On the classification of of supersymmetric backgrounds

A spinorial geometry approach

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Crete, June 2005

M-theory

J. Gillard, U. Gran and G.P. hep-th/0410155

U. Gran, G.P. and D. Roest, hep-th/0503046

IIB supergravity

J. Gutowski, U. Gran and G.P. hep-th/0501177, hep-th/0505074

J. Gutowski, U. Gran, D. Roest and G.P. in preparation

MOTIVATION

Supersymmetric solutions of supergravity theories have applications in

- String Theory, e.g. branes, intersecting branes, black holes
- AdS/CFT, gravity/Yang-Mills correspondences
- Geometry, e.g. spinorial geometry, special geometric structures

Topics

Spinors and instantons

- Self-duality and bounds
- A spinorial bound
- Instantons and spinorial geometry

Spinorial geometry and supersymmetry

The spinorial geometry of IIB supergravity

- $Spin(9,1)$ spinors from forms
- Systematics of IIB spinorial geometry
- Maximal and half-maximal H-backgrounds

- Current status
- Gauge symmetry and orbits
- Backgrounds with a $Spin(7) \ltimes \mathbb{R}^8$-, $SU(4) \ltimes \mathbb{R}^8$- and G_2- invariant Killing spinor
- Maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$- and $SU(4) \ltimes \mathbb{R}^8$-backgrounds

Conclusions and outlook

SPINORS AND INSTANTONS

Instanton bound

$$S = \int_{\mathbb{R}^4} d\text{vol tr}|F|^2 = \frac{1}{2}\int_{\mathbb{R}^4} d\text{vol tr } |F \pm \star F|^2$$
$$\mp\frac{1}{2}\int_{\mathbb{R}^4} \text{tr } F \wedge F \geq \mp\frac{1}{2}\int_{\mathbb{R}^4} \text{tr } F \wedge F \ ,$$

where $F = dA + A \wedge A$ is gauge field strength. The topological term which depends on the gauge bundle but not on the connection A. The sign is chosen such that the bound is not frustrated.

(anti)-Self-duality condition

Depending on the sign of the topological term, the bound is saturated iff

$$F \pm \star F = 0$$

- Since F is a two form, it takes values in $so(4) = \Lambda^2(\mathbb{R}^4) = so(3) \oplus so(3)$. The (anti)-self-duality condition projects F onto one or the other $so(3)$ subalgebra of $so(4)$.

A spinorial bound

Define the spinor

$$\eta = F_{\mu\nu}\Gamma^{\mu\nu}\epsilon \ ,$$

where ϵ is also a spinor and Γ^μ are gamma matrices. Then

$$\begin{aligned} 0 \leq \frac{1}{2}\text{tr } \eta^\dagger\eta &= -\frac{1}{2}\text{tr } (\epsilon^\dagger F_{\mu\nu}\Gamma^{\mu\nu}F_{\rho\sigma}\Gamma^{\rho\sigma}\epsilon) \\ &= \text{tr } |F|^2\epsilon^\dagger\epsilon - \frac{1}{2}\star\text{tr } (F \wedge F)\ \epsilon^\dagger\Gamma_{(5)}\epsilon \\ &= \text{tr } |F|^2 \pm \frac{1}{2}\star\text{tr } (F \wedge F) \end{aligned}$$

for $\epsilon^\dagger\epsilon = 1$ and $\Gamma_{(5)}\epsilon = \mp\epsilon$.

Supersymmetry condition

The spinorial bound is saturated iff $\eta = 0$, i.e. iff

$$F_{\mu\nu}\Gamma^{\mu\nu}\epsilon = 0$$

for ϵ a normalized, and so non-vanishing, Weyl spinor.

- The supersymmetry condition implies that ϵ is invariant under the element $F_{\mu\nu}\Gamma^{\mu\nu}$ of $spin(4) = su(2) \oplus su(2)$. If $F_{\mu\nu}\Gamma^{\mu\nu}$ is an element of the first $su(2)$, then ϵ is in the fundament rep of the other $su(2)$ and vice-versa.
- The supersymmetry condition is equivalent to the (anti)-self-duality condition because $su(2) = so(3)$.

