de Sitter space and Holography

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Plan of talk

1. Historical review and motivation

2. de Sitter space and its properties

3. dS/CFT correspondence

4. dS/dS correspondence

1. Historical review and motivation

- In 1917 the cosmological constant Λ was introduced by Einstein to get static homogeneous universe in the present of matter.

- In 1920 Slipher's works showed that the light from galaxies was redshifted indicating that they are moving away from us.

- In 1922 a matter dominated expanding universe without cosmological constant was constructed by Friedmann.

- In 1923 in a letter to Weyl, Einstein says

  " If there is no quasi-static world, then away with cosmological term."

- In 1927 the conclusive discovery was made by Hubble

A linear expansion law relating redshift to distance which made Friedmann models the standard geometrical framework.

• The recent astronomical observations indicate that the cosmological constant in our universe is not zero.

Not only it is not zero but also its contribution is quite important and in fact it is responsible for almost 73% of the energy of the universe (dark energy).

The cosmological constant is important after all.

This means our universe might currently be in the de Sitter phase.

• Beside from this observation, another motivation to study de Sitter space comes from inflation era in which we assume that the universe was also described by de Sitter phase.

• Another interesting feature of dS space is that it has cosmological horizon in which one can associate a temperature and entropy

$$S = \frac{A}{4G}$$

Like black hole, we would like to understand this entropy for dS as well.

Since the horizon of dS space is observer dependent, it is even difficult to say where the quantum microstates that we would like to count are supposed to be.

1. de Sitter space and its properties

1.1 Coordinates

Let's start with the Einstein-Hilbert action coupled to matters

$$S = \frac{1}{16\pi G}\int d^dx\sqrt{-g}(R - 2\Lambda) + S_m$$

To get the de Sitter space we consider the case with $\Lambda > 0$.

The Einstein equation reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -g_{\mu\nu}\Lambda + 8\pi G T_{\mu\mu}$$

We may consider the matter action as follows

$$S_m = \int d^dx\sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$$

In order not to have an additional contribution from the potential to the cosmological constant at the classical level we assume that

$$min\,(V(\phi)) = 0$$

For pure dS space where the energy-momentum tensor of the matter field vanishes, one may consider the spacetime as a solution of the Einstein equation for an empty spacetime with a positive constant vacuum energy

$$T^{vac}_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} \quad \rightarrow \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -g_{\mu\nu}\Lambda$$

Therefore the only non-trivial component of the Einstein equation is

$$R = \frac{2d}{d-2}\,\Lambda > 0$$

An interesting observation about the dS$_d$ space is its embedding into the flat $d+1$ dimensional space time. In $d+1$ dimensional Minkowski spacetime the Einstein equation is trivially satisfied

$$\begin{aligned} 0 &= \, ^{d+1}R \equiv g^{AB} R_{AB} \qquad for\, A, B = 0, ..., d \\ &= R + R_{dd} \end{aligned}$$

If we set $R_{dd} = -\frac{2d}{d-2}\,\Lambda$(which means positive constant curvature of embedded space) one gets

$$R = \frac{2d}{d-2}\,\Lambda \qquad \text{(dS Space)}$$

In fact the dS$_d$ space can be realized as a hypersurface described by the following algebraic equation in flat $d+1$ dimensional Minkowski space

$$-X_0^2 + X_1^2 + \cdots + X_d^2 = l^2$$

The dS metric is the induced metric from the flat space

$$dS^2 = \eta^{AB} dX_A dX_B|_{\eta^{AB} X_A X_B = l^2}$$

From the constraint one gets

$$dX_d = \frac{\eta_{\mu\nu} X^\mu dX^\nu}{\sqrt{l^2 - \eta_{\mu\nu} X^\mu X^\nu}}$$

and therefore the metric reads

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{X_\mu X_\nu}{l^2 - \eta_{\mu\nu} X^\mu X^\nu}$$

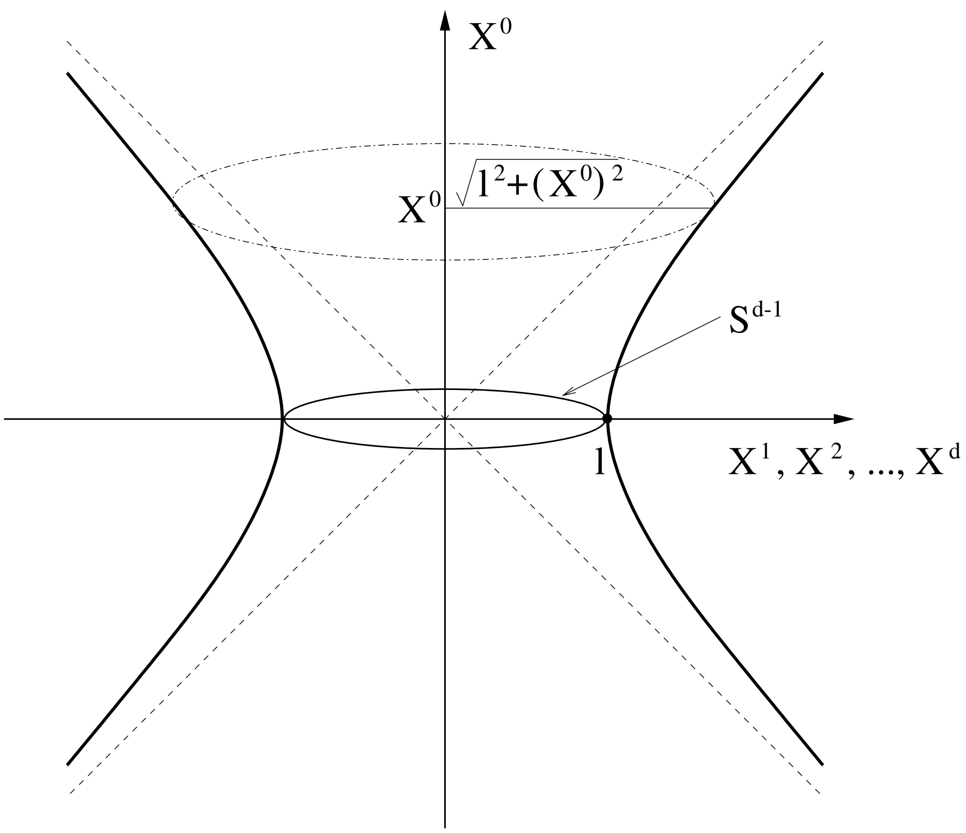
It is then easy to write the Einstein equation for this metric. Doing so we find

$$R = \frac{d(d-1)}{l^2}$$

Comparing with the Einstein equation we had which was written in terms of the cosmological constant one finds

$$\Lambda = \frac{(d-1)(d-2)}{2l^2}$$

Therefore the dS_d in the flat $d+1$ dimensional Minkowski space is a hyperboloid



Let us know study different coordinates systems with will be used later. The different coordinates systems are good for different purpose.

Global coordinates: $\tau, \theta_i, \; i = 1, \cdots, d-1$

This is the simplest one and can be found by just looking at the constraint. In fact different coordinates systems correspond to the different way one can solve the constraint.

Let's decompose the constraint as follows

$$-X_0^2 + (X_1^2 + \cdots + X_d^2) = l^2$$

so the solution is

$$X_0 = l \sinh \frac{\tau}{l}, \qquad X_i = l\omega_i \cosh \frac{\tau}{l}, \quad \text{for} \;\; i = 1, ..., d$$

for $-\infty < \tau < \infty$. Here $\sum_i \omega_i^2 = 1$ parameterize a S^{d-1} sphere

$$\begin{array}{lcll} \omega_1 & = & \cos\theta_1 & 0 \leq \theta_1 < \pi \\ \omega_2 & = & \sin\theta_1 \cos\theta_2 & 0 \leq \theta_2 < \pi \\ \cdots & = & \cdots & \\ \omega_d & = & \sin\theta_1 \cdots \sin\theta_{d-1} & 0 \leq \theta_{d-1} < 2\pi \end{array}$$

The metric is given by

$$dS^2 = -d\tau^2 + l^2 \cosh^2 \frac{\tau}{l}\, d\Omega_{d-1}^2$$

In this coordinates dS_d space looks like a $d-1$ sphere which starts out infinity large at $\tau = -\infty$ then shrinks to its minimum finite size that $\tau = 0$ and then grows again to infinite size at $\tau = \infty$.

