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Phases of Quantum Gravity in AdS_3 and Linear Dilaton Backgrounds

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We show that string theory in AdS_3 has two distinct phases depending on the radius of curvature $R_{AdS} = \sqrt{k} l_s$. For $k > 1$ (*i.e.* $R_{AdS} > l_s$), the $SL(2,\mathbb{C})$ invariant vacuum of the spacetime conformal field theory is normalizable, the high energy density of states is given by the Cardy formula with $c_{\rm eff} = c$, and generic high energy states look like large BTZ black holes. For $k < 1$, the $SL(2,\mathbb{C})$ invariant vacuum as well as BTZ black holes are non-normalizable, $c_{\rm eff} < c$, and high energy states correspond to long strings that extend to the boundary of AdS_3 and become more and more weakly coupled there. A similar picture is found in asymptotically linear dilaton spacetime with dilaton gradient $Q = \sqrt{\frac{2}{k}}$. The entropy grows linearly with the energy in this case (for $k > \frac{1}{2}$). The states responsible for this growth are two dimensional black holes for $k > 1$, and highly excited perturbative strings living in the linear dilaton throat for $k < 1$. The change of behavior at $k = 1$ in the two cases is an example of a string/black hole transition. The entropies of black holes and strings coincide at $k = 1$.

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1. Introduction

It is widely believed that in weakly coupled string theory in flat spacetime, a generic high energy state with particular values of mass, angular momentum and other charges looks from afar like a black hole with those charges. As the energy of the state increases, the curvature at the horizon decreases, and the black hole description becomes more and more reliable.

Conversely, as the mass of a black hole decreases, string (α') corrections in the vicinity of the horizon grow and the black hole picture becomes less and less appropriate. When α' effects at the horizon are large, *e.g.* if the string frame curvature at the horizon is much larger than the string scale, a more useful description of the state is in terms of weakly coupled strings and D-branes. The transition between the black hole picture valid for small horizon curvature, and the perturbative string picture valid for large curvature, which occurs when the curvature at the horizon is of order the string scale, is smooth. In particular, the black hole and perturbative string entropies match, up to numerical coefficients independent of the string coupling and charges.[1] This is known as the string/black hole correspondence principle [1,2].

The above discussion relies crucially on the fact that the curvature at the horizon of a black hole depends on its mass, and goes to zero for large mass. In this paper we will study situations where this is not the case. Two prototypical examples which we will discuss are the BTZ black hole in AdS_3, and the $SL(2,\mathbb{R})/U(1)$ black hole in $\mathbb{R}^{1,1}$ with asymptotically linear (spacelike) dilaton. In the former case, the curvature is the same everywhere, and is proportional to the cosmological constant of the underlying anti-de-Sitter spacetime. In the latter, the mass of the black hole determines the value of the dilaton at the horizon, while the curvature there as well as the gradient of the dilaton depend solely on the cosmological constant. Thus, for these black holes varying the mass does not change the size of the α' corrections.

It is natural to ask whether there is an analog of the string/black hole transition in these cases. We will see below that the answer is affirmative. To study the transition, we will take a slightly different approach than that of flat spacetime. Instead of varying the energy of the string or black hole, we will study the dependence of the high energy spectrum on the cosmological constant, which controls the α' corrections.

[1] Both pictures receive large corrections in the transition region, so it is difficult to compute and compare these numerical coefficients.

For AdS_3, this means varying the radius of curvature of the anti-de-Sitter space, which is related to the cosmological constant Λ via

$$\Lambda = -\frac{1}{R_{AdS}^2} \ . \tag{1.1}$$

According to the AdS/CFT correspondence [3], the spacetime dynamics is in this case given by a two dimensional conformal field theory with central charge [4]

$$c = \frac{3R_{AdS}}{2l_p} \ , \tag{1.2}$$

where l_p is the three dimensional Planck length. The expression for the central charge, (1.2), is valid when R_{AdS} is much larger than l_p, l_s ($l_s = \sqrt{\alpha'}$ is the string length). We will usually assume below that the theory is in the semiclassical regime, $R_{AdS} \gg l_p$, such that the central charge c (1.2) is very large. We expect our main results to be independent of that assumption.

The entropy of the two dimensional spacetime CFT behaves at high energies as

$$S = 2\pi\sqrt{\frac{c_{\rm eff} L_0}{6}} + 2\pi\sqrt{\frac{\bar{c}_{\rm eff} \bar{L}_0}{6}} \ , \tag{1.3}$$

where L_0 and $\bar{L}_0$ are the left and right moving spacetime scaling dimensions. They are related to energy and angular momentum in AdS_3 as follows:

$$ER_{AdS} = L_0 + \bar{L}_0 - \frac{c}{12}; \quad J = L_0 - \bar{L}_0 \ . \tag{1.4}$$

Modular invariance of the spacetime CFT, which is generally expected to be a property of quantum gravity in AdS_3, implies that the effective central charge $c_{\rm eff}$ is given by [5,6]

$$c_{\rm eff} = c - 24\Delta_{\rm min} \ , \tag{1.5}$$

where $\Delta_{\rm min}$ is the lowest scaling dimension in the spacetime theory. Unitarity of the theory implies that $\Delta_{\rm min} \geq 0$, so that $c_{\rm eff} \leq c$; $c_{\rm eff} = c$ iff the $SL(2,\mathbb{C})$ invariant vacuum is normalizable, in which case $\Delta_{\rm min} = 0$. Unitary CFT's with $c_{\rm eff} < c$ are known to exist. Examples include Liouville theory (see *e.g.* [7,8] for reviews) and the Euclidean CFT on the cigar, $SL(2,\mathbb{R})/U(1)$.

To study string theory in AdS_3 with string scale curvatures, we define the dimensionless ratio

$$\sqrt{k} = \frac{R_{AdS}}{l_s} \ . \tag{1.6}$$

We will see that the structure of the theory depends crucially on the magnitude of k. For $k > 1$, the $SL(2,\mathbb{C})$ invariant vacuum of the spacetime CFT is normalizable; hence the entropy behaves as (1.3) with $c_{\rm eff} = c$ (1.5). Generic high energy states look like BTZ black holes with mass and angular momentum given by (1.4). They are non-perturbative from the point of view of weakly coupled string theory in AdS_3.

For $k < 1$, the $SL(2,\mathbb{C})$ invariant vacuum, as well as BTZ black holes are non-normalizable; hence, $c_{\rm eff} < c$. The high energy spectrum is dominated by highly excited states of long strings, which become more and more weakly interacting as they approach the boundary. These states are well described by perturbative string theory in AdS_3.

The transition between the black hole and string phases occurs at the point where the radius of curvature, R_{AdS}, is equal to l_s (1.6), in agreement with general expectations from the correspondence principle. At the transition point, the black hole and perturbative string entropies match exactly.

A similar structure is found for the asymptotically linear dilaton case. The high energy behavior of the entropy is in general Hagedorn, but the nature of the high energy states depends on the linear dilaton slope, Q. Below a certain critical value of Q, the generic high energy states correspond to $SL(2,\mathbb{R})/U(1)$ black holes, or their charged cousins. Above this value, the black holes become non-normalizable, and the high energy states correspond to highly excited perturbative strings living in the linear dilaton region. Again, at the transition between the non-perturbative (black hole) and perturbative (string) phases, the entropies match, for both charged and uncharged states.

The fact that the AdS_3 and linear dilaton backgrounds exhibit very similar string/black hole transitions is not accidental. To see that, consider a background of the form $\mathbb{R}^{1,1} \times \mathbb{R}_\phi \times \mathcal{N}$, where $\mathbb{R}_\phi$ is the real line along which the dilaton varies linearly and $\mathcal{N}$ is a compact unitary CFT. String theory in this background has in general a Hagedorn density of states and exhibits the transition described above as a function of Q, the gradient of the dilaton. One can add to the background Q_1 fundamental strings stretched in $\mathbb{R}^{1,1}$, and study the low energy dynamics of the combined system. This theory is related by the AdS/CFT correspondence to string theory in $AdS_3 \times \mathcal{N}$, with string coupling $g_s^2 \sim 1/Q_1$, and curvature (1.6) $k = 2/Q^2$. The string/black hole transition in $AdS_3 \times \mathcal{N}$ occurs at $k = 1$ for all Q_1. Thus, it is natural to expect that the same transition should also occur at $k = 1$ (or $Q = \sqrt{2}$) for $Q_1 = 0$, the original linear dilaton background with no strings attached. We will see that this is indeed the case.

The AdS_3 and linear dilaton theories are also related via matrix theory [9,10]. The low energy theory on the fundamental strings living in the linear dilaton throat can be thought of as the discrete lightcone description of the throat, with the discretization parameter equal to Q_1. This relation implies that the transitions seen in the two cases should occur at the same value of k.

2. AdS_3

In this section we will study string theory on AdS_3 times a compact space. A large class of such backgrounds can be constructed as follows[2]. Consider type II string theory on

$$\mathbb{R}^{1,1} \times \mathbb{R}_\phi \times S^1 \times \mathcal{M} \ . \tag{2.1}$$

Here, $\mathbb{R}_\phi$ is the real line labeled by ϕ, with a dilaton linear in ϕ

$$\Phi = -\frac{Q}{2}\phi \ . \tag{2.2}$$

In particular, the string coupling $g_s = \exp\Phi$ goes to zero as $\phi \to \infty$, and diverges when $\phi \to -\infty$. The worldsheet central charge of ϕ is

$$c_\phi = 1 + 3Q^2 \ . \tag{2.3}$$

$\mathcal{M}$ is a compact CFT with $N = 2$ worldsheet supersymmetry and central charge

$$c_{\mathcal{M}} = 9 - 3Q^2 \ . \tag{2.4}$$

In the class of backgrounds (2.1) one can perform a chiral GSO projection which leads to a spacetime supersymmetric theory, with $N = 2$ supersymmetry in $\mathbb{R}^{1,1}$ [11,6].

One can think of (2.1) as the near-horizon geometry of an $NS5$-brane extended in $\mathbb{R}^{1,1}$ and wrapped around a singular four-cycle in a Calabi-Yau fourfold [12]. An AdS_3 background can be obtained by adding to it Q_1 fundamental strings whose worldvolume lies in $\mathbb{R}^{1,1}$. Taking the near-horizon limit of the strings leads to the geometry

$$AdS_3 \times S^1 \times \mathcal{M} \ . \tag{2.5}$$

[2] We will restrict here to a certain class of supersymmetric backgrounds, but most of what we say can be generalized to any stable linear dilaton or AdS_3 background.