Instantons and spinorial geometry

$Spin(4)$ spinors from forms

Take $V = \mathbb{R}^4 = \mathbb{R} < e_1, \dots, e_4 >$, and $U = \mathbb{R} < e_1, e_2 >$, where $e_1, \dots, e_4$ orthonormal basis. Equip $U \otimes \mathbb{C}$ with the hermitian inner product

$$< z^i e_i, w^j e_j >= \sum_{i=1}^{2} (z^i)^* w^i \ .$$

The space of Dirac spinors is $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$ and the Gamma matrices act as

$$\Gamma_i \eta = e_i \wedge \eta + e_i \lrcorner \eta \ ,$$
$$\Gamma_{i+2} \eta = i e_i \wedge \eta - i e_i \lrcorner \eta \ , \qquad i = 1, 2$$

Properties

- $\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\delta_{\mu\nu}, \ \mu, \nu = 1, \dots, 4$

- Weyl spinors: $\Delta_c^+ = \Lambda^{\rm even}(U\otimes\mathbb{C})$ and $\Delta_c^- = \Lambda^{\rm odd}(U\otimes\mathbb{C})$. Therefore if $\eta\in\Delta_c^+$,

$$\eta = a1 + be_{12}\ , \qquad a,b\in\mathbb{C}$$

and if $\eta\in\Delta_c^-$,

$$\eta = ae_1 + be_2\ .$$

- The Dirac inner product is $< \cdot,\cdot >$.
- Introduce the Hermitian Gamma matrices ($\Gamma^{\bar\alpha} = \delta^{\bar\alpha\beta}\Gamma_\beta$)

$$\Gamma_\beta = \frac{1}{\sqrt{2}}(\Gamma_\beta - i\Gamma_{2+\beta})\ , \quad 1\leq\beta\leq 2$$

- Observe that $\Gamma^\alpha 1 = 0$ and that

$$1\ , \quad \Gamma^{\bar\alpha}1\ , \quad \Gamma^{\bar\alpha\bar\beta}1$$

is a Hermitian basis in the space of spinors.

Supersymmetry condition and a linear system

Since $SU(2) \subset Spin(4))$ has one type of orbit in Δ_c^+, it can always be arranged, up to a possible $SO(4)$ rotation of F, such that $\epsilon = 1$. In such case, supersymmetry condition can be written as

$$F_{\mu\nu}\Gamma^{\mu\nu}1 = F_{\bar{\alpha}\bar{\beta}}\Gamma^{\bar{\alpha}\bar{\beta}}1 + \delta^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}}1 = 0$$

Using the Hermitian basis, one finds the linear system

$$F_{\bar{\alpha}\bar{\beta}} = 0 \; , \qquad \delta^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}} = 0$$

For a real connection A, $F_{\alpha\beta} = (F_{\bar{\alpha}\bar{\beta}})^* = 0$, and these are the Donaldson equations which are equivalent to the self-duality condition.

- If A is real, the number of linearly independent solutions of the supersymmetry condition is two

$$\epsilon = 1$$
$$\epsilon = e_{12}$$

- If A is complex and $F_{\alpha\beta} \neq 0$, then the number of linearly independent solutions is one, $\epsilon = 1$.

SPINORIAL GEOMETRY AND SUPERSYMMETRY

Gillard, Gran, GP, hep-th/0410155

The Killing spinor equations of supergravity theories are the vanishing conditions of the supersymmetry transformation of the fermions restricted on the bosonic sector the theories. These give rise to a parallel transport equation of the supercovariant connection $\mathcal{D}$ and some algebraic conditions $\mathcal{A}$, i.e.

$$\mathcal{D}_M\epsilon = 0$$
$$\mathcal{A}\epsilon = 0$$

$\mathcal{D}$ and A depend of the bosonic fields of the supergravity theory.