In this coordinates system

- $\frac{\partial}{\partial \theta_{d-1}}$ is the only Killing vector
- $\frac{\partial}{\partial \tau}$ is NOT a Killing vector
- This breaks conservation of the energy so the Hamiltonian is not well-defined

Conformal coordinates: $T, \theta_i, \quad i = 1, \cdots, d-1$

Looking at the metric in the global coordinates one may write

$$\begin{aligned} dS^2 &= \cosh^2\frac{\tau}{l}\left(-\frac{d\tau^2}{\cosh^2\frac{\tau}{l}} + l^2 d\Omega_{d-1}^2\right) \\ &= \frac{1}{\cos^2\frac{T}{l}}\left(-dT^2 + l^2 d\Omega_{d-1}^2\right) \end{aligned}$$

where

$$\cosh\frac{\tau}{l} = \frac{1}{\cos\frac{T}{l}}, \qquad -\frac{\pi}{2} < \frac{T}{l} < \frac{\pi}{2}$$

There is one-to-one correspondence between this and the global coordinates and therefore it covers entire space.

In this coordinates system

- $\frac{\partial}{\partial \theta_{d-1}}$ is the only Killing vector
- $\frac{\partial}{\partial \tau}$ is NOT a Killing vector
- This breaks conservation of the energy so the Hamiltonian is not well-defined

Any null geodesic with respect to the conformal metric is also null in the conformally rescaled metric

$$d\tilde{S}^2 = \cos^2 \frac{T}{l} dS^2 = -dT^2 + l^2 d\Omega_{d-1}^2$$

Therefore this coordinates system is useful for studying the causal structure of the dS space.

Planer (Inflationary) coordinates: $t, x^i, \quad i = 1, \cdots, d-1$

Consider the constraint and decompose into two parts

$$(-X_0^2 + X_d^2) + (X_1^2 + \cdots + X_{d-1}^2) = l^2$$

One may also consider a situation

$$-X_0^2 + X_d^2 = l^2 - x^2 e^{-2t/l}$$

$$X_1^2 + \cdots + X_{d-1}^2 = x^2 e^{-2t/l} \qquad (x^2 = x_i x^i)$$

which may solve as follows

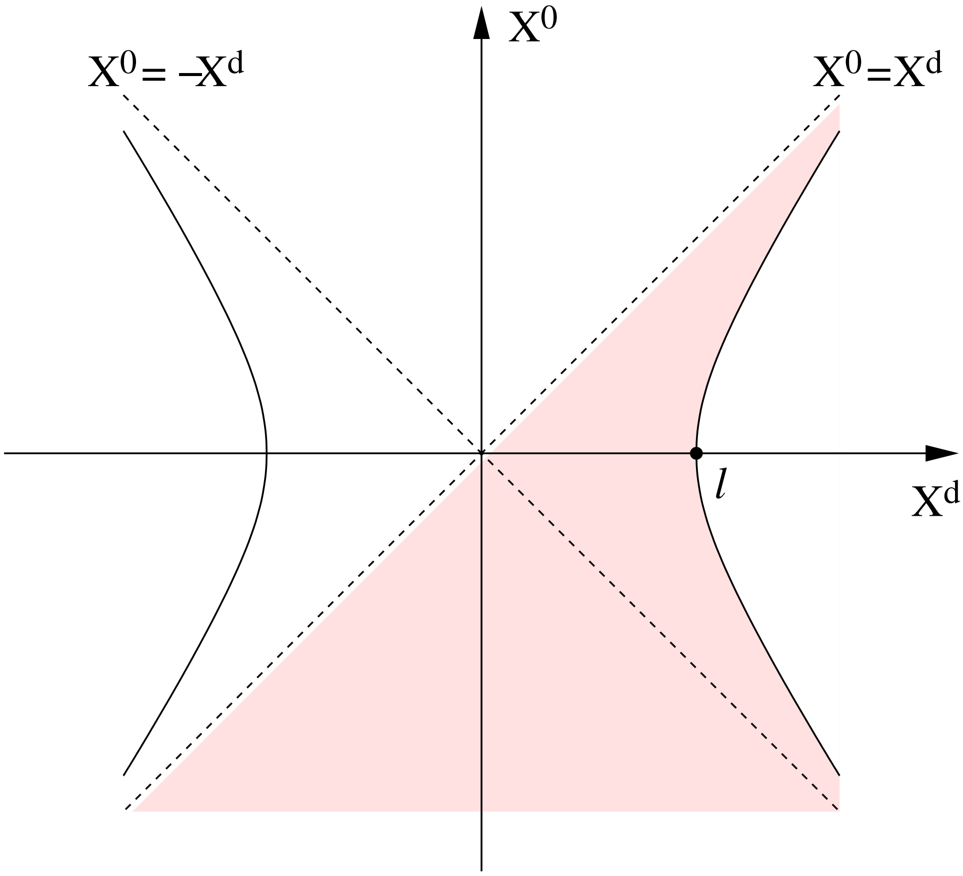
$$\begin{aligned} X_0 &= l \sinh \frac{t}{l} - \frac{x^2}{2} e^{-t/l} \\ X_0 &= l \cosh \frac{t}{l} - \frac{x^2}{2} e^{-t/l} \\ X_i &= x_i e^{-t/l} \end{aligned}$$

for $-\infty < x_i < \infty$ and $-\infty < t < \infty$.

Let's consider $X_0 - X_d$ which turns out to be

$$X_0 - X_d = -le^{-t/l} \leq 0$$

Therefore this coordinates does not cover the entire spacetime. In fact it just cover only half of it which in our convention it is given by the following figure.



The corresponding metric is

$$dS^2 = -dt^2 + e^{-2t/l} dx_i dx^i$$

In this coordinates system

- $\frac{\partial}{\partial x_i}$ for $i = 1, .., d-1$ are the Killing vectors
- $\frac{\partial}{\partial t}$ is NOT a Killing vector
- This breaks conservation of the energy so the Hamiltonian is not well-defined

Static coordinates: $t, r, \theta_i,\ i = 1, ...d-2$

There is another way to solve the constraint in which it is decomposed as follows