The radius of curvature of AdS_3, R_{AdS}, is related to the level of the $SL(2,\mathbb{R})$ current algebra of AdS_3, k, via the relation (1.6). The worldsheet central charge of the CFT on AdS_3 is given by[3]

$$c_{AdS} = 3 + \frac{6}{k} \ . \tag{2.6}$$

Comparing (2.1) and (2.5) we see that adding the fundamental strings and taking their near-horizon limit leads to the replacement

$$\mathbb{R}^{1,1} \times \mathbb{R}_\phi \to AdS_3 \ . \tag{2.7}$$

The parameters defining the two models are related via

$$Q = \sqrt{\frac{2}{k}} \ . \tag{2.8}$$

The relation between the linear dilaton solution (2.1) and the AdS_3 one (2.5) is a generalization of that between the CHS solution describing $NS5$-branes in flat space [13] and the near-horizon geometry of strings and fivebranes [3]. Just like there, the growth of the string coupling as one moves towards the core of the fivebrane, $\phi \to -\infty$ in (2.1), is compensated by the decrease of the coupling near the fundamental strings. The combination of the two effects leads to a solution with constant string coupling

$$g_s^2 \sim \frac{1}{Q_1} \ . \tag{2.9}$$

Thus, weakly coupled string theory in AdS_3 requires a large number of fundamental strings, $Q_1 \gg 1$.

A familiar example of (2.5) is the system of k $NS5$-branes wrapped around a four-manifold $\mathcal{M}_4$ ($\mathcal{M}_4 = T^4$ or $K3$), and fundamental strings stretched in the uncompactified direction of the fivebranes. In this case one has $\mathcal{M} = \frac{SU(2)_k}{U(1)} \times \mathcal{M}_4$. The GSO projection acts as a $\mathbb{Z}_k$ orbifold on $S^1 \times \frac{SU(2)_k}{U(1)}$, such that the background (2.5) becomes

$$AdS_3 \times S^3 \times \mathcal{M}_4 \ . \tag{2.10}$$

the levels of $SL(2,\mathbb{R})$, (2.6), and $SU(2)$ are both equal to the number of fivebranes, k, which is an integer ≥ 2. Thus, in the class of backgrounds (2.10), the question what happens for $k < 1$ does not arise.

[3] Both here and in (2.3) we omitted the contribution of the worldsheet fermions.

Nevertheless, there are cases where $k < 1$ in (2.5). For example, we can take $\mathcal{M}$ to be the $N = 2$ minimal model with $c_{\mathcal{M}} = 3 - \frac{6}{n}$. This corresponds [12] to a singularity of the form $z_1^n + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0$ in a Calabi-Yau fourfold, or equivalently to a fivebrane wrapped around the four dimensional surface $z_1^n + z_2^2 + z_3^2 = 0$, an A_{n-1} ALE space. Substituting into (2.4), (2.6), (2.8), we find that in this case

$$k = \frac{n}{n+1} < 1 \ . \tag{2.11}$$

Increasing the central charge of $\mathcal{M}$ beyond $c_{\mathcal{M}} = 3$ takes one to $k > 1$ [12].

The spacetime dynamics in an AdS_3 background of the form (2.5) corresponds to an $N = 2$ superconformal field theory with spacetime central charge [14-17,12]

$$c = 6kQ_1 \ . \tag{2.12}$$

If the string theory in AdS_3 is weakly coupled, the spacetime CFT has a large central charge, proportional to $1/g_s^2$ (2.9). In this semiclassical regime, the radius of curvature of AdS_3 in Planck units is large, $R_{AdS} \gg l_p$, (1.2).

The first question we would like to address concerns the normalizability of the ground state of the spacetime CFT. From perturbative studies of string theory in AdS_3 it is known [15] that the spacetime Virasoro algebra takes the form

$$T(x)T(y) \simeq \frac{3kI}{(x-y)^4} + \frac{2T(y)}{(x-y)^2} + \frac{\partial_y T}{x-y} \ , \tag{2.13}$$

where the operator I which determines the spacetime central charge, $c = 6kI$, is given by

$$I = \frac{1}{k^2} \int d^2z J(x;z) \bar{J}(\bar{x};\bar{z}) \Phi_1(x,\bar{x};z,\bar{z}) \ , \tag{2.14}$$

in the notations of [15]. I is an operator which commutes with the spacetime Virasoro generators. In particular, it has spacetime scaling dimension zero. Acting with it on the vacuum of the worldsheet theory creates a state proportional to the $SL(2,\mathbb{C})$ invariant vacuum of the spacetime theory. Applying (2.14) to the in-vacuum in radial quantization, *i.e.* sending $x \to 0$, we find that the spacetime in-vacuum corresponds to the worldsheet state

$$I|0\rangle \sim J_{-1}^+ \bar{J}_{-1}^+ |j = 0; m = \bar{m} = -1\rangle \ . \tag{2.15}$$

For the out-vacuum (obtained by sending $x \to \infty$) one gets a similar expression with all J^3, $\bar{J}^3$ charges reversed.

The state (2.15) is a descendant of a principal discrete series state with $j=0$, $|m|=1$. Such states are normalizable iff the unitarity condition [18,19]

$$-\frac{1}{2}<j<\frac{k-1}{2} \tag{2.16}$$

is satisfied. For $k>1$, $j=0$ is in the range (2.16), while for $k<1$, it is not. This means that the $SL(2,\mathbb{C})$ invariant vacuum of the spacetime CFT is normalizable in the former case, and is not normalizable in the latter. Hence, due to (1.5), for $k>1$ we expect the spacetime theory to have $c_{\rm eff}=c=6kQ_1$, while for $k<1$ we expect it to have $c_{\rm eff}<c$. We will calculate $c_{\rm eff}$ for $k<1$ shortly.

It was pointed out in [20,21] that the Cardy entropy (1.3), with $c_{\rm eff}=c$ given by (1.2), is equal to the Bekenstein-Hawking entropy of large BTZ black holes, with mass and angular momentum (1.4). The above discussion implies that while for $k>1$ the high energy entropy in the spacetime CFT corresponding to string theory in AdS_3 coincides with the Bekenstein-Hawking one, for $k<1$ the two are different. Are we to conclude that for $R_{AdS}<l_s$ the string analysis is in disagreement with the Bekenstein-Hawking prediction?

In fact, it is not. For $k<1$, where $c_{\rm eff}$ and c are different, BTZ black holes are non-normalizable states and do not belong to the Hilbert space of physical states. To see that, recall[4] that a BTZ black hole can be thought of as an orbifold of AdS_3 by the discrete group $\mathbb{Z}$ generated by a hyperbolic element of $SL(2,\mathbb{R})$,

$$g\to h_L g h_R\ , \tag{2.17}$$

with $g\in SL(2,\mathbb{R})$, $h_L=\exp\left[\pi(r_+-r_-)\sigma_3/R_{AdS}\right]$, $h_R=\exp\left[\pi(r_++r_-)\sigma_3/R_{AdS}\right]$. Here $r_\pm$ are the inner and outer horizons; they are related to the mass and angular momentum of the black hole via

$$8l_pM=\frac{r_+^2+r_-^2}{R_{AdS}^2};\quad 8l_pJ=\frac{2r_+r_-}{R_{AdS}}\ . \tag{2.18}$$

Since the orbifold (2.17) acts on AdS_3 as a finite left and right moving scale transformation, the operator I (2.14) is invariant under (2.17) – it is in the untwisted sector of the orbifold. Thus, one can repeat the discussion of the AdS_3 vacuum for this case.

The ground state in a sector with a BTZ black hole is again proportional to $I|0\rangle$, (2.15), where now $|0\rangle$ is the worldsheet vacuum of the orbifold theory (2.17). For $k>1$,

[4] For recent reviews see *e.g.* [22,23].

the ground state $I|0\rangle$ is normalizable, while for $k < 1$ it is not, due to (2.16). Thus, in the latter case the BTZ black hole is not in the spectrum of the theory.

To recapitulate, we find that string theory in AdS_3 has two phases. In one, corresponding to $R_{AdS} > l_s$ ($k > 1$, (1.6)), the $SL(2,\mathbb{C})$ invariant vacuum is normalizable, the Cardy formula (1.3) with $c_{\text{eff}} = c$ is valid and agrees with the Bekenstein-Hawking entropy for large BTZ black holes, which are normalizable as well. In the other, corresponding to $R_{AdS} < l_s$ ($k < 1$), the $SL(2,\mathbb{C})$ invariant vacuum is non-normalizable, $c_{\text{eff}} < c$, but there is no contradiction with the Bekenstein-Hawking analysis, since BTZ black holes are non-normalizable in this regime.

It remains to calculate c_{eff} for $k < 1$, and to identify the states in string theory on AdS_3 that give rise to the Cardy entropy (1.3) in this case. This is the problem we turn to next.

As mentioned in the introduction, one might expect that when the curvature exceeds the string scale and BTZ black holes cease to exist as normalizable states, the high energy spectrum should be dominated by weakly coupled highly excited perturbative string states. To see whether this is indeed the case, we need to recall a few facts about the perturbative string spectrum in AdS_3.

Low lying states in the background (2.5) belong to principal discrete series representations.[5] The worldsheet mass-shell condition for these states relates the quadratic Casimir of $SL(2,\mathbb{R})$ to the excitation level N [14]:

$$-\frac{j(j+1)}{k} + N = \frac{1}{2} \ . \tag{2.19}$$

The scaling dimension of the corresponding Virasoro primary in the spacetime CFT is

$$h = j + 1 \ . \tag{2.20}$$

The unitarity condition (2.16) implies that there is an upper bound on the excitation level N for which (2.19) can be solved. For higher excitation levels and spacetime scaling dimensions, one needs to consider long string states, which wind around the circle near the boundary of AdS_3 [19]. In a sector with given winding w, the worldsheet mass-shell condition becomes

$$-\frac{j(j+1)}{k} - mw - \frac{k}{4}w^2 + N = \frac{1}{2} \ , \tag{2.21}$$

[5] Principal continuous series states correspond to "bad" tachyons, which are absent in the supersymmetric theories discussed here.

and the spacetime scaling dimension is

$$h = |m + \frac{k}{2}w| \ . \tag{2.22}$$

The quantum number j takes the form $j = -\frac{1}{2} + ip$; $p \in \mathbb{R}$ is the momentum of the long string in the radial direction of AdS_3. There are also physical states of the form (2.21) with j real in the range (2.16), which correspond to strings that do not make it all the way to the boundary [19].

As a first step towards examining the dynamics of long strings, consider a single string with winding number $w = 1$. It can be thought of as extended in the $\mathbb{R}^{1,1}$ in (2.1). The remaining, transverse, directions of space, $\mathbb{R}_\phi \times S^1 \times \mathcal{M}$, parametrize the target space of the low energy CFT living on the string. The central charge of that CFT is $c_l = 6k$ [14,15]. The scalar field ϕ parametrizing the location of the string in the radial direction of AdS_3 is described near the boundary by an asymptotically linear dilaton CFT, with $\Phi = -\frac{Q_l}{2}\phi$ as $\phi \to \infty$. The linear dilaton slope Q_l is given by [24]:

$$Q_l = (1-k)\sqrt{\frac{2}{k}} \ . \tag{2.23}$$

Since this result is important for our purposes, we briefly review its derivation.