The main ingredients of the spinorial geometry approach to solve the Killing spinor equations are

- A description of spinors in terms of forms
- The use of the gauge group of the Killing spinor equations to bring the Killing spinors into a canonical or normal form
- A basis in the space of spinors which is used to expand the Killing spinor equations

The gauge group of the Killing spinor equations are the local transformations which preserve the form of the Killing spinor transformations.

- The holonomy of the supercovariant connection is different but includes the gauge group of the supercovariant derivative

IIB SUPERGRAVITY

Schwarz, West, Howe

Fields

$$g \ \ (\text{metric}) \ , \qquad G \ \ (\text{complex } 3-\text{form}) \ ,$$
$$F \ (\text{selfdual } 5-\text{form}), \quad P \ (\text{complex } 1-\text{form})$$

Killing spinor equations

$$\mathcal{D}_M \epsilon = 0$$

$$\mathcal{A}\epsilon = P_M \Gamma^M (C\epsilon)^* + \frac{1}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon = 0$$

where ϵ is chiral complex spinor and C is a charge conjugation matrix.

Supercovariant derivative

$$
\begin{aligned}
\mathcal{D}_M \epsilon = \tilde{\nabla}_M \epsilon &- \frac{1}{96}(\Gamma_M{}^{N_1 N_2 N_3} G_{N_1 N_2 N_3} \\
&-9\Gamma^{N_1 N_2} G_{M N_1 N_2})(C\epsilon)^* \\
&+\frac{i}{480}\Gamma^{N_1 \dots N_5}\Gamma_M \epsilon F_{N_1 \dots N_5}
\end{aligned}
$$

where

$$
\tilde{\nabla}_M = D_M + \frac{1}{4}\Omega_{M,AB}\Gamma^{AB}\ , \ \ D_M = \partial_M - \frac{i}{2} Q_M
$$

and Q is a $U(1)$ connection which depends on the scalars.

- $\mathcal{D}$ is a connection of the spinor bundle and it is *not* induced from the tangent bundle of the spacetime because of the terms that contain the fluxes

- The Killing spinor equations are linear over the real numbers and so the number of Killing spinors are counted over the reals
- The gauge group of the Killing spinor equations is $Spin(9,1) \times U(1)$ while the holonomy group of the supercovariant connection is $SL(32,\mathbb{R})$
- In many cases of interest the Killing spinors are invariant under a subgroup $H \subset Spin(9,1)$ which includes a Berger type of group like $SU(4)$, $SU(3)$, G_2 and $Spin(7)$

The application of the spinorial geometry method to solve the Killing spinor equations of IIB supergravity has the following consequence:

- The Killing spinor equations of (any) IIB supersymmetric background reduce to a linear system for the fluxes, geometry and spacetime derivatives of the functions that determine the Killing spinors. A linear system also determines the field equations that are implied by the Killing spinor equations.

U. Gran, D. Roest and G.P. hep-th/0503046

$Spin(9,1)$ SPINORS FROM FORMS

Take $U = \mathbb{R} < e_1, \dots, e_5 >$, $e_1, \dots, e_5$ orthonormal basis. The $Spin(9,1)$ Dirac spinors are $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$ and the chiral (Weyl) spinors are $\Delta_c^+ = \Lambda^{\text{even}}(U \otimes \mathbb{C})$ and $\Delta_c^- = \Lambda^{\text{odd}}(U \otimes \mathbb{C})$.

The gamma matrices are represented on Δ_c as

$$\begin{aligned} \Gamma_0\eta &= -e_5 \wedge \eta + e_5 \lrcorner \eta \ , \ \Gamma_5\eta = e_5 \wedge \eta + e_5 \lrcorner \eta \\ \Gamma_i\eta &= e_i \wedge \eta + e_i \lrcorner \eta \ , \qquad i = 1, \dots, 4 \\ \Gamma_{5+i}\eta &= ie_i \wedge \eta - ie_i \lrcorner \eta \ . \end{aligned}$$

The Dirac inner product on the space of spinors Δ_c is defined as

$$D(\eta, \theta) = < \Gamma_0\eta, \theta > \quad ,$$

where

$$< z^a e_a, w^b e_b > = \sum_{a=1}^{5} (z^a)^* w^a \ ,$$

on $U \otimes \mathbb{C}$ and then extended to Δ_c, and $(z^a)^*$ is the standard complex conjugate.