$$(-X_0^2 + X_d^2) + (X_1^2 + \cdots + X_{d-1}^2) = l^2$$

with

$$-X_0^2 + X_d^2 = l^2 - r^2$$

$$X_1^2 + \cdots + X_{d-1}^2 = r^2$$

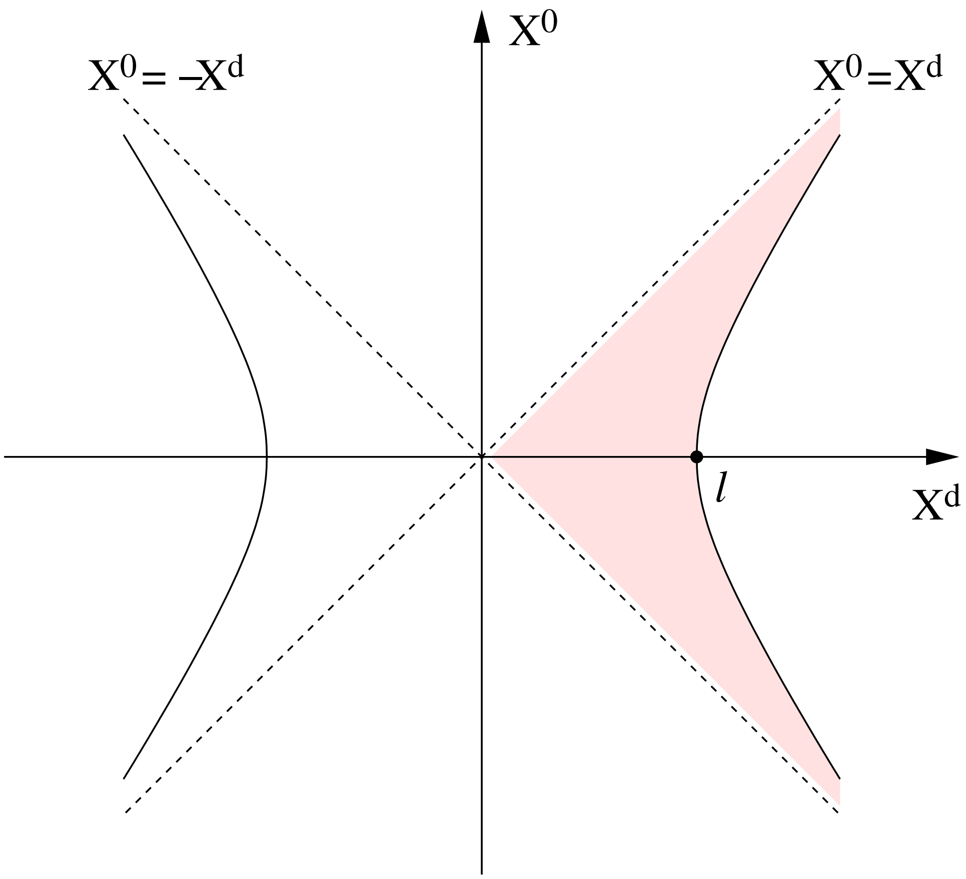
which can be solved by the following parametrization

$$\begin{aligned} X_0 &= \sqrt{l^2 - r^2} \sinh \frac{t}{l} \\ X_d &= \sqrt{l^2 - r^2} \cosh \frac{t}{l} \\ X_i &= r\omega_i, \quad i = 1, ...d-1 \end{aligned}$$

It is easy to see that

$$\begin{aligned} X_0 + X_d &= \sqrt{l^2 - r^2} e^{t/l} \geq 0 \\ X_0 - X_d &= -\sqrt{l^2 - r^2} e^{-t/l} \leq 0 \end{aligned}$$

Therefore this coordinates system can not cover whole spacetime. In fact it just cove quarter of it which in our convention is given by the following figure



The corresponding metric is given by

$$dS^2 = -(1 - \frac{r^2}{l^2})dt^2 + \frac{dr^2}{1 - \frac{r^2}{l^2}} + r^2 d\Omega_{d-2}^2$$

In this coordinate system

- $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \theta_{d-2}}$ are Killing vectors
- Hamiltonian is well-defined in the static coordinates but unitarity is threatened by the existence of the horizon at $r = l$. Note that in this coordinates $0 \leq r < l$.

There is a more general solution with the same symmetry as the static coordinates. Consider the following ansatz

$$dS^2 = -e^{B(r)}A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega_{d-2}^2$$

From Einstein equation one finds

$$\frac{dB}{dr} = 0, \quad \frac{d}{dr}[(1-A)r^{d-3}] = \frac{d-1}{l^2}r^{d-2}$$

which can be solved as follows

$$B(r) = B_0, \qquad A(r) = 1 - \frac{r^2}{l^2} - \frac{2M}{r^{d-3}}$$

where M is the integration constant. B_0 can also be absorbed by a rescaling of t.

For $M = 0$ this is dS space in the static coordinates. For non zero M it is a black hole in dS space with mass M which is called Schwarzschild-dS space.

1.2 Causal structure (Penrose diagram)

Using a conformal factor $\Omega^2 = \cos^2 T$ one may bring in the points at infinity to a finite position so that the whole spacetime is shrink into a finite region called Penrose diagram.

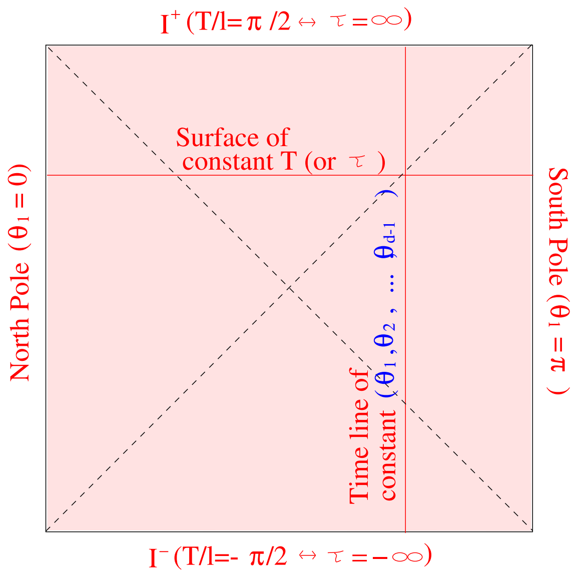
Since a conformal map does not change the causal structure, this may use to study the causal structure of the dS space, though the distances are highly distorted.

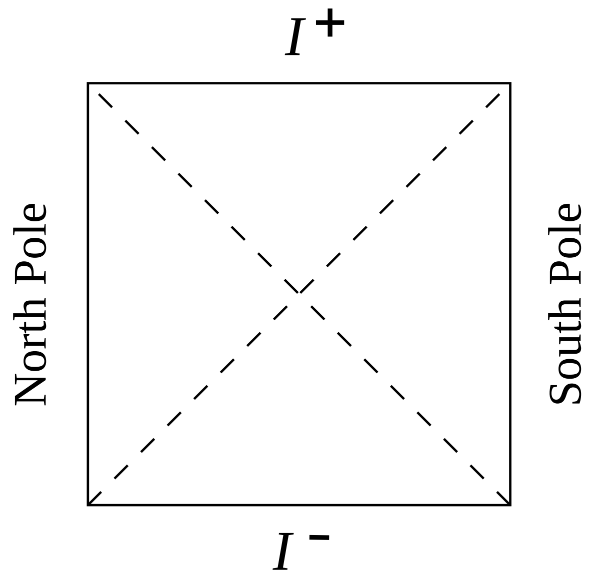
We would like to study causal structure in various coordinates in terms on the Penrose diagram.

$$d\tilde{S}^2 = \cos^2 T dS^2 = -dT^2 + l^2 d\Omega_{d-1}^2$$

One may write this as follows

$$-dT^2 + d\theta_1^2 + \sin^2\theta_1 \; d\Omega_{d-2}^2, \quad 0 < \theta_1 < \pi$$





1) North and South poles are timelike line.

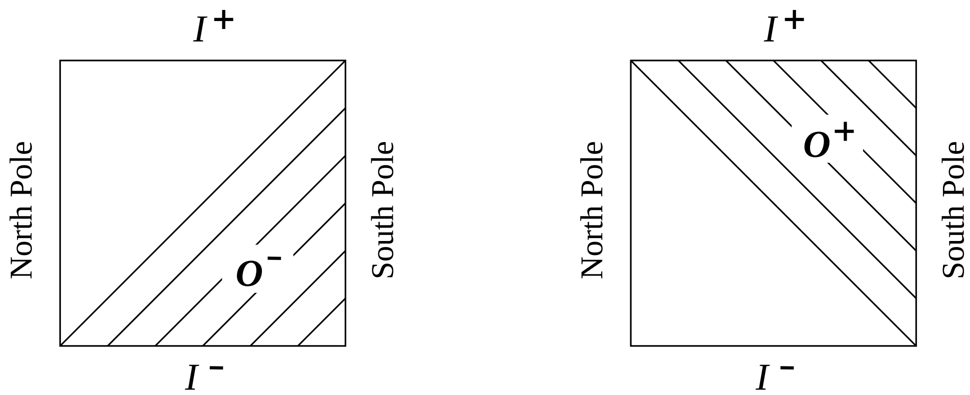
2) Every point in the interior represents an S^{d-2}.

3) A horizontal slice is an S^{d-1}.

4) I^- and I^+ are past and future null infinity. They are the surfaces where all null geodesics originate and terminate.

5) The dashed lines are the past and future horizons of an observer at the south pole.

Therefore no single observer can see entire spacetime.



