Spinning strings and integrable spin chains in the AdS/CFT correspondence

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Plan

1. Introduction and the generalized BMN limit

2. Spinning string solutions on $\mathbb{R} \times S^3$

   - Point particle
   - Folded and closed String

3. Dual Gauge Theory

   - Dilatation operator at one-loop and Heisenberg integrable spin chain
   - Coordinate Bethe Ansatz
   - Thermodynamic limit of Bethe equations
   - Higher-loops and three loop discrepancies
   - Non-planar sector and string interactions

4. Outlook

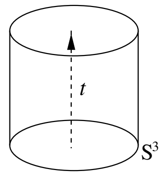
Introduction

AdS/CFT correspondence

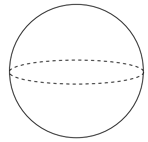
- Superstring in $AdS_5 \times S^5$ background: $X^m = X^m(\tau,\sigma)$, $Y^m = Y^m(\tau,\sigma)$

$$I = \sqrt{\lambda} \int d\tau\, d\sigma \left[G_{mn}^{(\mathrm{AdS}_5)}\, \partial_a X^m \partial^a X^n + G_{mn}^{(\mathrm{S}_5)}\, \partial_a Y^m \partial^a Y^n + \text{fermions} \right]$$

$$ds^2_{\mathrm{AdS}_5} = d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, d\Omega_3$$



$$ds^2_{\mathrm{S}_5} = d\gamma^2 + \cos^2\gamma\, d\phi_3^2 + \sin^2\gamma\, (d\psi^2 + \cos^2\psi\, d\phi_1^2 + \sin^2\psi\, d\phi_2^2)$$



- **Quantization unsolved!**

- $\sqrt{\lambda} = \frac{R^2}{\alpha'}$ R: Radius of the $AdS_5 \times S^5$ background

  Classical limit: $\sqrt{\lambda} \to \infty$, Quantum fluctuations ("σ-model loops"): $\mathcal{O}(1/\sqrt{\lambda})$
  $\Rightarrow$ yields free string theory ($g_S = 0$)

- Plus: String genus expansion in string coupling constant $g_S \ll 1$

$\mathcal{N} = 4$ Super Yang-Mills

- Field content: Gluons A_μ, 6 scalars Φ_i, 4 gluinos ψ^A_α:

$$S = \frac{N}{\lambda} \int d^4x \, \mathrm{Tr} \left[\tfrac{1}{4} F^2_{\mu\nu} + \tfrac{1}{2} (D_\mu \Phi_i)^2 - \tfrac{1}{4} [\Phi_i, \Phi_j][\Phi_i, \Phi_j] + \text{ferm.} \right]$$

(all fields in adjoint rep. $\rightarrow N \times N$ matrices)

- Access: Perturbation theory in λ & $1/N^2$, relation to string coupling: $g_S = \frac{\lambda}{N}$

  Free strings ($g_S = 0$): Planar Yang-Mills ($1/N \rightarrow 0$, λ free parameter):

  planar gauge theory: $\lambda \ll 1$ tree-level strings: $\sqrt{\lambda} \gg 1$

- $\mathcal{N} = 4$ Super Yang-Mills is quantum conformal theory, g_{YM} is not renormalized!

- **AdS/CFT conjecture**: Both theories are **equivalent**!

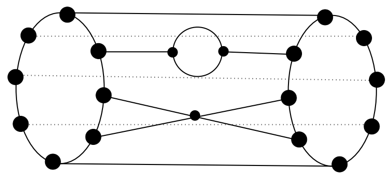
String spectrum E = Scaling dimensions Δ

- $AdS_5 \times S^5$ string spectrum

$$\widehat{H}\,|\psi\rangle_{\rm String} = E(\lambda)\,|\psi\rangle_{\rm String} \qquad E(\lambda) = ?$$

- Central observables in gauge theory: Correlation functions of composite operators
  e.g. $\mathcal{O}_\alpha(x) = \mathrm{Tr}[\,\Phi_{i_1}\,\Phi_{i_2}\ldots\Phi_{i_n}\,]$
  Two-point functions determined by scaling dimensions $\Delta(\lambda, N)$

$$\langle \mathcal{O}_\alpha(x)\,\mathcal{O}_\beta(y)\rangle = \frac{\delta_{\alpha\beta}}{(x-y)^{2\,\Delta(\lambda,N)}}$$



May be computed perturbatively in gauge theory: Loops (λ) + genera $(1/N^2)$

$$\boxed{\Delta = \Delta^0 + \lambda(\Delta_0^1 + \tfrac{1}{N^2}\Delta_1^1 + \ldots) + \lambda^2(\Delta_0^2 + \tfrac{1}{N^2}\Delta_1^2 + \ldots) + \ldots \quad \stackrel{!}{=} E(\lambda, g_s)}$$