A Majorana inner product is

$$B(\eta, \theta) =< B(\eta^*), \theta > \quad , \qquad B = \Gamma_{06789}$$

Observe that the inner product B is in addition Pin invariant and skew-symmetric $B(\eta, \theta) = -B(\theta, \eta)$.

The Majorana reality condition can be chosen as

$$\eta = -\Gamma_0 B(\eta^*) = \Gamma_{6789}\eta \ .$$

$C = \Gamma_{6789}$ is the charge conjugation matrix.

Example

Consider the complex chiral spinor $a1 + be_{1234}$, $a, b \in \mathbb{C}$. The associated real spinor of positive chirality is

$$\eta = a1 + a^* e_{1234} \ .$$

The two Majorana spinors are $1 + e_{1234}$ and $i1 - ie_{1234}$.

- The spinors $1+e_{1234}$ and $i(1+e_{1234})$ are invariant under the $Spin(7) \ltimes \mathbb{R}^8$ subgroup of $Spin(9,1)$
- The spinors $1+e_{1234}$, $i(1+e_{1234})$, $1-e_{1234}$ and $i(1-e_{1234})$ are invariant under the $SU(4) \ltimes \mathbb{R}^8$ subgroup of $Spin(9,1)$

Pseudo Hermitian basis

A creation annihilation basis for Dirac spinors is

$$\Gamma_{\bar{\alpha}} = \frac{1}{\sqrt{2}}(\Gamma_\alpha + i\Gamma_{\alpha+5}) \ ,$$
$$\Gamma_{\pm} = \frac{1}{\sqrt{2}}(\Gamma_5 \pm \Gamma_0) \ ,$$
$$\Gamma_{\alpha} = \frac{1}{\sqrt{2}}(\Gamma_\alpha - i\Gamma_{\alpha+5}) \ .$$

Observe that

$$\Gamma_{\bar{\alpha}}1 = \Gamma^{\alpha}1 = \Gamma_{+}1 = \Gamma^{-}1 = 0 \, , \qquad \Gamma^{B} = g^{BA}\Gamma_{A}$$

The representation Δ_c can be constructed by acting on 1 with the creation operators $\Gamma^{\bar{\alpha}}, \Gamma^{+}$. A basis in Δ_c is

$$\eta = \sum_{k=0}^{5} \frac{1}{k!} \, \phi_{\bar{a}_1 \ldots \bar{a}_k} \, \Gamma^{\bar{a}_1 \ldots \bar{a}_k} 1 \ , \qquad \bar{a} = \bar{\alpha}, + \ .$$

SYSTEMATICS OF IIB SPINORIAL GEOMETRY

Gutowski, Gran, Roest, GP

In the pseudo-Hermitian basis, the Killing spinor equations turn into a linear system for the fluxes, geometry and the spacetime derivatives of the functions that determine the Killing spinors.

The most general spinor of IIB supergravity can be written as

$$\epsilon = p1 + qe_{1234} + u^i e_{i5} + \frac{1}{2}v^{ij}e_{ij} + \frac{1}{6}w^{ijk}e_{ijk5}$$

where $p, q, u^i, v^i, v^{ij}, w^{ijk}$ are complex. Then

$$\mathcal{D}_A\epsilon = \partial_A p1 + p_0\mathcal{D}_A 1 + p_1\mathcal{D}_A(i1) + \dots$$

where $p = p_0 + ip_1$ and similarly for the rest. It turns out that $\mathcal{D}(i1)$ can be easily recovered from $\mathcal{D}_A 1$ and similarly for the rest. So derive the linear system associated with the supercovariant derivative on any spinor it suffices to express

$$\mathcal{D}_A 1 \ , \quad \mathcal{D}_A e_{ij} \ , \quad \mathcal{D}_A e_{i5} \ ,$$
$$\mathcal{D}_A e_{ijk5} \ , \quad \mathcal{D}_A e_{1234}$$

in the pseudo-Hermitian basis, and similarly for the algebraic Killing spinor equation