1) An observer in the south pole cannot see anything past the diagonal line stretching from the north pole at I^- to south pole at I^+:O^- region .

2) O^+ is the only part that an observer in south pole can send a signal to.

3) The planer coordinate we were talking about covers O^- region. (In our notation)

4) Changing $t \to -t$ the coordinates will cover O^+ region.

5) $O^- \bigcap O^+$ is the part accessible to the observer on the south pole (causal diamond). This is the part with is covered by static coordinates.

Let us to arrive at Penrose diagram from static coordinates. Starting from, t, r, θ_i one defines

$$U = -e^{x^-/l}, \qquad V = e^{-x^+/l}$$

where

$$x^{\pm} = t \pm \frac{l}{2} \ln \frac{1 + r/l}{1 - r/l}$$

Therefore we have

$$UV = -\frac{1 - r/l}{1 + r/l}$$

The metric in this coordinates (Kruskal) is given by

$$ds^2 = \frac{l^2}{(1 - UV)^2} \left[-4dUdV + (1 + UV)^2 d\Omega^2_{d-2} \right]$$

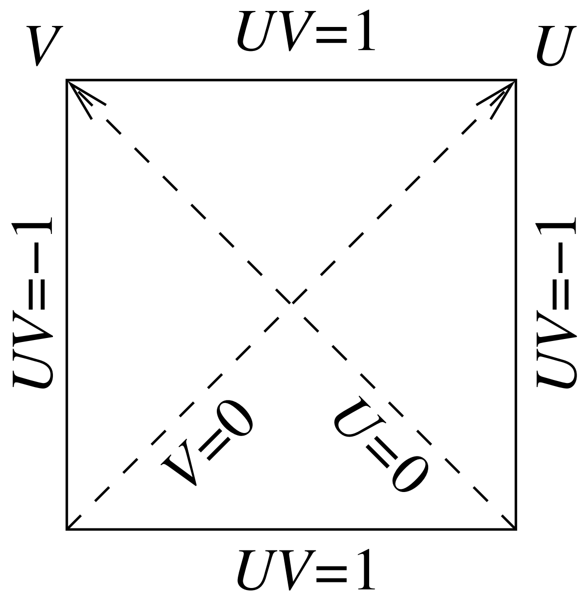
This coordinates cover all of the dS space

$$UV = -1, \qquad \frac{r}{l} = 0, \quad \text{Poles}$$

$$UV = 0, \qquad \frac{r}{l} = 1, \quad \text{Horizons}$$

$$UV = 1, \qquad \frac{r}{l} = \infty, \quad \mathrm{I}^{\pm}$$

The entire region of the dS space is drawn by a Penrose digram which is a square given by $|UV| = 1$.



The static patch is the region with $U > 0$ and $V < 0$.

$$\frac{\partial}{\partial t} = \frac{U}{l}\frac{\partial}{\partial U} - \frac{V}{l}\frac{\partial}{\partial V}$$

So the norm of the Killing vector is

$$|\frac{\partial}{\partial t}|^2 = \frac{4UV}{(1-UV)^2}$$

$$\begin{array}{ll} UV = 0, & \text{null} \\ UV > 0, & \text{spacelike} \\ UV < 0, & \text{timelike} \end{array}$$

1.3 Thermal property

Consider a circle with radius l : $X_1^2 + X_2^2 = l^2$.

$$X_1 = l\cos\theta, \qquad X_2 = l\sin\theta$$

The geodesic distance between two points on the circle is given by

$$d(\theta, \theta') = l(\theta - \theta')$$

Let's consider a quantity $P(X, X') = \frac{1}{l^2}\delta^{ij}X_iX_j'$ which is

$$\frac{1}{l^2}\delta^{ij}X_iX_j' = \cos\theta\cos\theta' + \sin\theta\sin\theta' = \cos(\theta - \theta')$$

So

$$d(\theta, \theta') = l\cos^{-1}P(X, X')$$

One may generalize it for higher sphere or dS space. In the case of the dS space given by $\eta^{\mu\nu}X_\mu X_\nu = l^2$ one has

$$P(X, X') = \frac{1}{l^2}\eta^{\mu\nu}X_\mu X_\nu', \qquad d(1,2) = l\cos^{-1}P(X, X')$$

For example

- In static coordinates:

$$l^2 P(X,X') = -\sqrt{(r^2-l^2)(r'^2-l^2)}\cosh\frac{t-t'}{l} + rr'\cos\Theta$$

where Θ is the geodesic distance of two points on the unit S^{d-2}.

- In planer coordinates:

$$l^2 P(X,X') = -l^2\cosh\frac{t-t'}{l} + \frac{1}{2}e^{\frac{-t-t'}{l}}\delta_{ij}(x^i-y^i)(x^j-y^j)$$

Consider a scalar field in dS space

$$S = -\frac{1}{2}\int d^dx\sqrt{-g}[(\nabla\phi)^2 + m^2\phi^2]$$

The Green function $G(X,Y) = \langle 0|\phi(X)\phi(Y)|0\rangle$ obeys

$$(\nabla^2 - m^2)G(X,Y) = 0$$

Since dS space is maximally symmetric, the Green function depends on X and Y only through $P(X,Y)$.

For any function $f(P)$ one can see

$$l^2(\nabla^2 - m^2)f(P) = (1-P^2)\frac{d^2 f}{dP^2} - Pd\frac{df}{dP} - m^2 l^2 f$$

Therefore one has

$$\left[(1-P^2)\frac{d^2}{dP^2} - Pd\frac{d}{dP} - m^2 l^2\right] G(P(X,Y)) = 0$$

Which as solution in terms of hypergeometric functions

$$G = \mathrm{const.}F(h_+, h_-, \frac{d}{2}, \frac{1+P}{2})$$

where

$$h_\pm = \frac{1}{2}\left[(d-1) \pm \sqrt{(d-1)^2 - 4m^2 l^2}\right]$$

Since the above equation is symmetric under $P \to -P$ there is another solution

$$G = \mathrm{const.}F(h_+, h_-, \frac{d}{2}, \frac{1-P}{2})$$

One parameter family of dS invariant Green function corresponding to a linear combination of these solutions.

$$G_\alpha(X,Y) = \langle\alpha|\phi(X)\phi(Y)|\alpha\rangle$$

An observer moving along a timelike geodesic sees a thermal bath of particles when the scalar field is in the vacuum state.

de Sitter space is naturally associated with a temperature.

One way to see this:

Consider an observer sitting in south pole. In this case

$$P(X, X') = -\cosh\frac{t-t'}{l} = -\cosh\frac{\tau}{l}$$

On the other hand the Green function is a function of P

$$G(P(X,Y)) = G(\cosh\frac{\tau}{l})$$

The Green function is periodic in imaginary time under $\tau \to \tau + 2\pi i l$ and therefore is thermal Green function and the temperature in given by the inverse of the period

$$T_{dS} = \frac{1}{2\pi l}$$

One can also associate entropy which is the same as black hole and is given by

$$S = \frac{A}{4G}$$

where A is the area of the horizon.

Why dS is difficult?

1) dS space is inconsistent with supersymmerty: there is no supergroup that includes the isometries of dS space and has unitary representation.

2) We have not been able to embed it in string theory (up to KKLT model).

3) Horizon is observer dependent: difficult to see where the quantum microstate we would like to count are in fact supposed to be.

3. dS/CFT correspondence

From what we have learned in AdS/CFT correspondence one may hope that some kind of holography can also be applied here and could help us to understand the quantum gravity on dS.

There is a naive observation:

Consider a AdS space with radius l, under $l \to il$ one gets

$$\begin{array}{ccc} \Lambda & \longrightarrow & -\Lambda \\ AdS & \longrightarrow & dS \\ SO(2,d) & \longrightarrow & SO(1,d+1) \end{array}$$

Gravity on dS is dual to a Euclidean CFT.

One can make this statement more precise which is in fact what is known as dS/CFT correspondence.

How to define conserved charges?

The deviation of the metric and other fields near spatial infinity from the vacuum provides a way to define conserved charges like mass, angular momentum....