The fact that the CFT living on a single long string has central charge $c_l = 6k$ places a constraint on Q_l:

$$\left(\frac{3}{2} + 3Q_l^2\right) + \frac{3}{2} + c_{\mathcal{M}} = 6k \ . \tag{2.24}$$

The term in brackets is the contribution of ϕ and its worldsheet superpartner; the other two contributions on the l.h.s. are due to S^1 and $\mathcal{M}$, respectively. Using (2.4), (2.8) we have $c_{\mathcal{M}} = 9 - \frac{6}{k}$; substituting in (2.24) we find

$$Q_l^2 = \frac{2}{k}(k-1)^2 \ . \tag{2.25}$$

In taking the square root of (2.25), there is a sign ambiguity. To see that the correct choice is (2.23), one can, for example, compare the dimensions of chiral operators computed in [12] to those computed in the CFT on the long string, $\mathbb{R}_\phi \times S^1 \times \mathcal{M}$. This is discussed in appendix A.

The most important features of (2.23) are its overall sign, and the fact that this sign changes at $k = 1$. For $k > 1$, the case discussed in [24], the sign is such that the string coupling grows when one approaches the boundary of AdS_3, $\phi \to \infty$. This means that highly excited perturbative strings in AdS_3 interact strongly and are unstable near the boundary (like widely separated quarks and gluons in QCD).

In particular, consider a state with winding number w consisting of w highly excited long strings near the boundary. One expects large interactions among the strings to lead to the formation of a bound state, whose properties can be very different from those of free strings. A natural candidate for such a state is a large BTZ black hole, with the same energy and angular momentum as the original configuration of strings. In this regime ($k > 1$), the perturbative description in terms of highly excited strings satisfying (2.21) is not useful. A better description of the highly excited state is as a BTZ black hole, which is a non-perturbative state from the point of view of string theory in AdS_3.

For $k < 1$, the situation is reversed. The strongly coupled bound state, the BTZ black hole, is no longer normalizable. Correspondingly, the interactions among highly excited perturbative strings are now *small* near the boundary, since the gradient of the dilaton on the long strings (2.23) has the opposite sign. Therefore, the generic high energy state with winding number w consists of w weakly interacting long strings. The higher the energy, the smaller the interactions among the strings. Note that it is crucial for the consistency of the picture that the linear dilaton on long strings (2.23) change sign precisely at the same value of k at which the BTZ black hole ceases to be normalizable. The fact that this is indeed the case supports the overall picture.

Since for $k < 1$ the high energy states consist of weakly interacting strings, it should be possible to compute the high energy entropy, and in particular $c_{\rm eff}$ (1.3), perturbatively. A quick way to get the answer is to note that if on a single long string we have the CFT $\mathbb{R}_\phi \times S^1 \times \mathcal{M}$, then on w weakly interacting strings we expect to find the symmetric product CFT

$$\left(\mathbb{R}_\phi \times S^1 \times \mathcal{M}\right)^w / S_w \ . \tag{2.26}$$

The fact that the interactions among the strings go to zero as one approaches the boundary of AdS_3 should translate in the spacetime CFT to the statement that (2.26) describes accurately the high energy (large $L_0, \bar{L}_0$) behavior of the theory of w strings. It may be modified significantly at low energies.

To calculate the high energy density of states of (2.26) one simply adds the effective central charges of the component CFT's in (2.26):

$$c_{\rm eff} = w\left(\frac{3}{2} + \frac{3}{2} + c_{\mathcal{M}}\right) = 6w(2 - \frac{1}{k}) \ . \tag{2.27}$$

The winding number w is bounded from above by Q_1, the total number of strings used to make the background (2.5). Thus, the high energy density of states is

$$c_{\text{eff}} = 6Q_1(2 - \frac{1}{k}) \ . \tag{2.28}$$

This should be compared with the central charge of the spacetime CFT, $c = 6Q_1 k$ (2.12). Note that one always has $c_{\text{eff}} \leq c$, with $c_{\text{eff}} = c$ iff $k = 1$. This is required by unitarity and modular invariance of the spacetime CFT (see the discussion after (1.5)).

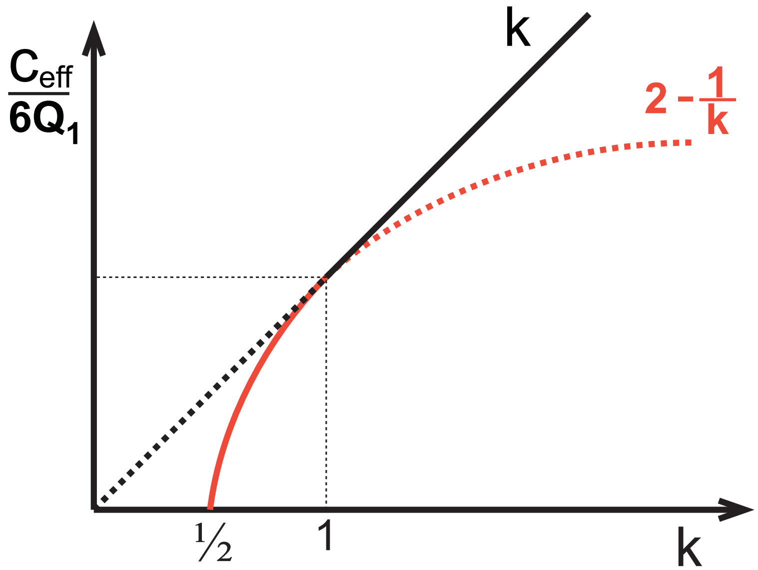

Figure 1: The behavior of c and c_{eff} as a function of k. The solid lines represent the dominant contributions to the high energy entropy, which are due to BTZ black holes for $k > 1$ (black) and fundamental strings for $k < 1$ (red). The dashed lines represent states that do not exist in the two phases: black holes for $k < 1$ (black) and weakly coupled highly excited strings for $k > 1$ (red).

The behavior of the central charge c and c_{eff} as a function of k is plotted in figure 1. The dashed red line for $k > 1$ represents the contribution of highly excited perturbative string states in AdS_3. As explained earlier, these states are in fact not there due to the large interactions near the boundary, and the spectrum in this regime is instead dominated by large BTZ black holes, whose contribution is represented by the solid black line in the figure. For $k < 1$ the dashed black line corresponds to the contribution of large BTZ black holes. These are non-normalizable in this regime; hence, their contribution is absent. The spectrum is dominated by perturbative strings (solid red line). Note that in the class of constructions (2.5), one is restricted to $k \geq \frac{2}{3}$, since otherwise $\mathcal{M}$ has a negative central charge and the theory is non-unitary. In fact, it is easy to see that in string theory in AdS_3 one can never reach the point $k = \frac{1}{2}$, at which c_{eff} vanishes.

Another way to arrive at (2.27) is the following. We saw before that in the sector with winding w, the worldsheet mass-shell condition takes the form (2.21). The spacetime scaling dimension (2.22) can thus be written as[6]

$$h = \frac{kw}{4} + \frac{1}{w}\left[-\frac{j(j+1)}{k} + N - \frac{1}{2}\right] . \tag{2.29}$$

The bulk of the high energy density of states comes from the entropy associated with the excitation level, N. To estimate it, we can assume that the term proportional to N dominates the r.h.s. of (2.29), so that

$$h \simeq \frac{N}{w} . \tag{2.30}$$

The density of states grows with N in a way familiar from perturbative string theory [5]. The effective worldsheet central charge of (2.5) is $c_{\text{eff}}^{(\text{ws})} = 6 + c_{\mathcal{M}} - 3$; the 6 comes from $AdS_3 \times S^1$, while the -3 accounts for the two towers of bosonic and fermionic oscillators removed by gauge invariance. Substituting (2.4) we find that $c_{\text{eff}}^{(\text{ws})} = 6(2 - \frac{1}{k})$. Using the fact that the spacetime scaling dimension h is related to N via (2.30), we see that the perturbative string entropy, $2\pi\sqrt{c_{\text{eff}}^{(\text{ws})} N/6}$, gives rise to a spacetime entropy (1.3) with $c_{\text{eff}} = w c_{\text{eff}}^{(\text{ws})}$, in agreement with (2.27).

The spectrum of long strings gives the same effective central charge as (2.26), but does the spacetime CFT approach the symmetric product as $L_0, \bar{L}_0 \to \infty$? We will leave a detailed analysis of this question to future work.[7] Here, we note that the dimension formula (2.29) can be written in the following suggestive way. Consider a physical state in the sector with winding number one, which satisfies (2.29) with $w = 1$. Denote its spacetime scaling dimension by h_1. According to (2.29), for every such state there is a state in the sector with winding w, related to it by spectral flow, whose dimension h_w is given by:

$$h_w = \frac{h_1}{w} + \frac{k}{4}\left(w - \frac{1}{w}\right) . \tag{2.31}$$

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[6] We take both m and w to be positive.

[7] For a discussion of the spectrum in the $w \neq 0$ sectors and the relation to symmetric orbifolds, see [25].

This is precisely the type of expression one finds in the w-twisted sector of a symmetric orbifold, whose building block has $c = 6k$ [26,27,25].

We see that a number of seemingly independent things happen at $k = 1$:

(1) The BTZ black holes, which are normalizable for $k > 1$, cease to be so for $k < 1$. The same is true for the $SL(2,\mathbb{C})$ invariant vacuum of the spacetime theory.

(2) The gradient of the linear dilaton on a long string living near the boundary of AdS_3 (2.23) changes sign at $k = 1$. For $k > 1$, long strings are strongly coupled near the boundary, while for $k < 1$ they are weakly coupled there.

(3) The black hole entropy, which is given by the Cardy formula (1.3) with $c_{\rm eff} = c = 6Q_1 k$ (2.12), coincides (see figure 1) with the perturbative string entropy, which has $c_{\rm eff} = 6Q_1(2 - \frac{1}{k})$ (2.28).

We argued above that these phenomena are in fact related. They are all manifestations of a sharp transition, which occurs at $k = 1$, between a strongly coupled phase where the high energy states are large black holes, and a weakly coupled phase in which they are weakly interacting fundamental strings. The matching of the entropies suggests that the dynamics of large BTZ black holes changes smoothly into that of highly excited fundamental strings as we vary k.

We finish this section with a few comments. So far, we discussed type II string theory in AdS_3. One could ask whether the matching that we found of black hole and string entropies extends to other cases, such as the heterotic string. Heterotic strings in AdS_3 were studied in [28], where it was shown that in this case the spacetime CFT inherits the chiral nature of the worldsheet one. While the left moving[8] central charge is still given by (2.12), $c = 6Q_1 k$, the right moving one is equal to $\bar{c} = 6Q_1(k+2)$. In the black hole phase this means that the Cardy formula is (1.3), with $c_{\rm eff} = c$ and $\bar{c}_{\rm eff} = \bar{c}$.