- A similar argument applies to the integrability conditions of the Killing spinor equations

$$\mathcal{R}_{AB} = [\mathcal{D}_A, \mathcal{D}_B]\epsilon = 0 \ , \quad [\mathcal{D}_A, \mathcal{A}]\epsilon = 0$$

  to re-express them as a linear system which can be used to determine the field equations and Bianchi identities that are implied by the Killing spinor equations.

- For arbitrary spinors, the resulting linear systems may be rather complicated. However, they can be simplified in many cases.

MAXIMAL AND HALF-MAXIMAL H-BACKGROUNDS

J. Gutowski, U. Gran and GP, hep-th/0505074

For any subgroup $H \subset Spin(9,1)$, there is a basis in the space, Δ^H, of H-invariant spinors of the type $(\eta_p, i\eta_p)$, where η_p are Majorana spinors. Set $\eta_{m+p} = i\eta_p$.

The maximal H-backgrounds admit $N = 2m = \dim\Delta^H$ Killing spinors ϵ_i given by

$$\epsilon_i = \sum_{j=1}^{N} f_{ij}\eta_j \ , \qquad \det f \neq 0 \ ,$$

where $f = (f_{ij})$ is a matrix of real functions.

It turns out that the Killing spinor equations factorize, i.e. the terms containing the fluxes P, G separate from the rest. The Killing spinor equations become

$$\begin{aligned} P_A \Gamma^A \, \eta_p &= 0 \;, \quad p = 1, \ldots, m \\ \Gamma^{ABC} G_{ABC} \, \eta_p &= 0 \;, \quad p = 1, \ldots, m \;. \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \; & \sum_{j=1}^{N} [(f^{-1} D_M f)_{pj} \; \eta_j - i(f^{-1} D_M f)_{m+pj} \; \eta_j] \\ & + \nabla_M \eta_p + \frac{i}{48} \Gamma^{N_1 \ldots N_4} \eta_p F_{N_1 \ldots N_4 M} = 0 \\ & \sum_{j=1}^{N} [(f^{-1} D_M f)_{pj} \; \eta_j + i(f^{-1} D_M f)_{m+pj} \; \eta_j] \\ & + \frac{1}{4} G_{MBC} \Gamma^{BC} \eta_p = 0 \end{aligned}$$

The solution of the Killing spinor equations leads to an equation of the form

$$f^{-1} df + C = 0$$

where C is interpreted as the restriction of the supercovariant connection on the subbundle of the Killing spinors $\mathcal{K}$

$$0 \to \mathcal{K} \to \mathcal{S} \to \mathcal{S}/\mathcal{K} \to 0$$

where $\mathcal{S}$ is the spin bundle. The necessary condition for a solution is

$$F(C) = 0$$

- Even though C must be a trivial connection, it cannot always be trivialized with a $Spin(9,1) \times U(1)$ transformation. This in turn implies that one cannot always find a gauge transformation to eliminate the dependence of the Killing spinors on the functions f

In the half-maximal case , $N = m = \frac{1}{2}\mathrm{dim}\Delta^H$ and the Killing spinors can be written as

$$\epsilon_i = \sum_{j=1}^{m} z_{ij}\eta_j \ , \qquad i = 1, \ldots, m = N \ ,$$

where z complex matrix.

- The generic case corresponds to $\det z \neq 0$ but there are a lot of special cases for which $\det z = 0$ but ϵ_i are linearly independent over the real numbers
- The Killing spinors equations simplify but do not always factorize as in the maximal cases.