Equivalently the conserved charges can be computed from the *asymptotic symmetries* of a space time.

For example the eigenvalue of an asymptotic timelike Killing vector will give the mass.

There are two basic problems to apply this definition for dS space (spacetime which is asymptotically dS):

1. There is no spatial infinity.

2. There is no globally defined asymptotic timelike Killing vector.

Fortunately there is a way to proceed generalizing the Brown-York construction to define stress tensor on the Euclidean boundary and by using this quantity to define mass or other charges for spaces which are asymptotically dS.

In d dimensions the Einstein equations with positive cosmological constant can be derived from the action

$$
\begin{aligned}
I &= I_{\text{bulk}} + I_{\text{GH}} \\
&- \frac{1}{16\pi G}\int_{\mathcal{M}} d^d x\sqrt{-g}(R+2\Lambda) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{h}K
\end{aligned}
$$

here

- I_{GH} is Gibbons-Hawking surface term which is needed to get a well-defined Euler-Lagrange variation.

- $\mathcal{M}$ is d-dimensional Manifold with Newton's constant G with spatial Euclidean boundary $\partial\mathcal{M}$.

- $g_{\mu\nu}$ is the bulk metric.

- $h_{\mu\nu}$ and K are induced metric and the trace of the extrinsic curvature of the boundary. The extrinsic curvature is defined by $K_{\mu\nu} = -\nabla_{(\mu}n_{\nu)}$ where n_ν is outward pointing unit vector.

- A useful length scale in the model is given by

$$
l = \sqrt{\frac{(d-1)(d-2)}{2\Lambda}}
$$

For example in the vacuum dS solution, l is the radius of dS space.

In general this action is divergent when evaluated on a solution of the equations of motion due to infinite volume of the spacetime.

For example in the case of dS space and in the inflationary coordinates one has ($d = 3$)

$$I \sim \frac{1}{8\pi G} \int d^2 x e^{2t/l} \left(\frac{-1}{l} \right)$$

which diverges as $t \to \infty$.

The divergence can be canceled by adding local boundary counterterms that do not affect the equations of motion.

In our case we have

$$I_{\text{total}} = I + \frac{1}{8\pi G} \int_{\partial \mathcal{M}} d^2 x \sqrt{h} \frac{1}{l}$$

For dS space with two boundaries at $t \to \pm\infty$, $\partial\mathcal{M}_{\pm}$, we have

$$I_{\text{total}} = I + \frac{1}{8\pi G} \int_{\partial \mathcal{M}_+} d^2 x \sqrt{h} \frac{1}{l} + \frac{1}{8\pi G} \int_{\partial \mathcal{M}_-} d^2 x \sqrt{h} \frac{1}{l}$$

which has the same solution as the previous action but is finite.

One can generalize it for a general space which is asymptotically dS. For $d =, 3, 4, 5$ the counterterms are given by

$$I_{\rm ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}_+} d^2x\sqrt{h}\mathcal{L}_{\rm ct} + \frac{1}{8\pi G}\int_{\partial\mathcal{M}_-} d^2x\sqrt{h}\mathcal{L}_{\rm ct}$$

where

$$\mathcal{L}_{\rm ct} = \frac{d-2}{l} - \frac{l^2}{2(d-3)}\mathcal{R}$$

- The second term is present for $d > 3$.

- $\mathcal{R}$ is the intrinsic curvature of the boundary.

- The calculations are preformed by cutting off the dS space at finite time and then sending the surface to infinity.

The total action is then

$$I_{\rm total} = I_{\rm bulk} + I_{\rm GH} + I_{\rm ct}$$

We can now compute the Euclidean boundary stress tensor which measures the response of the spacetime to changes of the boundary metric (Brown-York prescription).

Consider a spacetime with the metric

$$ds^2 = g_{ij}dx^i dx^j$$

One may rewrite it as follows

$$ds^2 = g_{ij}dx^i dx^j = -N_t^2 dt^2 + h_{\mu\nu}(dx^\mu + V^\mu dt)(dx^\nu + V^\nu dt)$$

So $h_{\mu\nu}$ is the metric induced on surface with fixed time.

Suppose u^μ is the future pointing unit normal to this surface, then the extrinsic curvature is given by

$$K_{\mu\nu} = -h_\mu^i \nabla_i u_\nu$$

The stress tensor associated to the boundary is given by

$$\begin{aligned} T^{\mu\nu} &= -\frac{2}{\sqrt{h}} \frac{\delta I_{\text{total}}}{\delta h_{\mu\nu}} \\ &= \frac{1}{8\pi G}\left[K^{\mu\nu} - Kh^{\mu\nu} - \frac{d-2}{l}h^{\mu\nu} - \frac{l\mathcal{G}^{\mu\nu}}{d-3}\right] \end{aligned}$$

where $\mathcal{G}^{\mu\nu}$ is the Einstein tensor of h.

In global coordinates (on the boundary $t \to -\infty$)

$$T_{\theta\theta} = -\frac{l}{16\pi G}, \qquad T_{\phi\phi} = -\frac{l}{16\pi G}\sin^2\theta$$

• One can always decompose the metric $h_{\mu\nu}$ on the equal time surfaces as follows

$$h_{\nu\nu}dx^{\mu}dx^{\nu} = N_{\rho}d\rho^{2} + \sigma_{ab}(d\phi^{a} + U^{a}d\rho)(d\phi^{b} + U^{b}d\rho)$$

where ϕ^{a} are angular variables parameterizing closed surfaces.

• Suppose the boundary has an isometry generated by a Killing vector ξ^{μ} . One can show that $T_{\mu\nu}\xi^{\nu}$ is divergenceless and therefore we can define conserved charge associated to ξ^{μ}.

• Consider n^{μ} be the unit normal on a surface of fixed ρ. Then the conserved charge is defined

$$Q = \oint_{\Sigma} d^{d-2}\phi\sqrt{\sigma}n^{\mu}\xi^{\nu}T_{\mu\nu}$$

Physically this means that a collection of observers on the hypersurface whose metric is $h_{\mu\nu}$ would all observe the same value of Q provided this surface had an isometry generated by ξ^{μ}.

Since there is no globally timelike Killing vector, it is difficult to see how the mass can be defined in dS space. There is however an proposal for this using the above construction.

Consider the case where ρ is the coordinate associated with the asymptotic Killing vector that is timelike inside the static patch but spacelike at I^{-}.

The mass of an asymptotically dS space is

$$M = \oint_{\Sigma} d^{d-2}\phi\sqrt{\sigma}N_{\rho}n^{\mu}n^{\nu}T_{\mu\nu}$$

One may also define momenta as

$$J_a = \oint_{\Sigma} d^{d-2}\phi\sqrt{\sigma}\sigma_{ab}n_{\nu}T^{b\nu}$$

Consider the Brown-York stress tensor $T^{\mu\nu}$ for a spacetime which is asymptotically dS. One may ask the following questions:

1. What would be the boundary conditions for the metric if we want the stress tensor to be finite?

2. What is the most general diffeomorphism which preserves this boundary conditions?

The first question can be answered by perturbing dS space and computing $T^{\mu\nu}$ and then one may answer to the second question which will be

The conformal group of the $(d-1)$-dimensional Euclidean space.

This is one of the main hints of the dS/CFT correspondence which says

Quantum gravity on dS$_d$ is dual to a $(d-1)$-dimensional Euclidean conformal field theory residing on the past boundary I^- of dS$_d$.

This CFT may be non-unitary!

To see how this works, let's consider an explicit example; dS$_3$.