In the perturbative string phase, the left-movers still satisfy (2.21), (2.22), (2.29), while for the right-movers one has the mass-shell condition

$$-\frac{j(j+1)}{k} - \bar{m}w - \frac{k+2}{4}w^2 + \bar{N} = 1 \ . \tag{2.32}$$

For large excitation level $\bar{N}$ one has, as in (2.30), $\bar{h} \simeq \bar{N}/w$. To calculate the effective worldsheet central charge of the right-movers recall that they are described by the worldsheet CFT

$$AdS_3 \times S^1 \times \mathcal{M} \times \Gamma \ , \tag{2.33}$$

[8] We take the left-movers to be supersymmetric and the right-movers bosonic.

where Γ is a CFT with central charge 13, which completes the total central charge of (2.33) to 26:

$$3 + \frac{6}{k} + 1 + c_{\mathcal{M}} + 13 = 26 \ . \tag{2.34}$$

The corresponding effective worldsheet central charge is $\bar{c}_{\rm eff}^{\rm (ws)} = 3 + 1 + c_{\mathcal{M}} + 13 - 2 = 24 - \frac{6}{k}$. Multiplying by Q_1 we conclude that the analog of (2.28) for the right-movers in the heterotic string is

$$\bar{c}_{\rm eff} = 6Q_1(4 - \frac{1}{k}) \ . \tag{2.35}$$

We see that, like in the type II case, $\bar{c}_{\rm eff}$ is always smaller than or equal to $\bar{c} = 6Q_1(k+2)$; the two coincide at the string/black hole transition point $k = 1$, in agreement with the general picture presented above.

Another natural question concerns the relation (1.5) for $k < 1$. In that case, we found $c = 6Q_1 k$ (2.12), and $c_{\rm eff} = 6Q_1(2 - \frac{1}{k})$ (2.28). According to (1.5), this means that

$$\Delta_{\rm min} = \frac{Q_1}{4}\frac{(k-1)^2}{k} = Q_1 \frac{Q_l^2}{8} \ . \tag{2.36}$$

This is puzzling since, especially for large Q_1, there are many normalizable states of the form (2.21), (2.22), whose dimensions are smaller than (2.36).

One possible resolution of this conundrum is the following. It was pointed out in [29] that in perturbative string theory in AdS_3, the spacetime central charge is in fact not proportional to the identity operator, but takes different values in different sectors of the theory. In other words, the spacetime CFT appears to have a non-trivial vacuum structure.

The situation is analogous to studying the extreme IR limit of an $N = 2$ Landau-Ginzburg model of a chiral superfield Φ with superpotential

$$W = \sum_{i=1}^{n} a_i \Phi^i \ . \tag{2.37}$$

For generic values of the $\{a_i\}$, the IR limit is a direct sum of massive theories. By fine tuning the coefficients one can reach multicritical points for which

$$W = a_n \prod_j (\Phi - \Phi_j)^{n_j} \ , \qquad n_j \geq 2 \ , \qquad \sum_j n_j = n \ , \tag{2.38}$$

and the theory is a direct sum of minimal models

$$\oplus_j \mathcal{M}_{n_j} \ . \tag{2.39}$$

Thus, it exhibits non-trivial vacuum structure. In particular, the torus partition sum is given by a sum of the partition sums of the individual $N = 2$ minimal models, $\mathcal{Z} = \sum_j \mathcal{Z}_{n_j}$.

In our case, it is possible that the spacetime CFT for $k < 1$ splits into sectors labeled by winding numbers $(w_1, w_2, \cdots)$, with $\sum_j w_j = Q_1$. In a given sector, the spacetime CFT is a direct sum of the individual theories corresponding to the different w_j. Each of those has central charge $c^{(w_j)}(= 6w_j k)$, and $c_{\text{eff}}^{(w_j)}(= 6w_j(2 - \frac{1}{k}))$. The torus partition sum is given by a sum over different components,

$$\mathcal{Z}(\tau, \bar{\tau}) = \sum_j \mathcal{Z}_{w_j}(\tau, \bar{\tau}) \ . \tag{2.40}$$

The partition sums $\mathcal{Z}_{w_j}$ are separately modular invariant. Thus, instead of a single minimal dimension operator (2.36), we have a separate minimal dimension operator for each w, with

$$\Delta_{\text{min}}^{(w)} = \frac{1}{24}\left(c^{(w)} - c_{\text{eff}}^{(w)}\right) = \frac{w}{4}\frac{(k-1)^2}{k} = w\frac{Q_l^2}{8} \ . \tag{2.41}$$

Clearly, more work is required to see whether this is the right picture. As a first step, in appendix B we show that perturbative string states in the sector with winding w indeed have spacetime scaling dimensions which are bigger than $\Delta_{\text{min}}^{(w)}$ (2.41).

In this section we studied string theory in AdS_3 with only the $NS\ B_{\mu\nu}$ field turned on. This is usually viewed as a special locus in the moduli space of string theory in AdS_3, along which the spacetime CFT is singular [24]. The singular nature of the spacetime CFT is reflected in the appearance of a continuum of scaling dimensions above a gap, associated with long strings. To eliminate the continuum, one can often turn on a modulus that creates a potential for long strings, preventing them from exploring the region near the boundary, where they are strongly coupled.

For example, for the background $AdS_3 \times S^3 \times \mathcal{M}_4$ (2.10), the modulus in question is the $N = 4$ Liouville superpotential on the long strings, which grows as one approaches the boundary of AdS_3, and thus repels the strings from this region [24]. This modulus is described by a RR vertex operator in perturbative string theory in AdS_3 [14].

It is important to emphasize that the above picture is only valid for $k > 1$. In the weakly coupled phase $k < 1$, similar RR perturbations grow as one moves *away* from the boundary of AdS_3, and thus do not change the singular nature of the spacetime CFT. In essence, for $k < 1$ the spacetime CFT is *always* singular, *i.e.* it always has a continuum above a gap in its spectrum of scaling dimensions.

Note that this behavior is consistent with the general picture described above: for $k > 1$, perturbative strings in AdS_3 are strongly coupled at high energies, and the strength of their interactions can be modified by changing the moduli of the spacetime CFT. For $k < 1$, perturbative strings are weakly coupled at high energies and thus no marginal or relevant perturbations can make their interactions strong there.

3. Linear dilaton

In the previous section we have seen that the low energy physics in linear dilaton backgrounds such as (2.1), in the presence of Q_1 fundamental strings in the throat, is qualitatively different for $k < 1$ and $k > 1$. This was demonstrated for $Q_1 \gg 1$, but we expect it to be true for all Q_1. It is natural to ask whether something similar happens for $Q_1 = 0$, *i.e.* in the absence of strings in the throat. In this case, the theory under consideration is the Little String Theory (LST) of the throat [30,12], and one would like to know whether the nature of the states which dominate its high energy behavior changes as k passes through the value $k = 1$, or as the gradient of the dilaton Q passes through $Q = \sqrt{2}$. We will next show that this is indeed the case.

The high energy density of states in LST exhibits Hagedorn growth (see *e.g.* [31,32] for recent discussions). For $k > 1$, this comes from the Bekenstein-Hawking density of states of $SL(2,\mathbb{R})_k/U(1)$ black holes. These are obtained by considering the string frame dilaton gravity action

$$\mathcal{S} = \int d^2x\sqrt{-g}e^{-2\Phi}\left(\mathcal{R} + 4g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 4\Lambda\right) , \tag{3.1}$$

where, as before, Λ is related to the level of $SL(2,\mathbb{R})$, k, by (1.1), (1.6). The equations of motion of (3.1) have the solution[9] [33,34]

$$\begin{aligned} &ds^2 = f^{-1}d\phi^2 - f dt^2 , \qquad f = 1 - \frac{2M}{r} , \\ &Qe^{-2\Phi} = Qe^{Q\phi} = r . \end{aligned} \tag{3.2}$$

Here (t,ϕ) provide a parametrization of the two dimensional space $\mathbb{R}_t \times \mathbb{R}_\phi$ in (2.1); (t,r) are analogs of Schwarzschild coordinates. The value of the dilaton at the horizon of the black hole, $\Phi(r = 2M) = \Phi_0$, is determined by its mass M: $\exp(-2\Phi_0) = 2M/Q$. The

[9] Here and below we often set $\alpha' = 2$.

gradient of the dilaton Q is related to the level k via (2.8). Note that, as mentioned in the introduction, the mass determines the strength of quantum corrections near the horizon, but the α' effects are insensitive to it.

The Bekenstein-Hawking entropy of these black holes is obtained in the usual way by Wick rotating (3.2) to Euclidean space and calculating the circumference of the Euclidean time direction near the boundary at $\phi = \infty$. After Wick rotation, the metric (3.2) can be written as [35,33]

$$\begin{aligned} ds^2 =& k l_s^2 \left(d\rho^2 + \tanh^2 \rho d\tau^2 \right) \\ \Phi =& \Phi_0 - \log \cosh \rho \ , \end{aligned} \tag{3.3}$$

where τ is periodic, with period 2π. The corresponding entropy is

$$S_{\rm bh} = 2\pi l_s M \sqrt{k} \ . \tag{3.4}$$

From the string/black hole correspondence principle and the discussion of the previous section one would expect that when α' corrections to the geometry (3.2), (3.3) become of order one, the black hole should become non-normalizable and the system should make a transition to a string phase.

This leads to the question what is a useful measure of the α' corrections. The simplest guess is the curvature at the horizon [1,2]. In two dimensions, every spacetime is maximally symmetric. Thus, the Riemann tensor is completely determined by the scalar curvature, and one may try to check when the latter becomes of order the string scale. For (3.2), the scalar curvature is

$$\mathcal{R}(r) = \frac{2}{k} \frac{2M}{r} \ . \tag{3.5}$$

At the horizon, $\mathcal{R}(r = 2M) = \frac{2}{k}$, and one may expect a transition to a string phase at $k \sim 1$. In fact, it turns out that the curvature is not the most useful guide for estimating the size of string corrections. One way to see that is to consider the generalization of (3.5) to charged black holes. We will see below that in that case, as one varies the mass to charge ratio, the curvature at the outer horizon changes continuously and even vanishes for a particular ratio. This does not necessarily mean that string corrections are small there!

Indeed, the metric is not the only field that is excited in the black hole geometry. In the uncharged case (3.2) we also have a non-trivial dilaton, and for charged black holes a gauge field as well. We are looking for a measure of the size of the α' corrections which takes all these fields into account.

It is natural to propose that the relevant quantity is the cosmological constant Λ in (3.1). This parameter determines both the gradient of the dilaton, and the curvature of the metric, and it plays the same role here as in the AdS_3 discussion of section 2. Furthermore, while the curvature (3.5) depends on position, the cosmological constant does not, so we do not have to face the issue where in the black hole spacetime to evaluate it.

A nice feature of the above proposal is that the black hole (3.2) is a coset of $SL(2,\mathbb{R})$, while the BTZ black hole discussed in section 2 is an orbifold of the same space. Therefore, it is natural to expect that the α' corrections should be determined by the underlying $SL(2,\mathbb{R})$ group manifold, and should therefore only depend on k, (1.1), (1.6), (2.8). It also makes it easy to generalize to other examples related to $SL(2,\mathbb{R})$, such as the charged black holes that will be discussed below.