CURRENT STATUS

The results obtained so far are summarized in the table below. N is the number of supersymmetries and H is the stability subgroup of the Killing spinors.

H	N = 1	N = 2	N = 3	N = 4	N = 6	N = 8	N = 16	N = 32
$Spin(7) \ltimes \mathbb{R}^8$	$\surd$	$\surd$	–	–	–	–	–	–
$SU(4) \ltimes \mathbb{R}^8$	$\surd$	$\surd$		$\surd$	–	–	–	–
G_2	$\surd$	$\odot$		$\surd$	–	–	–	–
$Sp(2) \ltimes \mathbb{R}^8$	–		$\odot$		$\odot$	–	–	–
$(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$	–			$\odot$		$\odot$	–	–
$SU(3)$	–			$\odot$		$\odot$	–	–
$\mathbb{R}^8$	–					$\odot$	$\odot$	–
$SU(2)$	–					$\odot$	$\odot$	–
1	–						$\odot$	$\surd$
							(1)	

Table:

$\surd$ solved cases.

$\odot$ cases that can be tackled.

– do not occur.

GAUGE SYMMETRY AND ORBITS

The Killing spinor equations of IIB supergravity are covariant under $Spin(9,1)$ gauge transformations, i.e.

$$U^{-1}\mathcal{D}(e,F,G)U = \mathcal{D}(e',F',G')$$

where (e',F',G') are related to (e,F,G) with a Lorentz transformation.

- To solve $\mathcal{D}\epsilon = \mathcal{A}\epsilon = 0$ for one Killing spinor ϵ suffices to take ϵ to be a representative of the orbits of $Spin(9,1)$ in $\Delta^+_{16}\oplus\Delta^+_{16}=\Delta^+_c$.

There are three types of orbits of $Spin(9,1)$ in Δ^+_{16} with stability subgroups $SU(4)\ltimes\mathbb{R}^8$, $Spin(7)\ltimes\mathbb{R}^8$ and G_2. Representatives are

$$\epsilon = (f-g_2+ig_1)1 + (f+g_2+ig_1)e_{1234}$$

$$\epsilon = (f + ig)(1 + e_{1234})$$
$$\epsilon = f(1 + e_{1234}) + ig\Gamma^{+}(e_1 + e_{234}) \ ,$$

respectively, where f, g, g_1, g_2 are spacetime functions.

Remarks

- The Killing spinor of IIB backgrounds with one supersymmetry depends on more than one spacetime function while the Killing spinor of eleven-dimensional $N = 1$ backgrounds depends on one function.
- In eleven-dimension there are two types of orbits while in IIB there are three.

BACKGROUNDS WITH A $Spin(7) \ltimes \mathbb{R}^8$ $SU(4) \ltimes \mathbb{R}^8$ and G_2 KILLING SPINOR

U. Gran, J. Gutowski and GP, hep-th/0501177, hep-th/0505074

Consider the $Spin(7) \ltimes \mathbb{R}^8$ case

The spacetime of $N = 1$ supersymmetric backgrounds with Killing spinor $(f + ig)(1 + e_{1234})$ admits a null, self-parallel, Killing vector field and has an $Spin(7) \ltimes \mathbb{R}^8$ structure.

Proof

One uses the pseudo Hermitian basis in the space of spinors to express some of the fluxes in terms of the geometry and to find that the geometry is restricted as

$$\Omega_{+,\alpha+} = \Omega_{\alpha,+}{}^{\alpha} = \Omega_{+,\alpha}{}^{\alpha} = 0 ,$$

$$\Omega_{\alpha,+\beta} - \frac{1}{2}\epsilon_{\alpha\beta}{}^{\bar{\gamma}_1\bar{\gamma}_2}\Omega_{\bar{\gamma}_1,+\bar{\gamma}_2} = 0 \ ,$$

$$\Omega_{+,\bar{\alpha}\bar{\beta}} - \frac{1}{2}\epsilon_{\bar{\alpha}\bar{\beta}}{}^{\gamma_1\gamma_2}\Omega_{+,\gamma_1\gamma_2} = 0 \ .$$

