F_{\mu\nu}F^{\mu\nu} + i\frac{\theta}{2\pi}\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right] .$$

leads holographically to the 2-pt function $(\mathcal{J}_0)_{ij}$ with

$$\tau = \frac{\theta}{4\pi^2} + i\frac{2}{e^2}$$

Use the ADM form of the (Euclidean) AdS_4 metric (with radius set to 1)

$$ds^2 = d\rho^2 + \gamma_{ij}dx^i dx^j\,, \quad \gamma_{ij} = e^{2\rho}\eta_{ij}\,,\ \ \gamma = \mathrm{det}\gamma_{ij}\,,\ \ i,j = 1,2,3\,.$$

I will show that a *canonical* duality transformation in the bulk, induces the S-transformation in the boundary 2-pt function.

Method: Hamilton-Jacobi Holography

The variation of the *on shell* bulk action with respect to the canonical variable A_i gives the canonical momentum Π^i at the "boundary" ρ_0.

Π^i is interpreted as the *regularized* 1-pt function in the presence of external sources - the latter are the values of A_i at the boundary.

Finally, to reach the asymptotic boundary at $\rho \to \infty$ one invokes a further technical step, [sometimes called holographic renormalization], such as to obtain finite 1-pt functions from which all correlation functions of the boundary CFT can be found.

Schematically we have

$$\frac{1}{\sqrt{\gamma}} \frac{\delta I}{\delta A_i(\rho_0, x_i)}\Big|_{on-shell} = \Pi^i(\rho_0, x_i) \sim_{\rho_0 \to \infty} \langle \mathcal{J}^i(x_i) \rangle_{A_i}$$

The canonical form of the action is

$$
\begin{aligned}
I &= -\int d\rho \int d^3x \sqrt{\gamma}\left[\Pi^i \dot{A}_i - \mathcal{H}(\Pi^i, A_i)\right] , \\
\mathcal{H}(\tau, \bar{\tau}; \Pi^i, \mathcal{B}_i) &= \frac{1}{e^2}\frac{1}{\sqrt{\gamma}}\gamma_{ij}\left(\mathcal{E}^i\mathcal{E}^j - \mathcal{B}^i\mathcal{B}^j\right) \\
&= \frac{i}{\tau-\bar{\tau}}\frac{1}{\sqrt{\gamma}}\gamma_{ij}(\Pi^i - i\mathcal{B}^i\tau)(\Pi^j - i\mathcal{B}^j\bar{\tau}) , \\
\Pi^i &= \frac{2}{e^2}\mathcal{E}^i + i\frac{\theta}{4\pi^2}\mathcal{B}^i , \;\; \mathcal{E}^i = \sqrt{\gamma}E^i , \, \mathcal{B}^i = \sqrt{\gamma}B^i ,
\end{aligned}
$$

with $E^i = F^{i0}$ and $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ the usual electric and magnetic fields. It is crucial that we have solved the Gauss Law constraint $\partial_i\mathcal{E}^i = 0$.

Next we consider canonical transformations in the bulk from the set of variables (A_i, Π^i) to the new set $(\tilde{A}_i, \tilde{\Pi}^i)$ via a generating functional of the 1st kind

$$
\mathcal{F}[A_i, \tilde{A}_i] = \frac{i}{2}\int_{\rho=fixed} d^3x \; \sqrt{\gamma}A_i(\rho, x_i)\epsilon^{ijk}\tilde{F}_{jk}(\rho, x_i) .
$$

This canonical transformation leaves the Hamiltonian invariant $H = \tilde{H}$ if

$$\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{F}}{\delta A_i} = i\tilde{B}^i \equiv \Pi^i\,, \quad \frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{F}}{\delta \tilde{A}_i} = iB^i \equiv -\tilde{\Pi}^i$$

Under the above transformations the Hamiltonian retains its from up to a $\tau \to -1/\tau$ transformation i.e.

$$\tilde{H} = H\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}, \tilde{\Pi}^i, \tilde{\mathcal{B}}_i\right)$$

[The above is the $\theta \neq 0$ extension of an old observation by Deser, Teitelboim (76) regarding the duality invariance of the action of electromagnetism.]

The two *dual* bulk actions, one written in terms of (A_i, Π^i) and the other in terms of $(\tilde{A}_i, \tilde{\Pi}^i)$, which give respectively at $\rho = \infty$,

$$\langle \mathcal{J}_i \rangle_{A_i} = i\tilde{B}_i = -\epsilon_{ijk} p_j \tilde{A}_k\,, \;\; \langle \tilde{\mathcal{J}}_i \rangle_{\tilde{A}_i} = -iB_i = \epsilon_{ijk} p_j A_k.$$

Since the Hamiltonians are *dual* as above, one expects that the parameters in the above 2-pt functions will be related by $\tau \to -1/\tau$.