↑ here: $\Delta^0 = n = \#$ of scalars

Gauge theory scaling dimensions $=$ energies of string states

$$\Delta = \Delta^0 + \lambda(\Delta_0^1 + \tfrac{1}{N^2}\Delta_1^1 + \ldots) + \lambda^2(\Delta_0^2 + \tfrac{1}{N^2}\Delta_1^2 + \ldots) + \ldots \stackrel{!}{=} E(\lambda, g_S)$$

- In this talk:

  ⋆ **String side:**

  Semiclassical methods to determine $E(\lambda)$ in free ($g_S = 0$) string theory $\Rightarrow$ prediction for Δ_0^n.

  ⋆ **Gauge Theory:**

  Determination of Δ (planar & non-planar) in perturbative gauge theory $\Rightarrow$ predictions for interacting ($g_S \neq 0$) strings (Δ_h^1).

Symmetry structure:

String: Isometry group: AdS_5 S^5

$$\boxed{\underset{\text{Conformal group}}{\overset{AdS_5}{SO(2,4)}} \times \underset{R\text{-symmetry group}}{\overset{S^5}{SO(6)}}}$$

Gauge theory: Conformal group R-symmetry group

$\Rightarrow$ String states $|\psi\rangle_{\text{String}}$ are classified by values of corresponding Cartan generators ("quantum numbers" or charges):

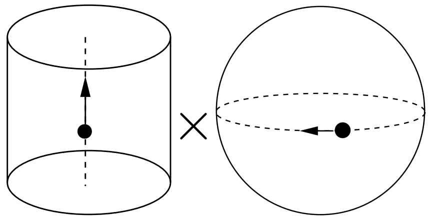
$$[\underbrace{E\,;\,S_1\,,\,S_2}_{AdS_5}\,;\,\underbrace{J_1\,,\,J_2\,,\,J_3}_{S^5}]$$

Dual gauge theory operators carry identical charges $\Rightarrow$ enables identification.

In this talk: Excitations of $\boxed{E, J_1, J_2}$. $\Leftrightarrow$ strings moving on $\mathbb{R} \times S^3$

Solution 1: Rotating point particle on S^5

$$t = \kappa\,\tau \quad \rho = 0 \quad \gamma = \frac{\pi}{2} \quad \phi_1 = \kappa\,\tau \quad \phi_2 = \phi_3 = \psi = 0$$



Solves eqs. of motion & Virasoro constraint (here $S_1, S_2, J_2, J_3 = 0$)

$$E = \sqrt{\lambda} \int_0^{2\pi} \tfrac{d\sigma}{2\pi}\, \dot{X}_0 = \sqrt{\lambda}\,\kappa \qquad \boxed{E = J} \quad \text{classical}$$

$$J_1 = \sqrt{\lambda} \int_0^{2\pi} \tfrac{d\sigma}{2\pi}\, (Y_1\,\dot{Y}_2 - Y_2\,\dot{Y}_1) = \sqrt{\lambda}\,\kappa =: J$$

Quantum fluctuations around solution: $\boxed{X^\mu = X^\mu_{\text{sol}}(\tau) + \tfrac{1}{\lambda^{1/4}}\, x^\mu(\tau, \sigma)}$

$\Rightarrow$ Energy: $\quad E = \sqrt{\lambda}\,\kappa + E_2(\kappa) + \dfrac{1}{\sqrt{\lambda}}\, E_4(\kappa) + \ldots$

- **Limit of large quantum number:** $J \to \infty$ with $\kappa = J/\sqrt{\lambda}$ fixed. [Berenstein Maldacena Nastase]

$$E = J + E_2(\kappa) + \frac{1}{\sqrt{J}}\tilde{E}_4(\kappa) + \ldots$$

$\Rightarrow$ For $E - J$: One-loop contribution E_2 exakt!

- Quadratic fluctuations yield the action (in light-cone-gauge) $(i = 1, \ldots, 8)$

$$I_2 = \int d\tau d\sigma \left(\tfrac{1}{2}\,\partial_a x^i\,\partial^a x^i - \tfrac{\kappa^2}{2}\,x^i\,x^i + \text{fermions}\right)$$

Free, massive 2d theory $\Rightarrow$ easily quantized $\Leftrightarrow$ plane-wave geometry [Metsaev]