Another question that one might ask at this point is the following. While α' corrections are expected to grow when Q increases (or k decreases), it is known [36] that the metric and dilaton (3.2), (3.3), which are obtained by studying the lowest order in α' action (3.1), give rise to a solution of the classical string equations of motion[10], to all orders in Q. In what sense are α' corrections growing[11] as one increases Q?

The answer to this question is that for large Q, or small k, the sigma model description breaks down due to non-perturbative effects in Q or $1/k$. For example, to describe the Euclidean black hole (3.3) in the bosonic string, one needs to turn on, in addition to the metric and dilaton, certain winding modes of the tachyon. This was first shown in unpublished work by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov, and is reviewed in [38]; see also [39] for a recent discussion. In the fermionic string, which is the case relevant here, one has to turn on winding modes of the fermionic string tachyon [18]. The leading one is the worldsheet superpotential

$$\int d^2\vartheta e^{-\frac{1}{Q}\Psi} , \tag{3.6}$$

where Ψ is a chiral superfield whose bottom component is

$$\Psi = \phi + i\sqrt{k}(\tau_L - \tau_R) + \cdots . \tag{3.7}$$

[10] This is the case for fermionic strings; in the bosonic case, there are corrections, which were discussed in [37].

[11] This question could have been asked in AdS_3 as well, but we chose to discuss it here, since it is more familiar in this case.

The worldsheet superpotential (3.6) is non-perturbative in Q and becomes more and more important as Q grows. Its effects are visible in the structure of correlation functions in the Euclidean black hole geometry, as discussed in [29].

An interesting question is what is the analog of the superpotential (3.6) for the Minkowski $\frac{SL(2,\mathbb{R})}{U(1)}$ black hole (3.2). This question is rather confusing, since the mode (3.6) has winding number one around the Euclidean time direction, and is difficult to continue to Minkowski spacetime.

One might be able to make sense of this continuation by using the observation that T-duality exchanges the region outside the horizon of the Minkowski $\frac{SL(2,\mathbb{R})}{U(1)}$ black hole with the region behind the singularity [40,37]. One can try to think of the analytic continuation of the winding mode (3.6) as a finite energy excitation living "behind the singularity at $r=0$" in the extended black hole spacetime. An observer living far outside the horizon of such a black hole would not see this field turned on directly, but would presumably feel its effects on the physics.

Now that we have established that as k decreases, α' effects grow, and pointed out the close analogy to the $SL(2,\mathbb{R})$ case of section 2, we can turn to a discussion of the string/black hole transition. The most important effect of the superpotential (3.6) is that it implies that for $k<1$, the Euclidean $\frac{SL(2,\mathbb{R})}{U(1)}$ black hole is non-normalizable. Indeed, the perturbation (3.6) is normalizable for $\frac{1}{Q}>\frac{Q}{2}$, and non-normalizable otherwise. The transition point occurs at $Q^2=2$, or $k=1$.

Thus, for $k<1$ the Euclidean black hole (3.3) does not contribute to the canonical partition sum. The behavior (3.4), which is due to its contribution, is only valid for $k>1$.

The discussion of the previous section suggests that for $k<1$ the high energy spectrum should be dominated by highly excited perturbative strings living in the linear dilaton throat (2.1). The same calculation as that done there gives the entropy of these strings,

$$S_{\rm pert}=2\pi l_s M\sqrt{2-\frac{1}{k}}\ . \tag{3.8}$$

Equations (3.4) and (3.8) are direct analogs of the discussion of section 2. The former is the analog of the BTZ black hole entropy (1.3) with $c_{\rm eff}=6kQ_1$; the latter, of the perturbative string entropy with (2.28) $c_{\rm eff}=6(2-\frac{1}{k})Q_1$. The same comments as there apply here. In particular, $S_{\rm bh}\geq S_{\rm pert}$; the two are equal at the string/black hole transition point $k=1$. As in AdS_3, the transition from black holes to strings at $k=1$ occurs not because the string entropy becomes larger than the black hole one, but rather because the black holes cease to be normalizable.

A natural question is why the highly excited fundamental strings change their behavior at $k = 1$. In analogy to the AdS_3 discussion, we expect that for $k > 1$, as the energy of perturbative strings grows, they should become more strongly coupled and form bound states, the $\frac{SL(2,\mathbb{R})}{U(1)}$ black holes. For $k < 1$ their coupling should go to zero as their energy goes to infinity. In AdS_3 this was a consequence of the dynamics on long strings, and in particular of the behavior of the linear dilaton slope (2.23).

We will not analyze the analogous question in our case, except to say that the behavior should be the same here. Note that the central charge of the CFT on a long string in AdS_3, $c = 6k$, and the value of Q_l (2.23), are independent of the number of strings that make the AdS_3 background, Q_1. As Q_1 decreases, the string coupling in AdS_3 grows, but the fact that, for $k < 1$, long strings become free in the UV (*i.e.* near the boundary) does not change, since it is driven by the sign of (2.23). Hence, one expects that these results can be extrapolated to small Q_1, and in particular to $Q_1 = 0$. It would be nice to show this directly.

To recapitulate, we see that like in AdS_3, string theory in linear dilaton backgrounds, such as (2.1), has two distinct phases. For $k > 1$, the high energy entropy is given by (3.4) and is due to $\frac{SL(2,\mathbb{R})}{U(1)}$ black holes. For $k < 1$ these black holes become non-normalizable, and the entropy, while still Hagedorn (for $k > \frac{1}{2}$), is now given by (3.8) and is due to highly excited weakly coupled fundamental string states living in the linear dilaton throat.

The detailed formulae for the entropy (3.4), (3.8) bear a striking resemblance to those found in the AdS_3 case. From the worldsheet point of view, this is due to the fact that in both cases there is an underlying $SL(2,\mathbb{R})$ structure. From the spacetime point of view one can understand the agreement in the following way. One of the ways to study LST's corresponding to linear dilaton throats is using Matrix theory [9,10]. This is obtained by studying the low energy dynamics of Q_1 fundamental strings in the linear dilaton throat, which is exactly the theory discussed in section 2. Q_1 is interpreted as the discretized light-like momentum, P^+, while P^- corresponds to the energy in the CFT describing the strings in the throat. For uncharged states it is given by $L_0 - \frac{c}{24} = \bar{L}_0 - \frac{c}{24}$. Thus, the energy in LST is given by

$$\frac{\alpha'}{4} M^2 = Q_1 \left(L_0 - \frac{c}{24} \right) \ . \tag{3.9}$$

This provides a direct map between the CFT entropy (1.3) and the LST entropy (3.4), (3.8), using the formulae for c_{eff}, (2.12), (2.28). In particular, when the CFT of the strings in the throat makes the transition from the BTZ black hole phase to the perturbative string phase, the LST makes a similar transition from the $SL(2,\mathbb{R})/U(1)$ black hole phase to the perturbative string one.

In the perturbative string phase, the DLCQ prescription (3.9) with discretization parameter w, and the AdS_3 result (2.29), give rise to the spectrum

$$\frac{\alpha'}{4}M^2 = w\left[L_0^{(w)} - \frac{c^{(w)}}{24}\right] = -\frac{j(j+1)}{k} + N - \frac{1}{2} \ . \tag{3.10}$$

It is interesting that (3.10) is the same as the spectrum of perturbative strings living in the linear dilaton throat, with the usual relation between the momentum in the ϕ direction and j, $\beta = Qj$ (see appendix A).

Thus, we see that for $k < 1$, DLCQ maps weakly coupled highly excited fundamental strings in AdS_3 to weakly coupled strings in the linear dilaton throat. Similarly, for $k > 1$, when the strings are strongly coupled, DLCQ maps BTZ black holes in AdS_3 to $\frac{SL(2,\mathbb{R})}{U(1)}$ black holes in LST.

So far we discussed black holes that do not carry any charges. As a further check on the general picture, we will next show that the string/black hole transition at $k = 1$ works correctly also for a certain class of black holes and strings that are charged under one or two gauge fields.

3.1. One charge black holes

In this subsection we will test the string/black hole correspondence for two dimensional black holes that carry charge q under an Abelian gauge field A_μ. The black holes in question are solutions of two dimensional dilaton gravity (3.1) coupled to the gauge field,

$$\mathcal{S} = \int d^2x\sqrt{-g}e^{-2\Phi}\left(\mathcal{R} + 4g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{4}F^2 - 4\Lambda\right) \ , \tag{3.11}$$

where $F_{\mu\nu}$ is the field strength of A_μ. The charged black hole solution that we will consider is [41]

$$ds^2 = f^{-1}d\phi^2 - f dt^2 \ , \qquad Qe^{-2\Phi} = Qe^{Q\phi} = r \ , \qquad A = \frac{q}{r}\sqrt{2}dt \ , \tag{3.12}$$

where

$$f = 1 - \frac{2M}{r} + \frac{q^2}{r^2} \ . \tag{3.13}$$

The values of the dilaton at the inner and outer horizons of the black hole are related to the mass M and charge q of the black hole as follows:

$$r_\pm \equiv Qe^{-2\Phi_\pm} = M \pm \sqrt{M^2 - q^2} \ . \tag{3.14}$$

For $q = 0$, the solution (3.12) – (3.14) reduces to the uncharged one (3.2).

The curvature of the metric (3.12) is

$$\mathcal{R}(r) = \frac{2}{k}\left(\frac{2M}{r} - \frac{4q^2}{r^2}\right) . \tag{3.15}$$

In particular, its value at the event horizon,

$$\mathcal{R}(r_+) = \frac{2}{k}\frac{2Mr_+ - 4q^2}{2Mr_+ - q^2} , \tag{3.16}$$

changes continuously from $2/k$, for $q = 0$, to $-4/k$ in the extremal case $M = q$. As mentioned above, this is an example where the curvature *at* the horizon is not a particularly good guide to the size of α' corrections. One can still use the curvature for that purpose, by studying the geometry in the vicinity of $r = r_+$, but as we argued before, a simpler indicator of the size of string corrections is the cosmological constant Λ, which is sensitive to all the fields via the dilaton equation of motion,

$$\mathcal{R} - 4(\nabla_\mu\Phi\nabla^\mu\Phi - \nabla_\mu\nabla^\mu\Phi) - \frac{1}{4}F^2 = 4\Lambda = -\frac{2}{k} . \tag{3.17}$$

The string/black hole transition is expected to occur at $k = 1$, when the cosmological constant is equal to the string scale.

To realize the black hole solution (3.12) in string theory, consider a background of the form (2.1), and compactify the spatial direction of $\mathbb{R}^{1,1}$, replacing it by $\mathbb{R}_t \times S^1$. The two dimensional spacetime of (3.11) is $\mathbb{R}_t \times \mathbb{R}_\phi$, and the S^1 gives rise to two gauge fields, whose charges are the left and right-moving momenta (or momentum and winding) on the circle. Here, we are interested in black holes carrying only one charge, which we will choose to be the right-moving momentum, p_R. The resulting black hole background (3.12) can be described by a coset CFT, like the uncharged black hole (3.2). The coset in question is [42,43] $\frac{SL(2,\mathbb{R})_k \times U(1)_R}{U(1)}$, where $U(1)_R$ is the right-moving part of the S^1 mentioned above. The $U(1)$ symmetry that is being gauged acts on $SL(2,\mathbb{R}) \times U(1)_R$ as

$$(g, x_R) \simeq (h_L g h_R, x_R + \beta) , \tag{3.18}$$

where $(g, x_R) \in SL(2,\mathbb{R}) \times U(1)_R$ and

$$h_L = e^{\alpha\sigma_3} \ , \ \ h_R = e^{\alpha\cos(\psi)\sigma_3} \ , \ \ \beta = \alpha\sin(\psi) , \tag{3.19}$$

with $\tan^2(\psi/2) = r_-/r_+$.