Consider dS$_3$ space in the planar coordinates ($l = 1$)

$$ds^2 = -dt^2 + e^{-2t} dz d\bar{z}$$

Consider a perturbation around this metric $g_{ij} + \gamma_{ij}$ so that

$$ds^2 = g_{ij} dx^i dx^j = -dt^2 + e^{-2t} dz d\bar{z} + \gamma_{ij} dx^i dx^j$$

which can be recast into the following form

$$ds^2 = -N dt^2 + h_{\mu\nu}(dx^\mu + V^\mu dt)(dx^\nu + V^\nu dt)$$

Therefor the induced metric reads

$$h_{zz} = \gamma_{zz}, \qquad h_{z\bar{z}} = \frac{1}{2} e^{2t} + \gamma_{z\bar{z}}, \qquad h_{\bar{z}\bar{z}} = \gamma_{\bar{z}\bar{z}}$$

The outward pointing unit normal vector is also given by

$$n_\mu = (1 - \frac{\gamma_{tt}}{2}, 0, 0)$$

So we find

$$\begin{aligned} K_{zz} &= -\partial_z \gamma_{tz} + \frac{1}{2} \partial_t \gamma_{zz} \\ K_{z\bar{z}} &= -\frac{1}{2} e^{-2t} (1 + \frac{\gamma_{tt}}{2}) - \frac{1}{2} (\partial_{\bar{z}} \gamma_{tz} + \partial_z \gamma_{t\bar{z}} - \partial_t \gamma_{z\bar{z}}) \\ K &= -2 - \gamma_{tt} + 4 e^{2t} \gamma_{z\bar{z}} - 2 e^{2t} (\partial_{\bar{z}} \gamma_{tz} + \partial_z \gamma_{t\bar{z}} - \partial_t \gamma_{z\bar{z}}) \end{aligned}$$

Using the definition of Brown-York stress tensor one finds

$$
\begin{aligned}
T_{zz} &= \frac{1}{4G}\left[h_{zz} - \partial_z h_{tz} + \frac{1}{2}\partial_t h_{zz}\right] + \mathcal{O}(h^2) \\
T_{z\bar{z}} &= \frac{1}{4G}\left[\frac{1}{4}e^{-2t}h_{tt} - h_{z\bar{z}} + \frac{1}{2}(\partial_{\bar{z}}h_{tz} + \partial_z h_{t\bar{z}} - \partial_t h_{z\bar{z}}\right] \\
&\quad + \mathcal{O}(h^2)
\end{aligned}
$$

Requiring the stress tensor to be finite one leads to the boundary conditions

$$
\begin{aligned}
g_{z\bar{z}} &= \frac{e^{-2t}}{2} + \mathcal{O}(1), \\
g_{tt} &= -1 + \mathcal{O}(1), \\
g_{zz} &= \mathcal{O}(1), \\
g_{tz} &= \mathcal{O}(1).
\end{aligned}
$$

The most general diffeomorphism ξ which preserves this boundary conditions can be written as

$$
\xi = U(z)\partial_z + \frac{1}{2}U'(z)\partial_t + \mathcal{O}(e^{2t}) + C.C.
$$

From bulk dS_3 theory this is a diffeomorphism while from boundary point of it is two dimensional diffeomorphism of the complex plane and a Weyl transformation.

A three dimensional diffeomorphism is equivalent to a two dimensional conformal transformation.

The asymptotic symmetry group of gravity in dS_3 is the conformal group of the complex plane

This is a first hint that the dual theory is a Euclidean conformal field theory.

Using this diffeomorphism one may see how the stress tensor transform under this diffeomorphism

$$\delta_\xi T_{zz} = -U\partial T_{zz} - 2U' T_{zz} - \frac{1}{8G} U'''$$

So the central charge of the dual CFT is

$$c = \frac{3l}{2G}$$

One may reach to the same central charge using Wayl anomaly. We saw in global coordinates

$$T_{\theta\theta} = -\frac{l}{16\pi G}, \qquad T_{\phi\phi} = -\frac{l}{16\pi G}\sin^2\theta$$

So

$$K = h^{\mu\nu} K_{\mu\nu} = -\frac{1}{8\pi G l}$$

Comparing with

$$T = -\frac{c}{24\pi}$$

and using $R = 2/l^2$ one gets the same result as above.

One can compute correlation function of CFT operators that couple to bulk fields (much similar to AdS/CFT)

Consider a massive scalar field. In static coordinates one has

$$\frac{1}{r}\partial_r(rV(r)\partial_r\Phi) - \frac{1}{V(r)}\partial_t\Phi + \frac{1}{r^2}\partial_\phi^2 - m^2\Phi^2 = 0$$

where

$$V(r) = 1 - \frac{r^2}{l^2}$$

.

Near the boundary I^- which is at $r \to \infty$ we get

$$\Phi \sim r^{-h_\pm}$$

with

$$h_\pm = 1 \pm \sqrt{1 - m^2 l^2}$$

One may impose the boundary condition

$$\lim_{r\to\infty} \Phi(r,t,\phi) = r^{-h_-}\Phi_-(t,\phi)$$

Therefore the two point function of an operator $\mathcal{O}$ coupled to Φ on the boundary can be evaluated in the same way as AdS/CFT correspondence.

$$\lim_{r\to\infty}\int_{I^-} dt dt' d\phi d\phi' \; \frac{(rr')^2}{l^2} \left(\Phi(r,t,\phi)\overleftrightarrow{\partial}_{r_*} G(r,t,\phi;r',t',\phi')\overleftrightarrow{\partial}_{r'_*}\Phi(r',t',\phi')\right)$$

where $dr_* = (-V(r))^{-1/2}dr$.

Using the Green function we find

$$\langle \mathcal{O}(t,\phi)\mathcal{O}(t',\phi')\rangle = \frac{\text{const.}}{(\cosh\frac{\Delta t}{l} - \cos\Delta\phi)^{h_+}}$$

The dimension of the CFT operator is given by h_+ which could be complex for highly massive scalar field. So the CFT could be non-unitary!

We note however that in studying this correspondence one uses the dS in the planer patch. So it is nature to ask:

How does the holography for dS space work for other patches?

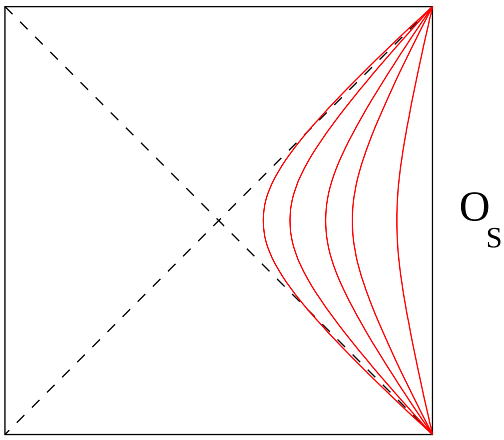
4. dS/dS correspondence

Statement of de Sitter Holography

The dS_d static patch is dual to two conformal field theories on dS_{d-1} cut off at an energy scale $1/R$ and coupled to each other as well as to $(d-1)$-dimensional gravity.

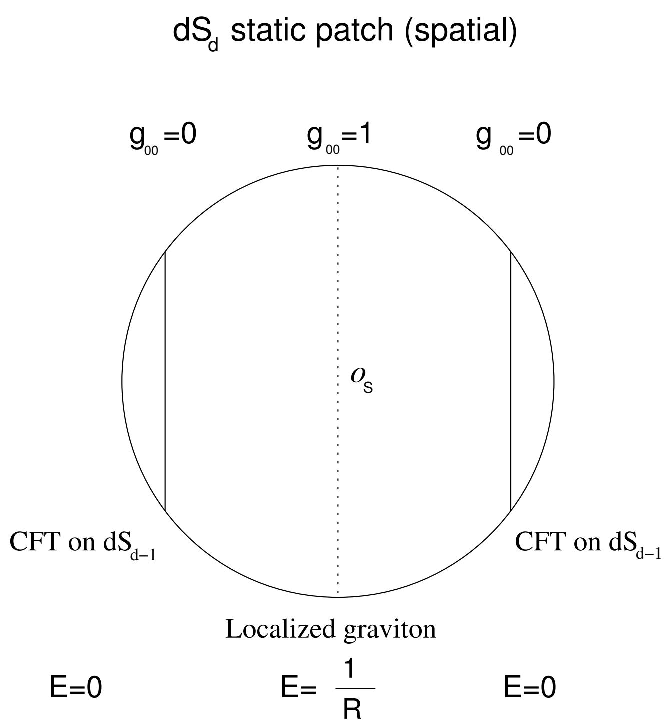
The static patch of d-dimensional dS space with radius R can be foliated by dS_{d-1} slices

$$ds^2 = \sin^2\left(\frac{w}{R}\right)\ ds^2_{ds_{d-1}} + dw^2$$



The warp factor $\sin^2(w/R)$ is maximal with the finite value at central slice $w = \pi R/2$ and dropping monotonically on each side until reaches zero at the horizon $w = 0, \pi$.