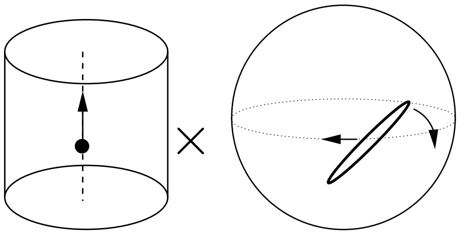
- **Spectrum:** $$\mathcal{H}_2 = \frac{1}{\sqrt{\lambda'}} \sum_{n=-\infty}^{\infty} \sqrt{1 + \lambda'\,n^2}\,\alpha_n^{i\,\dagger}\,\alpha_n^i$$

All loop prediction for gauge theory in the **BMN limit** $(N, J \to \infty$ with N/J^2 fixed$)$!

$$\lambda' := \frac{1}{\kappa^2} = \frac{\lambda}{J^2} \qquad\qquad [\alpha_m^i, \alpha_n^{j\,\dagger}] = \delta_{nm}\,\delta^{ij}$$

Spinning strings

2. Solution: Spinning, folded string



Ansatz:

$$t = \kappa\,\tau \quad \rho = 0 \quad \gamma = \frac{\pi}{2}$$

$$\phi_1 = \omega_1\,\tau \quad \phi_2 = \omega_2\,\tau \quad \phi_3 = 0 \quad \psi = \psi(\sigma)$$

- Charges $\boxed{J_1 \;\&\; J_2 \neq 0}$. Ansatz leads to string action [Frolov, Tseytlin]

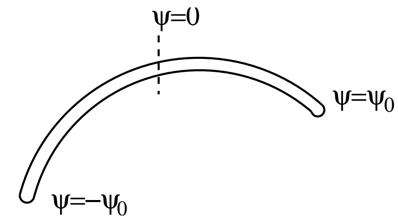
$$I = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau \int_0^{2\pi} d\sigma\left[\kappa^2 + \psi'^2 - \cos^2\psi\,{\omega_1}^2 - \sin^2\psi\,{\omega_2}^2\right]$$

With equation of motion

$$\psi'' + \sin\psi\,\cos\psi\,({\omega_2}^2 - {\omega_1}^2) = 0$$

Define ${\omega_{21}}^2 := {\omega_2}^2 - {\omega_1}^2 > 0$, yields "string pendulum" eq.

$$\frac{d\psi}{d\sigma} = \omega_{21}\sqrt{q - \sin^2\psi}$$



$q \leq 1$: Folded string $\qquad$ $q > 1$ Closed string (ψ' never vanishes)

- Virasoro constraints:

$$0 \stackrel{!}{=} \dot{X}^m X'_m + \dot{Y}^p Y'_p = 0 \quad \checkmark$$

$$0 \stackrel{!}{=} \dot{X}^m \dot{X}_m + \dot{Y}^p \dot{Y}_p + X^{m\prime} X'_m + Y^{p\prime} Y'_p \quad \Rightarrow \quad q = \frac{\kappa^2 - {\omega_1}^2}{{\omega_{21}}^2}$$

assuming ${\omega_{21}}^2 \neq 0$.

Classical energy and angular momentum

Generally

$$E = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \dot{X}^0 = \sqrt{\lambda}\,\kappa \qquad \mathcal{J}_{pq} = \sqrt{\lambda}\,, \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(X_p \dot{X}_q - X_q \dot{X}_p \right)$$

Here only $J_1 := \mathcal{J}_{12}$ and $J_2 := \mathcal{J}_{34}$ are non vanishing and become

$$J_1 = \sqrt{\lambda}\,\omega_1 \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2\psi(\sigma) \qquad J_2 = \sqrt{\lambda}\,\omega_2 \int_0^{2\pi} \frac{d\sigma}{2\pi} \sin^2\psi(\sigma)\,.$$

From this we learn that $\boxed{\sqrt{\lambda} = \frac{J_1}{\omega_1} + \frac{J_2}{\omega_2}}$

Recall Virasoro constr. $\boxed{q = \frac{\kappa^2 - {\omega_1}^2}{{\omega_{21}}^2}}$ Goal: Find $E = E(J_1, J_2)$.

- Use $d\sigma = \frac{d\psi}{\omega_{21}\sqrt{q-\sin^2\psi}}$ in definition of J_1:

$$J_1 = \frac{\sqrt{\lambda}\,\omega_1}{2\pi}\,4\int_0^{\psi_0} d\psi\,\frac{\cos^2\psi}{\omega_{21}\,\sqrt{q-\sin^2\psi}} = \frac{2\sqrt{\lambda}\,\omega_1}{\pi\,\omega_{21}}\,E(q)\,,$$

- Similarly

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^{\psi_0}\frac{d\psi}{\omega_{21}\,\sqrt{q-\sin^2\psi}} = \frac{4}{\omega_{21}}\,K(q)\,.$$