The entropy of the black hole is given in terms of its mass and charge by [44]

$$S_{\rm bh} = \pi\sqrt{2k}\left(M + \sqrt{M^2 - q^2}\right) \ . \tag{3.20}$$

The description in terms of a coset of $SL(2,\mathbb{R})$ suggests that the black hole exists as a normalizable state only for $k > 1$, while for $k < 1$ the generic high energy states with the same charge correspond to highly excited perturbative strings with $(p_L, p_R) = (0, q)$. The entropy of such strings is

$$S_{\rm pert} = 2\pi\sqrt{2 - \frac{1}{k}}\left(\sqrt{N} + \sqrt{\bar{N}}\right) \ ; \tag{3.21}$$

the left and right moving excitation numbers $(N, \bar{N})$ are given in terms of M and q by the on-shell condition[12]

$$M^2 = 2N = 2\bar{N} + q^2 \ . \tag{3.22}$$

Substituting (3.22) in (3.21) gives

$$S_{\rm pert} = \pi\sqrt{4 - \frac{2}{k}}\left(M + \sqrt{M^2 - q^2}\right) \ . \tag{3.23}$$

Comparing (3.23) to (3.20) we see that the situation is very similar to the uncharged case. The black hole entropy, $S_{\rm bh}$, is always larger or equal than the perturbative string one, $S_{\rm pert}$. The two coincide at the transition point $k = 1$, below which the black hole does not exist and perturative strings dominate. The uncharged case is reproduced for the special case $q = 0$.

3.2. Two charge black holes

In the previous subsection we studied states with only right-moving charge p_R, but no left-moving one. It is natural to generalize the discussion to states carrying both left and right-moving charges. On the perturbative string side, these are states with non-zero left and right-moving momenta (q_L, q_R). Their high energy entropy is given by

$$S_{\rm pert} = \pi\sqrt{4 - \frac{2}{k}}\left(\sqrt{M^2 - q_L^2} + \sqrt{M^2 - q_R^2}\right) \ . \tag{3.24}$$

[12] Up to additive corrections of order one.

The corresponding black holes can be described as follows. Start with three dimensional dilaton gravity coupled to the Neveu-Schwarz B-field,

$$\mathcal{S} = \int d^3X\sqrt{-G}e^{-2\Phi}\left(\mathcal{R} + 4G^{MN}\partial_M\Phi\partial_N\Phi - \frac{1}{12}H^2 - 4\Lambda\right) \ , \tag{3.25}$$

where the three form $H = dB$ is the field strength of B. This action has a black string solution,

$$\begin{aligned}
G_{xx} =& 1 + \frac{2q_Rq_L}{Mr} \ , \qquad Qe^{-2\Phi} = r \ , \\
G_{tt} =& -1 + \frac{M^2 + \sqrt{(M^2-q_R^2)(M^2-q_L^2)} - q_Rq_L}{Mr} \ , \\
G_{rr} =& \left[\left(Mr - M^2 + q_Rq_L\right)^2 - (M^2-q_R^2)(M^2-q_L^2)\right]^{-1}\frac{kM^2}{2} \ , \\
G_{tx} =& -\frac{q_L\sqrt{M^2-q_R^2} + q_R\sqrt{M^2-q_L^2}}{\left[2\sqrt{(M^2-q_R^2)(M^2-q_L^2)}\left(M^2 + \sqrt{(M^2-q_R^2)(M^2-q_L^2)} - q_Rq_L\right)\right]^{\frac{1}{2}}}G_{tt} \ , \\
B_{tx} =& \frac{q_L\sqrt{M^2-q_R^2} - q_R\sqrt{M^2-q_L^2}}{\left[2\sqrt{(M^2-q_R^2)(M^2-q_L^2)}\left(M^2 + \sqrt{(M^2-q_R^2)(M^2-q_L^2)} - q_Rq_L\right)\right]^{\frac{1}{2}}}G_{tt} \ .
\end{aligned} \tag{3.26}$$

Here, x is taken to be compact, M is the ADM mass, and (q_L, q_R) are related to the angular momentum, q_G, and the axion charge per unit length, q_B, by

$$q_G = q_L + q_R \ , \qquad q_B = q_L - q_R \ . \tag{3.27}$$

The black string has a singularity hidden behind inner and outer horizons, all of which are extended in x. The singularity is located at $r = 0$ and the horizons are at

$$r_\pm = M \pm \frac{\sqrt{M^2-q_L^2}\sqrt{M^2-q_R^2} - q_Lq_R}{M} \ . \tag{3.28}$$

Performing a Kaluza-Klein reduction down to two dimensions (t, r) gives a two dimensional black hole with two charges (3.27). The one charge solution of the previous subsection corresponds to $q_R = q$, $q_L = 0$. The static black string solution, (3.26) with $q_L + q_R = 0$, was studied in [45].

The black string background (3.26) can be described as the quotient CFT $\frac{SL(2,\mathbb{R})\times U(1)}{U(1)}$. The $U(1)$ symmetry that is being gauged acts on $SL(2,\mathbb{R}) \times U(1)_L \times U(1)_R$ as follows

$$(g, x_L, x_R) \simeq (h_Lgh_R, x_L + \beta_L, x_R + \beta_R) \ , \tag{3.29}$$

where $(g, x_L, x_R) \in SL(2, \mathbb{R}) \times U(1)_L \times U(1)_R$ and

$$h_L = e^{\alpha\sqrt{M^2 - q_L^2}\sigma_3} \ , \qquad h_R = e^{\alpha\sqrt{M^2 - q_R^2}\sigma_3} \ , \qquad (\beta_L, \beta_R) = \alpha(q_L, q_R) \ . \tag{3.30}$$

Some further discussion of this coset CFT, the black string solution (3.26), and the relation between them, appear in appendix C, where it is shown that the Bekenstein-Hawking entropy of the black string is given by

$$S_{\rm bs} = \pi\sqrt{2k}\left(\sqrt{M^2 - q_L^2} + \sqrt{M^2 - q_R^2}\right) \ . \tag{3.31}$$

This is a natural generalization of the uncharged (3.4), and one charge (3.20) entropies discussed above. Comparing to (3.24) we see that the relation between the black hole and string entropies, as well as the matching between them at $k = 1$ work in the same way as before.

4. Discussion

In this paper we argued that string theory in asymptotically AdS_3 and linear dilaton spacetimes exhibits an interesting phase structure as one varies the (negative) cosmological constant Λ that produces the background. For $|\Lambda| < |\Lambda_c|$, the theory is in a strongly coupled phase. Generic high energy states in this phase correspond to large black holes; they are non-perturbative from the weakly coupled string perspective. For $|\Lambda| > |\Lambda_c|$, the black holes become non-normalizable and the generic high energy states are highly excited weakly coupled perturbative string states. The critical cosmological constant Λ_c is of order the string scale.

The distinction between the two phases is very reminiscent of the string/black hole correspondence of [1,2]. The main difference is that in that case, the transition between the black hole and string pictures can be achieved by varying the energy of the state, holding all the parameters that define the background fixed. In our discussion, the transition is achieved by varying a parameter of the theory, the cosmological constant Λ.

Another (related) difference between the two cases, is that in [1,2] the matching between the entropies of strings and black holes in the transition region can only be done approximately, and both pictures suffer large corrections there. In our case, the transition between the black hole and string phases is sharp. For example, for AdS_3 one can define an order parameter, $(c - c_{\rm eff})/c$, which vanishes in the black hole phase, and is strictly positive in the perturbative string one. Note that in the string phase, the difference between $c_{\rm eff}$ and c is not small, even in the semiclassical limit $R_{AdS} \gg l_p$ (1.2). Indeed, for $k < 1$ one has

$$1 - \frac{c_{\rm eff}}{c} = \left(1 - \frac{1}{k}\right)^2 , \tag{4.1}$$

so $c_{\rm eff}$ is a finite fraction of c in the limit where both go to infinity. Thus, one expects in general to be able to see the difference between the two phases semiclassically.

The matching of the entropies of black holes and strings at the transition point can be done precisely. We demonstrated that the two agree for BTZ black holes with arbitrary mass and angular momentum in AdS_3, and for black holes carrying up to two charges in the linear dilaton case. The comparison was facilitated by the fact that the black hole metric does not receive any α' corrections in these cases, so that the entropy formula obtained to leading order in α' is in fact valid all the way to the transition point, where the black hole abruptly dissappears due to non-perturbative effects.

The resulting picture is also reminiscent of the phenomenon of confinement in gauge theory. Fundamental strings behave like partons; in the black hole phase, they are strongly interacting at large distances, and form bound states – the black holes. In the string phase, their interactions go to zero at large distances (near the boundary) and they behave like free partons in a non-confining theory.

While the discussion of this paper was in the framework of string theory, we expect the basic phenomena to be more general. This leads to a natural question. Suppose we start with three dimensional gravity with negative cosmological constant, coupled to matter, with the action

$$\mathcal{S} = \frac{1}{16\pi l_p} \int d^3x \left(\mathcal{R} + \frac{2}{R_{AdS}^2}\right) + \mathcal{S}_m , \tag{4.2}$$

where $\mathcal{S}_m$ is the matter action. From the data in (4.2) one can construct the spacetime central charge c (1.2), but where is the information about the phase the theory is in? In particular, if one is not doing string theory, l_s does not appear in (4.2).

A likely answer to this question is the following. Whether one is studying the action (4.2) in the context of string theory or any other consistent theory of quantum gravity, the theory will be in one of the two phases we have described[13]. The phase can be determined by computing the order parameter $(c - c_{\rm eff})/c$ mentioned above. If the theory is in a perturbative phase, with $c - c_{\rm eff} > 0$, it does not contain black holes as dynamical

[13] This should be true also beyond the semiclassical approximation $R_{AdS} \gg l_p$.

excitations (*e.g.* they cannot be formed by gravitational collapse of normalizable states in the theory), and the high energy density of states is dominated by the perturbative states constructed from the fields in the Lagrangian (4.2). If, on the other hand $c_{\text{eff}} = c$, the theory is in a non-perturbative phase and contains dynamical black holes.