$$\Omega_{\alpha,+\bar{\beta}} + \Omega_{\bar{\beta},+\alpha} = 0 \ , \quad \Omega_{\alpha,+\beta} + \Omega_{\beta,+\alpha} = 0 \ ,$$

$$2\partial_+ f + \Omega_{+,-+} f + Q_+ g = 0 \ ,$$

$$2\partial_+ g + \Omega_{+,-+} g - Q_+ f = 0 \ ,$$

$$\partial_-(f^2 + g^2) + \Omega_{-,-+}(f^2 + g^2) = 0 \ ,$$

$$\partial_{\bar{\alpha}}(f^2 + g^2) + (\Omega_{\bar{\alpha},-+} + \Omega_{-,\bar{\alpha}+})(f^2 + g^2) = 0 \ ,$$

The forms associated to the Killing spinor ϵ and $\tilde{\epsilon} = C\epsilon^*$ are three one-forms

$$\kappa(\epsilon,\epsilon) = (f+ig)^2(e^0 - e^5) \ ,$$
$$\kappa(\tilde{\epsilon},\tilde{\epsilon}) = (f-ig)^2(e^0 - e^5) \ ,$$

$$\kappa(\epsilon, \tilde{\epsilon}) = (f^2 + g^2)(e^0 - e^5) ,$$

and three five-forms

$$\begin{aligned}
\tau(\epsilon, \epsilon) &= (f + ig)^2(e^0 - e^5) \wedge \phi , \\
\tau(\tilde{\epsilon}, \tilde{\epsilon}) &= (f - ig)^2(e^0 - e^5) \wedge \phi , \\
\tau(\epsilon, \tilde{\epsilon}) &= (f^2 + g^2)(e^0 - e^5) \wedge \phi ,
\end{aligned}$$

where

$$\phi = \mathrm{Re}\chi - \frac{1}{2}\omega \wedge \omega ,$$

is the $Spin(7)$ invariant four-form.

- The vector field associated to $\kappa(\epsilon, \tilde{\epsilon})$ is null, self-parallel and Killing.
- There are no restrictions on the geometry that do not involve lightcone directions.
- The spacetime admits the expected forms for an $Spin(7) \ltimes \mathbb{R}^8$ structure.

- Examples of spacetimes with such geometries are certain Lorentzian extensions of one-parameter families of eight-dimensional manifolds with a generic $Spin(7)$ structure.
- The geometry of $N = 1$ $SU(4) \ltimes \mathbb{R}^8$-backgrounds is similar to that of $Spin(7) \ltimes \mathbb{R}^8$ backgrounds. The only difference is that now the spacetime admits a $SU(4) \ltimes \mathbb{R}^8$-structure instead.
- The spacetime of $N = 1$ G_2-backgrounds admits a time-like Killing vector field, and two other distinguished directions which are not Killing. In addition, it admits a weak-like G_2 structure.

MAXIMALLY SUPERSYMMETRIC $Spin(7) \ltimes \mathbb{R}^8$- and $SU(4) \ltimes \mathbb{R}^8$-BACKGROUNDS

hep-th/0505074

The spacetime of maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$- and $SU(4) \ltimes \mathbb{R}^8$-backgrounds admits a rotation free null Killing vector field X and the holonomy of the Levi-Civita connection is in $Spin(7) \ltimes \mathbb{R}^8$ and $SU(4) \ltimes \mathbb{R}^8$, respectively.

The above conditions imply that in both cases the metric can be written as

$$ds^2 = 2dv(du + \alpha dv + \beta_I dy^I) + \gamma_{IJ} dy^i dy^J$$

where $X = \frac{\partial}{\partial u}$, $\alpha(y, v)$, $\beta(y, v)$ and $\gamma(y, v)$.