The region near horizon (low energy in static patch) is isomorphic to d-dimensional AdS space foliated by dS_{d-1} slices and hence constitutes a CFT on dS_{d-1} at low energy.



- D-brane probes of this region exhibit the same physics: Both are equivalent to CFT on dS$_{d-1}$ for energy $0 < E \ll \frac{1}{R}$.

- Probes constructed from bulk gravitons range from energy 0 up to $1/R$ and upon dimensional reduction their spectrum exhibits the mass gap expected of $d-1$ dimensional CFT on dS.

- Dimensionally reducing to $d-1$ dimensional effective field theory also yields a finite $d-1$ dimensional Plank mass$\longrightarrow$ Lower dimensional theory has also gravity.

Let us consider pure gravity in d dimensional dS space. The differential equations of linearized gravity can be recast to a one dimensional quantum mechanical system with the following potential

$$V = \frac{1}{l}\left(\frac{(d-2)^2}{4} - \frac{d(d-2)}{4}\,\frac{1}{\cosh^2\frac{z}{l}}\right)$$

It is a volcano shape potential which has a single bound state at zero energy, separated from a continuum modes by a mass gap of order $\frac{1}{l^2}$.

The zero mode solution is

$$\psi = \left(\cosh^2\frac{z}{l}\right)^{(2-d)/2}$$

It is a normalizable solution to the wave equation with $E = 0$.

Therefore there is a $d-1$ dimensional graviton on the central slice at $z = 0$.

One may also write the lower dimensional Planck scale in terms of d dimensional theory

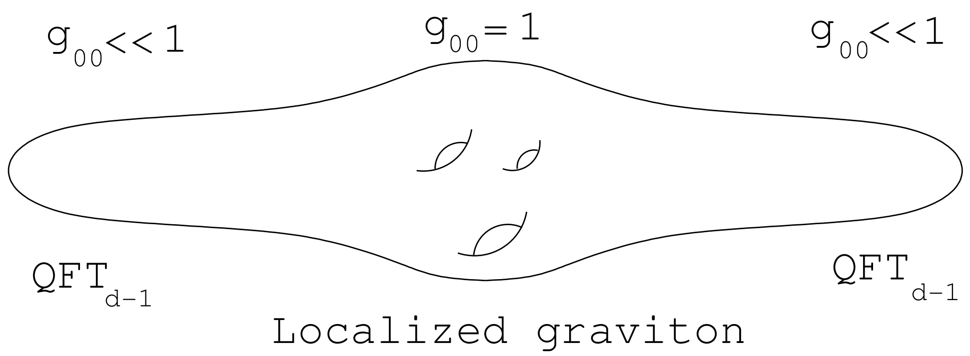
$$M_{d-1}^{d-3} = l M_d^{d-2}$$

• Lower dimensional Plank mass is consistent with that generated by a renormalization of Newton constant from S degrees of freedom cut off at the scale $1/R$.

$$\begin{aligned}\frac{1}{G_{N,d-1}} \sim M_{d-1}^{d-3} \sim RM_d^{d-2} &\sim (RM_d)^{d-2}\left(\frac{1}{R}\right)^{d-3} \\ &\sim S\left(\frac{1}{R}\right)^{d-3} \\ &\sim SM_{UV}^{d-3}\end{aligned}$$

The picture

Written in a dS_{d-1} slicing, dS_d has the form of a Randall-Sundrum system.



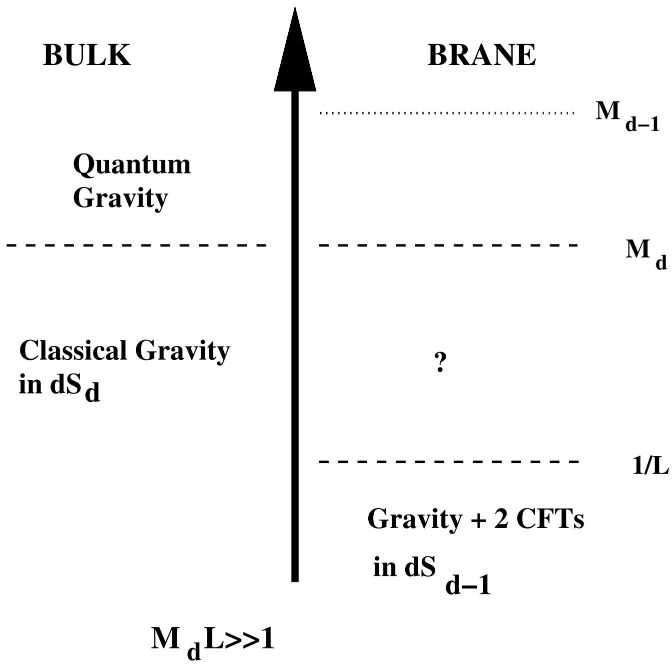
At higher energies $E \to \frac{1}{R}$ AdS and dS differ:

• In AdS, the warp factor diverges towards the UV $\to$ $d-1$ dimensional gravity decouples.

• In dS, the warp factor is bounded $\to$ a dynamical $d-1$ dimensional graviton.

In the RS construction one truncates the warp factor at a finite value of the radial coordinate by including a brane source with extra degree of freedom. In the dS case, the additional brane source is unnecessary.

A smooth UV brane at which the warp factor turns around is built in to the geometry.



• On d-dimensional gravity side, one has local effective field theory up to $M_d \gg 1/R$.

• Above M_d quantum gravity effects become important in the bulk and one has to UV complete the system.

• Using gravity side one can study the $d-1$ dimensional theory in the range of energy $1/R < E < M_d$.

One can use AdS/CFT correspondence to study some properties of this duality.

dS slicing of dS:

$$ds^2_{dS_d} = \frac{R^2}{\cosh^2(z)}(ds^2_{dS_{d-1}} + dz^2)$$

dS slicing of AdS:

$$ds^2_{AdS_d} = \frac{R^2}{\sinh^2(z)}(ds^2_{dS_{d-1}} + dz^2)$$

So

$$ds^2_{AdS_d} = \frac{1}{\tanh^2(z)} ds^2_{dS_d}$$

We can use this to map the physics of dS to dynamics in AdS, albeit with unusual actions. Namely this leads to

- Scalars with position dependent masses.

- Gravity with a position dependent Newton constant.

By applying the AdS/CFT dictionary to the resulting system, this allows us to make a direct comparison of UV behavior of the $d-1$ dual of dS$_d$ to the UV behavior of a strongly coupled CFT.

g_{mn}	$\to$	$f^2 g_{mn}$
X	$\to$	$f^{-\frac{d-2}{2}} X$
$\sqrt{-g}$	$\to$	$f^d \sqrt{-g}$
$-\sqrt{-g}(\partial X)^2$	$\to$	$-\sqrt{-g}(\partial X)^2 - \sqrt{-g}\frac{(d-2)}{2} X^2 (\nabla^2 \omega)$ $-\sqrt{-g}\frac{(d-2)^2}{4} X^2 (\nabla\omega)^2$
$\sqrt{-g}R$	$\to$	$f^{d-2}\sqrt{-g}(R - 2(d-1)(\nabla^2\omega)$ $-(d-2)(d-1)(\nabla\omega)^2)$
$-\sqrt{-g}\xi R X^2$	$\to$	$-\sqrt{-g}\xi R X^2 + 2\sqrt{-g}\xi(d-1)(\nabla^2\omega)$ $+\sqrt{-g}\xi(d-2)(d-1)(\nabla\omega)^2$
$-2\sqrt{-g}\Lambda$	$\to$	$-2 f^d \sqrt{-g}\Lambda$