An example of a theory of quantum gravity which does not come from string theory and is in the perturbative phase is Chern-Simons gravity [46,47]. In this case, the spacetime CFT is Liouville theory [48], or more precisely the conformal block of the identity of any CFT with central charge c; it has[14] $c_{\text{eff}} = 1$, and in particular $c_{\text{eff}} < c$. We expect this theory not to contain dynamical BTZ black holes, just like two dimensional string theory (or more generally, any asymptotically linear dilaton theory with $Q > \sqrt{2}$) is not expected to contain $\frac{SL(2,\mathbb{R})}{U(1)}$ black holes. There have been attempts in the past to calculate the entropy of BTZ black holes by studying three dimensional quantum gravity in the weakly coupled phase (for a recent review see [23]). From our general perspective it seems that these attempts are unlikely to succeed.

A particularly interesting special case of our discussion is theories with $k = 1$, which lie on the boundary between the black hole and string phases. The fact that these theories are special was noted already in [11], but our discussion clarified their significance – in these theories one expects black holes to be "almost normalizable" and to have the same properties as highly excited fundamental strings. From the point of view of the order parameter (4.1), the theories with $k = 1$ look like critical theories at a second order phase transition.

In the linear dilaton case, an example of a theory with $k = 1$ is the near-horizon geometry of the conifold

$$\mathbb{R}^{3,1} \times \mathbb{R}_\phi \times S^1 \ . \tag{4.3}$$

If we compactify $\mathbb{R}^{3,1} \to \mathbb{R}^{1,1} \times T^2$ and add Q_1 fundamental strings in the linear dilaton throat, we get, at low energies, a CFT dual to string theory on

$$AdS_3 \times S^1 \times T^2 \ , \tag{4.4}$$

again with $k = 1$. One sense in which this theory is critical is that the slope of the linear dilaton on long strings propagating near the boundary of AdS_3 (2.23) vanishes in this

[14] In supersymmetric systems one finds instead the superconformal block of the identity, for which c_{eff} is still of order one, and in particular, is much smaller than c in the semiclassical regime.

case. From the discussion of section 2 (see the discussion around eq. (2.26)), we expect the spacetime CFT corresponding to (4.4) to behave at high energies as

$$\left(\mathbb{R} \times S^1 \times T^2\right)^{Q_1} / S_{Q_1} \ . \tag{4.5}$$

It is intriguing that this CFT closely resembles the spacetime CFT corresponding to parallel fivebranes and strings (whose near-horizon geometry is given by (2.10)), if one formally sets the number of fivebranes k to one. This does not really make sense in that context, since a single fivebrane does not have a linear dilaton throat, but we see that the CFT (4.5) nevertheless makes an appearance, as the spacetime CFT corresponding to the critical, conifold, case.

The fact that the dilaton on long strings near the boundary is constant in this case suggests that one can think of (4.5) as providing a light-cone description of a theory that is Lorentz invariant in $5+1$ dimensions, at least at high energies. It would be interesting to investigate this possibility further.

Another interesting fact is that the backgrounds (4.3), (4.4) corresponding to the critical case $k = 1$ are $5 + 1$ dimensional. As we briefly reviewed earlier, asymptotically linear dilaton and AdS_3 backgrounds of the sort studied here should be thought of as theories of fivebranes wrapped around various cycles, with or without fundamental strings. It is possible that from that point of view, the transition we have observed is due to the fact that for $k > 1$, the fivebrane is embedded in a space which is more than $5 + 1$ dimensional, and there is "room" for transverse fluctuations. These fluctuations are described, at high energies, by black holes. For $k < 1$, the fivebrane is embedded in a target space which is lower dimensional than its own worldvolume, and thus does not have transverse fluctuations. Hence it is in a different, perturbative, phase.

This is analogous to studying the behavior of strings in a d dimensional target space. As is well known, for $d > 2$, the strings have transverse fluctuations, which are reflected in the appearance of a Hagedorn spectrum, whereas for $d < 2$ there is no Hagedorn spectrum, and the theory is essentially topological. The critical theory in that case is two dimensional string theory.

It is natural to ask whether any of the phenomena that we have found in AdS_3 can occur for AdS_{d+1} spacetimes with $d > 2$. This is relevant for ideas [49] to study black hole physics, whose natural home is at large R_{AdS}/l_s (large 't Hooft coupling in the dual super Yang-Mills theory for $d = 4$) by continuing them to small 't Hooft coupling.

For AdS_3, such a continuation is impossible for two reasons. One is that the ratio R_{AdS}/l_s is bounded from below by a number of order one (*e.g.* for the class of theories (2.5), $R_{AdS}/l_s \geq 2/3$). The second is that the black holes cease being normalizable at $R_{AdS}/l_s = 1$. Could similar things happen for $d > 2$?

At first sight the answer seems to be no. For $d = 4$, it is expected that R_{AdS}/l_s can vary freely from zero to infinity, since the 't Hooft coupling of the dual gauge theory has this property. Also, in contrast to AdS_3, the conformally invariant vacuum of quantum gravity on AdS_5 should exist as a normalizable state for all values of R_{AdS}, or 't Hooft coupling. The fate of large AdS black holes for small R_{AdS} is not clear, and it would be interesting to investigate it further.

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Appendix A. Determination of the sign of Q_l

As mentioned in the text, in taking the square root in (2.25) to determine the linear dilaton slope on a long string one encounters a sign ambiguity. The authors of [24] showed that in spacetimes of the form (2.10), which have $k > 1$, the correct sign is (2.23). Here we will show that for a class of models with $k < 1$ (2.23) is reproduced as well. In fact, the argument below is general, and can be used for all k to prove (2.23).

Consider string propagation in $AdS_3 \times S^1 \times \mathcal{M}_n$, where $\mathcal{M}_n$ is an $N = 2$ minimal model with $c(\mathcal{M}_n) = 3 - \frac{6}{n}$. This model was discussed in subsection 4.1 of [12]. Chiral operators in the spacetime CFT are described by RR vertex operators obtained by dressing chiral worldsheet operators V_{Q_V} in $\mathcal{M}_n$ with R-charge Q_V by certain operators in $AdS_3 \times S^1$ with

$$j = \frac{k}{2}(Q_V - 1) \ . \tag{A.1}$$

Their scaling dimension in the spacetime CFT is (see eq. (4.14) in [12]):

$$h_{\text{bottom}} = j + \frac{1}{2} \ . \tag{A.2}$$

As discussed in the text, the target space of the CFT living on a long string is

$$\mathbb{R}_\phi \times S^1_Y \times \mathcal{M}_n \ , \tag{A.3}$$

with the ϕ dependent dilaton

$$\Phi(\phi) = -\frac{Q_l}{2}\phi \ . \tag{A.4}$$

Our aim is to compute Q_l.

Chiral operators in the CFT on the long string are obtained by dressing the chiral operators V_{Q_V} in $\mathcal{M}_n$ with chiral operators in the $N=2$ Liouville CFT, $\mathbb{R}_\phi \times S^1_Y$:

$$e^{\beta(\phi+iY)}V_{Q_V} \ , \qquad \beta = Qj \ , \qquad Q = \sqrt{\frac{2}{k}} \ . \tag{A.5}$$

The value of β in (A.5) can be determined by comparing the behavior of the operator (A.5) at large ϕ to that of the operator corresponding to (A.1), (A.2) described in [12]. The latter goes at large ϕ like $\exp(Qj\phi)$; hence the former must have the same behavior. Comparing the scaling dimension of (A.5) to that of (A.2) one finds

$$-\frac{1}{2}\beta(\beta+Q_l) + \frac{\beta^2}{2} + \frac{Q_V}{2} = h_{\text{bottom}} \ . \tag{A.6}$$

Solving for Q_l leads to

$$Q_l = -(k-1)Q \ . \tag{A.7}$$

Appendix B. A check of the relation $h_w \geq \Delta^{(w)}_{\min}$

In this appendix we show that, for $k<1$, perturbative string states in the sector with winding number w have spacetime scaling dimensions which are bounded from below by $\Delta^{(w)}_{\min} = wQ_l^2/8$ (2.41).

In string theory on AdS_3 there are two kinds of states: those that belong to principal continuous series representations of $SL(2,\mathbb{R})$, with

$$j = -\frac{1}{2} + ip \ , \qquad p \in \mathbb{R} \ , \tag{B.1}$$

where p is the momentum along the radial direction of AdS_3, and those that belong to principal discrete series representations, for which j is real and obeys the unitarity condition (2.16).

Consider first the continuous states (B.1). These only exist in sectors with $w \neq 0$, since for $w = 0$ they correspond to "bad" tachyons and are projected out by the GSO projection. In the sector with winding w, which we take to be positive, as in section 2, their spacetime scaling dimensions (2.29) can be written as

$$h_w = \Delta_{\rm min}^{(w)} + \frac{1}{4}\left(w - \frac{1}{w}\right)\left(2 - \frac{1}{k}\right) + \frac{1}{w}\left(\frac{p^2}{k} + N\right) , \tag{B.2}$$

by using eqs. (B.1) and (2.41). Hence, it is clear that

$$h_w \geq \Delta_{\rm min}^{(w)} . \tag{B.3}$$

Equality requires $w = 1$, and $p = N = 0$. Note that $N = 0$ corresponds to the fermionic string tachyon, and one might think that it should be projected out by the chiral GSO projection. In fact, this is not the case, since the GSO projection acts differently for even and odd w [25]. For even w, the tachyon is projected out, while for odd w it is not. This is similar to the situation in the non-critical superstring [11], where tachyons with odd winding around the S^1 in (2.1) survive the GSO projection.

Thus, we see that for $w = 1$, the dimension $\Delta_{\rm min}^{(w)}$ lies at the bottom of a continuum of long string states. For $w > 1$, the single string states (B.2) satisfy $h_w > \Delta_{\rm min}^{(w)}$, and the state with dimension $\Delta_{\rm min}^{(w)}$ is a multi-string state consisting of w long strings which do not interact near the boundary of AdS_3, as explained in section 2.

Next we turn to principal discrete series states, focusing for concretenes on backgrounds of the form

$$AdS_3 \times S^1 \times \mathcal{M}_n , \tag{B.4}$$

where $\mathcal{M}_n$ is an $N = 2$ minimal model with $c(\mathcal{M}_n) = 3 - \frac{6}{n}$ (see section 2).

Consider first the sector with $w = 1$. Principal discrete series states with $w = 1$ are equivalent by spectral flow to short strings with $w = 0$ [19]. Thus, we should check that short string states satisfy $h_{\rm short} \geq \Delta_{\rm min}^{(1)}$. The lowest lying states in a sector with given R-charge are the chiral ones, so it is enough to perform the check for those. The dimensions of short string chiral operators in the spacetime CFT corresponding to (B.4) were found in [12] to be given by

$$h_i = \frac{i+1}{2(n+1)}, \quad i = 0, 1, ..., n-2 . \tag{B.5}$$

They are indeed larger than

$$\Delta^{(1)}_{\rm min} = \frac{(k-1)^2}{4k} = \frac{1}{4n(n+1)} \ . \tag{B.6}$$

Turning to sectors with $w > 1$, we use the observation that for any state with dimension h_i in the sector with $w = 1$, there is one in the sector with winding number w, which is related to it by spectral flow and has dimension (2.31)

$$h_i^{(w)} = \frac{h_i}{w} + \frac{k}{4}\left(w - \frac{1}{w}\right) \ . \tag{B.7}$$

Since all h_i satisfy $h_i > \Delta^{(1)}_{\rm min}$, we have

$$h_i^{(w)} > \frac{\Delta^{(1)}_{\rm min}}{w} + \frac{k}{4}\left(w - \frac{1}{w}\right) = \Delta^{(w)}_{\rm min} + \frac{1}{4}\left(w - \frac{1}{w}\right)\left(2 - \frac{1}{k}\right) \ . \tag{B.8}$$

Hence, we conclude that these states satisfy (B.3).