It turns out that all components of $\nabla\psi$ vanish apart from

$$\nabla_A \psi_{B_1 B_2 B_3 -} = \Omega_{A,-}{}^C \psi_{B_1 B_2 B_3 C}$$

where ψ is the $Spin(7)$-invariant or $SU(4)$-invariant forms. This implies that the holonomy of ∇ is contained in $Spin(7)\ltimes \mathbb{R}^8$ or in $U(4)\ltimes\mathbb{R}^8$, respectively.

- The spacetime can be constructed from a one-parameter family of a $Spin(7)$ or a Calabi-Yau manifold which in addition satisfies the condition that
$$\frac{1}{2}(d\beta)_{AB} + \partial_v e_{I[A} e^I{}_{B]}\ , \quad A, N = \alpha, \bar\alpha$$
lies in $Spin(7)$ or $SU(4)$, respectively.
- The maximally supersymmetric G_2-backgrounds (N=4) are diffeomeorphic to $\mathbb{R}^{1,2}\times B$, where B is a G_2 manifold and all the fluxes vanish.

CLASSIFICATION OF MAXIMALLY SUPERSYMMETRIC IIB BACKGROUNDS

J. Figueroa-O'Farrill, GP

The maximal supersymmetric solutions of IIB supergravity are locally isometric to $AdS_5 \times S^5$, Minkowski space $\mathbb{R}^{10,1}$ and the plane wave.

Proof

- Compute the supercovariant curvature, $\mathcal{R} = [\mathcal{D}, \mathcal{D}] = 0$, and set $\mathcal{R} = 0$ $(F(C) = \mathcal{R})$

- $\mathcal{R} = 0$ implies

(i) The spacetime is a symmetric space

(ii) The flux F is parallel with respect to ∇

(iii) The *new* Plücker relation

$$i_X i_Y i_Z (F)^L \wedge (F)_L = 0$$

- F in invariant. In addition the new Plücker relation implies that

$$F = K + {}^*K$$

where K is a simple five-form.

Then one has

- $M = AdS_5 \times S^5$, $\quad K^2 \neq 0$
- M plane wave , $\quad K^2 = 0$, $K \neq 0$
- M Minkowski space , $\quad F = K = 0$.

CONCLUSIONS AND OUTLOOK

- The spinorial geometry approach provides a systematic way to classify all supersymmetric backgrounds of supergravity theories and in particular those in eleven and ten dimensions.

- The Killing spinor equations and their integrability conditions turn into linear systems for the fluxes, geometry and spacetime derivatives of the functions that determine the Killing spinors. These are solved to find the conditions on the geometry of spacetime of supersymmetric backgrounds and to determine the field equations that are implied as intergrability conditions of the Killing spinor equations.

- Several supersymmetric IIB backgrounds were presented. These included maximally supersymmetric backgrounds with $Spin(7)\ltimes\mathbb{R}^8$-, $SU(4)\ltimes\mathbb{R}^8$-, G_2- and 1- invariant spinors, and all backgrounds with one supersymmetry.
- Many other cases remain to be understood.

H	N = 1	N = 2	N = 3	N = 4	N = 6	N = 8	N = 16	N = 32
$Spin(7)\ltimes\mathbb{R}^8$	$\surd$	$\surd$	$-$	$-$	$-$	$-$	$-$	$-$
$SU(4)\ltimes\mathbb{R}^8$	$\surd$	$\surd$		$\surd$	$-$	$-$	$-$	$-$
G_2	$\surd$	$\odot$		$\surd$	$-$	$-$	$-$	$-$
$Sp(2)\ltimes\mathbb{R}^8$	$-$		$\odot$		$\odot$	$-$	$-$	$-$
$(SU(2)\times SU(2))\ltimes\mathbb{R}^8$	$-$			$\odot$		$\odot$	$-$	$-$
$SU(3)$	$-$			$\odot$		$\odot$	$-$	$-$
$\mathbb{R}^8$	$-$					$\odot$	$\odot$	$-$
$SU(2)$	$-$					$\odot$	$\odot$	$-$
1	$-$						$\odot$	$\surd$

(2)