Table 1: Transformations under conformal rescaling; $f = \tanh z,\ \omega = \log f$

Under this conformal map one has:

- The bulk action for a free, massive scalar field in dS$_d$

$$S = \int d^d x \sqrt{-g} \left(-(\partial_\mu X) - (M^2 + \xi R) X^2 \right)$$

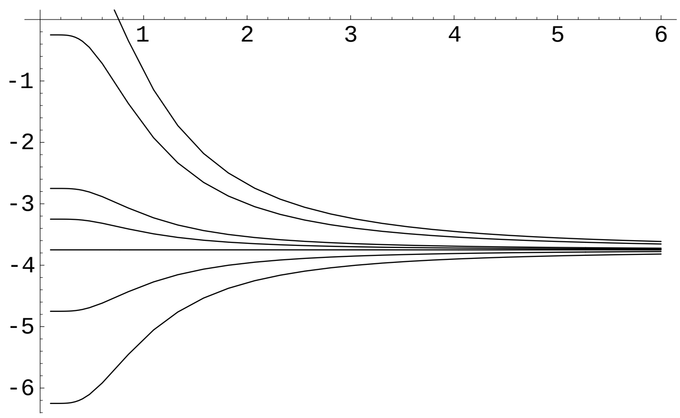
maps to scalar field in AdS with mass

$$M^2_{\rm total} = \tanh^2(z) \left(M^2 + \xi d(d-1) \right) - \frac{d(d-2)}{4} \left(1 + \tanh^2(z) \right)$$

where we used $R = -d(d-1)$ for the AdS$_d$.

In the UV $(z = 0)$ the original dS mass term M scales to zero, as does the original ξ term. Instead we get the universal result

$$M^2_{\rm total} = -\frac{d(d-2)}{4} \qquad \text{for all } \xi,\ M.$$



That is we get a conformally coupled scalar in AdS, independent of what values of the parameters M and ξ we started with in dS! The corresponding UV dimension of the dual operator is

$$\Delta_O = \begin{cases} \frac{d}{2} \\ \frac{d}{2} - 1 \end{cases}$$

This ensures that the $< OO >$ two point function for the second choice reduces to the usual $\frac{1}{|x|^{d-2}}$ behavior of a scalar field in d dimensions.

- Starting with the Einstein-Hilbert action in dS

$$S = M_d^{d-2} \int d^d x \sqrt{-g}(R - 2\Lambda)$$

where $\Lambda = \frac{1}{2}(d-1)(d-2)$, we get a new gravitational action in AdS

$$S = \int d^d x \sqrt{-g}(M_d f)^{d-2} \left(R + (d-2)(d-1)(\nabla\omega)^2 - 2f^2\Lambda\right)$$

Close to the boundary the graviton looks like a flat space graviton! The possible boundary behaviors are z^0, z^1, as opposed to z^0, z^4 in AdS. Still this couple to energy-momentum tensor with dimension $d-1$.

Conformal Anomaly

AdS/CFT instructs us to evaluate the bulk action on a given solution in order to calculate the boundary partition function for a given boundary metric. This quantity has divergences due to the infinite volume of AdS. One needs to add local counterterms.

However in even boundary dimensions (odd bulk dimension d) there are in addition $\log(z)$ terms and they represent the conformal anomaly.

For the standard Einstein Hilbert action on the dS$_{d-1}$ sliced AdS$_d$ background we get

$$S_{\mathrm{on-shell}} = \frac{-2(d-1)}{16\pi G_N} \int \frac{dz}{\sinh^d(z)}$$

Expanding in powers of z around $z = 0$ one finds

$$\begin{aligned}
\int \frac{dz}{\sinh^3(z)} &= -\frac{1}{2z^2} - \frac{\log(z)}{2} + \mathcal{O}(z^0) \\
\int \frac{dz}{\sinh^5(z)} &= -\frac{1}{4z^4} + \frac{5}{12z^2} + \frac{3}{8}\log(z) + \mathcal{O}(z^0) \\
\int \frac{dz}{\sinh^7(z)} &= -\frac{1}{6z^6} + \frac{7}{24z^2} - \frac{259}{720z^2} - \frac{5}{16}\log(z) + \mathcal{O}(z^0)
\end{aligned}$$

The $\log$ terms give the conformal anomaly evaluated on dS$_2$, dS$_4$ and dS$_6$ respectively.

Now let us repeat the same exercise for the gravitational action with position dependent G_N. The position dependence of M_d is $M_d(z) = \tanh(z) M_d$ which gives an extra factor of $\tanh^{d-2}(z)$. Up to terms that remain finite as $z \to 0$

$$
\begin{aligned}
S_{\text{on-shell}} &= \int \tanh^{d-2}(z) \frac{dz}{\sinh^d(z)} \\
&= \int \frac{1}{\sinh^2(z)} \frac{dz}{\cosh^d(z)} = \frac{-1}{z} + \mathcal{O}(z)
\end{aligned}
$$

For all d the only divergent term is a universal $-\frac{1}{z}$ and there are no logarithms. The conformal anomaly vanishes.

One possible interpretation:

- Lower dimensional gravity screens the central charge to be zero, just like is well known from 2d gravity on string theory worldsheets. In this scenario one does not even need a conformal field theory beyond scales $1/R$ since the gravitational dressing will also make any FT a CFT.

- In the same spirit the universal UV dimension of the scalar fields can be understood as gravitational dressing.

• Naively one would think that gravitational dressing should bring the operator dimension to $d-1$ so that one can add it to the action. But the $(d-1)/2 \pm 1/2$ we find is consistent as long as we only add products of the form $O^1_{d/2} O^2_{d/2-1}$ to the action. We know that the coupling of the two CFTs has to be achieved via its boundary interactions.

• Continuity across the UV brane in the original dS space means that the value of the field at the boundary in one AdS (dual to CFT 1) appears as a boundary condition in the second AdS (dual to CFT 2).

• The discussion of multi-trace operators uses precisely the product operator $\mathcal{O} = O^1_{d/2} O^2_{d/2-1}$ in order to achieve boundary conditions of the type we want, at least in a folded version of our duality: instead of one scalar field living in 2 copies of AdS there one has 2 decoupled scalar fields in one copy of AdS.

• Since we are dealing with gravity in addition to scalar fields, for us the 2 copies of AdS are more appropriate in order to avoid having 2 gravitons living in the same space.

Some comments on dS/dS correspondence

- The de Sitter space is a particularly symmetric example of a Randall-Sundrum system. Bulk gravity calculations allow one to probe some of the basic physics of induced gravity.

- We computed several quantities determining how the lower dimensional theory behaves at energies above the cutoff. This resulted in a holographic verification that the total central charge and heat capacity is zeroed out and that a simple asymptotic dressing of operator dimensions arises. These are both features familiar in 2d gravity plus matter systems. Direct couplings between the two CFTs are also required.

- The duality naturally can also be extended to situations with changing cosmological constant, such as inflation.

- Repeated application of our duality allows one to go to sufficiently low dimensions, that is two or one, so that gravity becomes non-dynamical and its effects reduce to constraints.

- The motivation for this dual formulation is ultimately to provide a framework for the physics of accelerated expansion in the real universe. Although at large radius and low energies the effective weakly coupled description remains the bulk d dimensional one, the description in terms of $d-1$ dimensional physics may shed light on the physics of inflation and dark energy.

This talk is based on the following papers

1. M. Spradlin, A. Strominger and A. Volovich, hep-th/0110007.

2. Y. Kim, C.Y. Oh and N. Park, hep-th/0212326.

3. D. Klemm and L. Vanzo, hep-th/0407255.

4. J. D. Brown and W. York, Phys. Rev. D47 (1993) 1407.

5. V. Balasubramanina, J. de Boer and D. Minic, hep-th/0110108.

6. M. Alishahiha, A. Karch, E. Silverestein and D. Tong, hep-th/0407125.

7. M. Alishahiha, A. Karch and E. Silverestein. hep-th/0504056.