Appendix C. $\frac{SL(2,\mathbb{R})\times U(1)}{U(1)}$

In this appendix we briefly review, following [43,50], the gauging of the $SL(2,\mathbb{R})\times U(1)$ WZW model by a family of anomaly free U(1) subgroups and the manner by which it leads to the rotating, charged black string (3.26). We emphasize those features of the gauging procedure that are less standard. We then calculate various properties of the black hole: its entropy, ADM mass and two charges.

The background is constructed as follows [43,50]. Let $(g,x) \in SL\,(2,\mathbb{R}) \times U\,(1)$ be a point on the product group manifold. The $U(1)$ gauge group is a subgroup of the $U(1)^2_L \times U(1)^2_R$ symmetry, which acts as

$$(g, x_L, x_R) \simeq (e^{\rho\sigma_3} g e^{\tau\sigma_3}, x_L + \rho', x_R + \tau') \ , \tag{C.1}$$

where $x_{L,R}$ are the left-moving and right-moving parts of x, respectively. The right-moving part of the gauged $U(1)$ corresponds to $\vec{\tau} \equiv (\tau,\tau') = \tau\hat{u}$, where

$$\hat{u} = \begin{pmatrix} \cos(\chi) \\ \sin(\chi) \end{pmatrix} , \tag{C.2}$$

is a fixed unit vector. The left-moving part corresponds to $\vec{\rho} \equiv (\rho,\rho') = R\vec{\tau}$, where R is the orthogonal 2×2 matrix

$$R = \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix} \ . \tag{C.3}$$

The fact that $|\vec{\rho}| = |\vec{\tau}|$ is necessary for an anomaly free gauging.

The gauged WZW action takes the form

$$\begin{aligned} S =& \frac{k}{4\pi}[\int_\Sigma Tr(g^{-1}\partial g g^{-1}\bar{\partial}g) - \frac{1}{3}\int_B Tr(g^{-1}dg)^3] + \frac{k}{2\pi}\int_\Sigma \partial x\bar{\partial}x \\ &+ \frac{k}{2\pi}\int d^2z[A\bar{\mathbf{J}}^T\hat{u} + \bar{A}\mathbf{J}^T R\hat{u} + 2A\bar{A}\,(R\hat{u})^T\, M\hat{u}] \ , \end{aligned} \tag{C.4}$$

where k is the level of $SL(2,\mathbb{R})$, Σ is the worldsheet, B is a three-manifold whose boundary is Σ, and $(A,\bar{A})$ is the $U(1)$ gauge field. The currents $\mathbf{J}^T$ and $\bar{\mathbf{J}}^T$ are the row vectors

$$\mathbf{J}^T = (Tr[\partial g g^{-1}\sigma_3], 2\partial x) \ , \qquad \bar{\mathbf{J}}^T = (Tr[g^{-1}\bar{\partial}g\sigma_3], 2\bar{\partial}x) \ . \tag{C.5}$$

The 2×2 matrix M in (C.4) is:

$$M = \begin{pmatrix} \frac{1}{2}Tr[g^{-1}\sigma_3 g\sigma_3] & 0 \\ 0 & 1 \end{pmatrix} + R \ . \tag{C.6}$$

The gauge field can be removed by a gaussian integration. Following eqs. (14)–(23) in [43], after fixing the gauge we obtain the worldsheet action and dilaton[15]

$$\begin{aligned} S =& \frac{k}{2\pi}\int d^2z\{\partial\theta\bar{\partial}\bar{\theta} + \partial x\bar{\partial}x - \sinh^2(\theta)\partial y\bar{\partial}y \\ &+ 2\Delta^{-1}\left[\sinh^2(\theta)\cos(\chi)\bar{\partial}y + \sin(\chi)\bar{\partial}x\right]\left[\sinh^2(\theta)\cos(\chi-\psi)\partial y - \sin(\chi-\psi)\partial x\right]\} \ , \\ e^{-2\Phi^{(3)}} =& e^{-2\Phi_0^{(3)}}\Delta \ , \end{aligned} \tag{C.7}$$

where $\Phi_0^{(3)}$ is a constant and

$$\Delta = 1 + \cosh^2(\theta)\cos(\psi) + \sinh^2(\theta)\cos(2\chi-\psi) \ . \tag{C.8}$$

It represents a three dimensional rotating black string with a non-trivial antisymmetric B field. To obtain the two charge two dimensional black hole, a KK reduction is done along the x coordinate [43]; this results in a modified two dimensional metric and dilaton as well as two gauge fields, A_G and A_B, whose origin is in the three dimensional metric and B field,

$$\begin{aligned} g_{yy} =& -2k\frac{\sinh^2(\theta)\cosh^2(\theta)[1+\cos(\psi)][1+\cos(2\chi-\psi)]}{\Delta[1+\cosh^2(\theta)\cos(2\chi-\psi)+\sinh^2(\theta)\cos(\psi)]} \ , \qquad g_{\theta\theta} = 2k \ , \\ A_G =& \frac{\sinh^2(\theta)\sin(\psi)}{1+\cosh^2(\theta)\cos(2\chi-\psi)+\sinh^2(\theta)\cos(\psi)}dy \ , \\ A_B =& 2k\frac{\sinh^2(\theta)\sin(2\chi-\psi)}{\Delta}dy \ , \\ e^{-2\Phi} =& e^{-2\Phi_0}\sqrt{\frac{\Delta\left(1+\cosh^2(\theta)\cos(2\chi-\psi)+\sinh^2(\theta)\cos(\psi)\right)}{(1+\cos(\psi))\,(1+\cos(2\chi-\psi))}} \ , \end{aligned} \tag{C.9}$$

[15] The coordinates $\{\theta, y, x\}$ cover only one out of various regions of the background; see [43].

where Φ_0 is the value of the dilaton at the outer horizon $\theta = 0$. A string mode which is charged under A_G (A_B) is a string mode with x momentum (winding).

Next we describe a few properties of the black hole. The ADM mass is evaluated from [51]

$$M = -\frac{1}{8\pi}\int_{\partial\Sigma_t} \sqrt{-\sigma}\left[N^1K - {N_0}^1K_0 - N^\mu P_{\mu\nu}r^\nu + e^{-\Phi}NB_{t\mu}H^{t\mu\nu}n_\nu\right], \tag{C.10}$$

where Σ_t is a family of spacelike surfaces labeled by t, $\partial\Sigma_t$ is the boundary of Σ_t, σ is the determinant of the induced metric on $\partial\Sigma_t$, $P^{\mu\nu}$ is the canonical momenta conjugate to $G_{\mu\nu}$, N and N^μ are the lapse function and shift vector, n^μ is the unit vector normal to $\Sigma_\infty = \bigcup_t \partial\Sigma_t$, ${}^1K$ is the one dimensional extrinsic curvature on $\partial\Sigma_t$, while N_0 and ${}^1K_0$ are the lapse function and the extrinsic curvature of the reference background which, in our case, is the pure linear dilaton background

$$\mathbb{R}_\phi \times \mathbb{R}_t \times S^1_x\ , \tag{C.11}$$

with $t \simeq y$ and $\phi \simeq \theta$.

One can use (C.10) in two ways. The first, by applying it directly to the three dimensional background (C.7) in the Einstein frame. The second, by adding a trivial fictitious $U(1)$ factor to the two dimensional string frame metric (C.9) and then passing to the Einstein frame in three dimensions[16]. This leads to the following expression for the mass:

$$M = -\frac{1}{2}\lim_{\theta\to\infty}\left(\sqrt{\frac{|g_{tt}|}{g_{\theta\theta}}}\partial_\theta e^{-2\Phi} - \sqrt{\frac{2}{k}}e^{-2\Phi}\right) = \sqrt{\frac{2}{k}}\frac{e^{-2\Phi_0}}{|\cos(\chi)+\cos(\psi-\chi)|}\ , \tag{C.12}$$

where $t \simeq y$ is normalized such that $\lim_{\theta\to\infty} g_{tt} = 1$.

The Bekenstein-Hawking entropy of (C.7), (C.9) is given by[17]

$$S_{\rm bh} = 2\pi e^{-2\Phi_0}. \tag{C.13}$$

[16] The ADM mass, as well as all other physical properties, are invariant under a KK reduction which is trivial in the string frame.

[17] The horizon area can be evaluated directly from the three dimensional background (C.7) or by adding to the two dimensional one (C.9) an extra circle in the string frame.

The computation of the charges is an application of the Gauss law, as in [41]. The charges are the integrals of J^y over θ, where

$$\begin{aligned} J^\mu_G =& \frac{1}{\sqrt{-2kg}} \partial_\nu \left(\sqrt{-g} e^{-2\Phi} G_{xx} F^{\mu\nu}_G \right) \ , \\ J^\mu_B =& \sqrt{-\frac{2k}{g}} \partial_\nu \left(\sqrt{-g} e^{-2\Phi} G^{-1}_{xx} F^{\mu\nu}_B \right) \ . \end{aligned} \tag{C.14}$$

The resulting charges are

$$\begin{aligned} q_G =& \frac{1}{\sqrt{2k}} \lim_{\theta\to\infty} \ \sqrt{-g}\ e^{-2\Phi} G_{xx} F^{\theta y}_G = 2M \sin(\frac{\psi}{2}) \cos(\frac{\psi}{2} - \chi) \ , \\ q_B =& \sqrt{2k} \lim_{\theta\to\infty} \ \sqrt{-g}\ e^{-2\Phi} G^{-1}_{xx} F^{\theta y}_B = 2M \sin(\frac{\psi}{2} - \chi) \cos(\frac{\psi}{2}) \ . \end{aligned} \tag{C.15}$$

The right and left charges are given by

$$\begin{aligned} q_R =& \frac{1}{2} \left(q_G - q_B \right) = M \sin(\chi) \ , \\ q_L =& \frac{1}{2} \left(q_G + q_B \right) = M \sin(\psi - \chi) \ . \end{aligned} \tag{C.16}$$

From eqs. (C.7), (C.12), (C.16) one can now obtain the three dimensional background in Schwarzschild-like coordinates (3.26) by the change of variables

$$r = \sqrt{\frac{2}{k}} e^{-2\Phi^{(3)}(\theta)} \ , \qquad t \simeq y \ , \tag{C.17}$$

where $\Phi^{(3)}(\theta)$ is the function given in eqs. (C.7), (C.8). Finally, from eqs. (C.12), (C.13) and (C.16), one obtains the entropy (3.31).

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