We can also prove the above statement: From the above 1-pt functions we can calculate the corresponding 2-pt functions by functionally differentiating with respect to A_i and $\tilde{A}_i$ as

$$
\begin{aligned}
\frac{\delta \widehat{A}_i(\bar{x})}{\delta A_j(\bar{x})} &= \mathcal{M}_i^j(\bar{x}-\bar{y}) = \int \frac{d^3p}{(2\pi)^3} e^{i\bar{p}(\bar{x}-\bar{y})} \mathcal{M}_i^j(\bar{p}) \\
J^{ik}(\bar{p}) &= -\epsilon^{ijl} p_j \mathcal{M}_l^k(\bar{p}) \\
\widehat{J}^{ik}(\bar{p}) &= \epsilon^{ijl} p_j (\mathcal{M}^{-1})_l^k(-\bar{p})
\end{aligned}
$$

and I get

$$
\epsilon_{i\alpha\beta} J^{ik}(\bar{p}) = -\left(p_\alpha \mathcal{M}_\beta^k(\bar{p}) - p_\beta \mathcal{M}_\alpha^k(\bar{p}) \right)
$$

Finally, using $\mathcal{M}_j^i(\bar{p})(\mathcal{M}^{-1})_k^j(\bar{p}) = \delta_k^i$ we obtain

$$
\widehat{J}_{ic}(\bar{p}) J^{ik}(\bar{p}) = -p^\alpha p_\beta \left[\mathcal{M}_\alpha^k(\bar{p})(\mathcal{M}^{-1})_c^\beta(\bar{p}) - \delta_\beta^\alpha \delta_c^k \right]
$$

We make the following ansatz for the the matrix $\delta \tilde{A}_i / \delta A_j$

$$\frac{\delta \tilde{A}_i}{\delta A_j} = C_1 \frac{1}{p^2} \Pi_{ij} + C_2 \epsilon_{ijk} \frac{p_k}{|p|} + (\xi - 1) \frac{p_i p_j}{p^2} \, ,$$

where ξ plays as usual the role of gauge fixing, necessary for its inversion. Then we find, independently of ξ

$$\langle \mathcal{J}_i \mathcal{J}_k \rangle \langle \tilde{\mathcal{J}}_k \tilde{\mathcal{J}}_j \rangle = -\Pi_{ij} \, ,$$

with Π_{ij} defined previously.

It is easy to verify that the parameters in the 2-pt functions $\langle \mathcal{J}_i \mathcal{J}_k \rangle$ and $\langle \tilde{\mathcal{J}}_k \tilde{\mathcal{J}}_j \rangle$ are related by the S-transformation o.ε.δ.

The above S-transformations, combined with the trivial transformation

$$\tau \to \tau + 1 \,,$$

form the SL(2,Z) group [Witten (03)]. This transformation is the boundary image of the bulk shift of the θ-angle

$$\theta \to \theta + 2\pi \,.$$

INTRIGUING REMARK:

The parameters in 2-pt functions of spin=1 currents in 3-d CFTs appear to correspond to measurable physical quantities i.e. Ohmic and Hall contactivities in Fractional Quantum Hall Systems. Their values at different critical points [i.e. different plateaus] are related one to the other with the action of $SL(2, Z)$*, or a subgroup if it [e.g. Dolan, Lütken, (02)]. This is a phenomenological observation that asks for an explanation.*

Outlook

- Think of some unknown 3-d CFT that has spin=1 currents. To detect them, we couple external $U(1)$ gauge fields to the currents and "measure" the current's (linear) response. Suppose then that the (linear) response must be given by the boundary values of the *electric field or canonical momenta* of a $U(1)$ field on AdS_4. Then, there should exist at least another fixed point where the (linear) response is given by the boundary value of the transformed electric field or canonical momenta.

- Study models with bulk electric and magnetic charges. This will lead to 3-d models with particle-vortex duality properties .

4. Overview and Way to Go: Duality Transformations in Linearized Higher-Spin Gauge Theories on AdS_4

- I suggested that Higher-Spin Holography [Type-H] is qualitative different form the standard Supergravity Holography [Type-S]. Only the former can yield a holographic description of free theories.

- I proposed to use Type-H Holography ”bottom-up” in order to get information for gravitational theories from non-gravitational ones. Of particular importance is the AdS_4/CFT_3 case.

- As an application of the above proposal, I used dynamical information of 3-d CFTs [double-trace deformations] to argue that one of the salient features of Type-H Holography is a generalization of electric/magnetic duality to higher-spin fields. It is then natural to suggest that this duality [S-duality?] property should be the guiding principle for connecting String Theory to Higher-Spins.

In the $U(1)$ theory, any bulk canonical transformation would correspond to some transformation of the parameters in the boundary correlation functions. However, only duality transformations give the boundary the S-transformation.

CONJECTURE [Leigh, Petkou (03)]: A similar phenomenon exists for all higher-spin fields on AdS_4 i.e. there exist canonical duality transformations that induce in the boundary the S-transformation on 2-pt functions of higher-spin currents. Moreover, these duality transformations play - at the linearized level - a similar role to the electric/magnetic dualities of the $U(1)$ theory; namely they are symmetries of the equations of motion and also of the bulk action [Deser, Teitelboim (76), Henneaux, Teitelboim (04), Deser (05)]. *In the sense that the action is invariant - up to a total ''time'' derivative - when the duality transformations are implemented in terms of the gauge-fixed canonical variables: the quadratic Hamiltonian is invariant and the kinetic term* $\Pi \cdot \dot{Q}$ *gives a total ''time'' derivative.*

In other words, to prove our conjecture we need to do a Hamiltonian analysis of the linearized higher-spin actions around AdS_4 and write them in a form similar to electromagnetism.

[Leigh, Petkou to appear (soon)]