With elliptic integrals $E(x) := \int_0^{\pi/2} d\psi\,\sqrt{1-x\,\sin^2\psi}$ and $K(x) := \int_0^{\pi/2} d\psi\,\frac{1}{\sqrt{1-x\,\sin^2\psi}}$

- $\Rightarrow$ Eliminate parameters of solution $\kappa, \omega_1, \omega_2$ in favor of $E, J_1, J_2; q$

Folded string equations: $q \leq 1$

$$\frac{4\,q\,\lambda}{\pi^2} = \frac{E^2}{K(q)^2} - \frac{{J_1}^2}{E(q)^2}\,, \qquad \frac{4\,\lambda}{\pi^2} = \frac{{J_2}^2}{(K(q)-E(q))^2} - \frac{{J_1}^2}{E(q)^2}$$

Assume analytic behaviour in BMN type limit $J := J_1 + J_2 \to \infty$ with λ/J^2 fixed:

$$q = q_0 + \frac{\lambda}{J^2}\,q_1 + \frac{\lambda^2}{J^4}\,q_2 + \ldots \qquad E = J\left(E_0 + \frac{\lambda}{J^2}\,E_1 + \frac{\lambda^2}{J^4}\,E_2 + \ldots\right).$$

Solve iteratively: $\boxed{E_0 = 1}$ $\Leftrightarrow$ consistent with gauge theory.

q_0 implicitly given by $\boxed{\frac{J_2}{J} = 1 - \frac{E(q_0)}{K(q_0)}\,.}$ and

$$\boxed{E_1 = \frac{2}{\pi^2}\,K(q_0)\Big(E(q_0) - (1-q_0)\,K(q_0)\Big)}$$

Similarly E_l: l-loop gauge theory prediction.

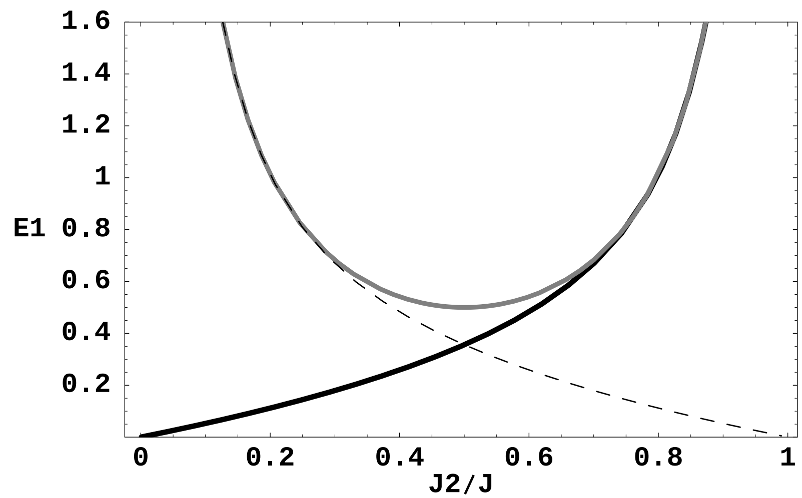
Circular string equations: $q > 1$

- In complete analogy finds $\dfrac{J_2}{J} = q_0 \left(1 - \dfrac{E(q_0^{-1})}{K(q_0^{-1})}\right)$

  With first two energy terms in the λ/J^2 expansion

$$E_0 = 1\,, \qquad E_1 = \frac{2}{\pi^2}\, E(q_0^{-1})\, K(q_0^{-1})\,,$$

- Folded and circular string one-loop energies: [Frolov, Tseytlin]



Further developments

Classical $AdS_5 \times S^5$ string is integrable system

- More general solutions with $(S_1, S_2, J_1, J_2, J_3)$ through ansatz reducing to Neumann integrable model [Arutyunov,Russo,Frolov,Tseytlin]

- Construction of underlying algebraic curve parametrizing the solutions

  ⋆ $\mathbb{R} \times S^3$ [Kazakov,Marshakov,Minahan,Zarembo]
  ⋆ $\mathbb{R} \times S^5$ [Beisert,Kazakov,Sakai]
  ⋆ $AdS_5 \times S^5$ [Beisert,Kazakov,Sakai,Zarembo]

  Leads to integral eqs. of Bethe type $\Leftrightarrow$ direct comparison to thernodynamic limit of gauge theory Bethe eqs. (to be discussed)

- Conjecture for quantum string Bethe eqs:

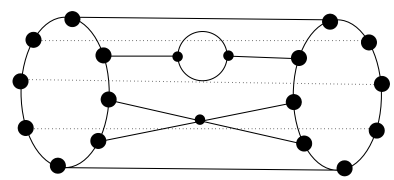
  ⋆ $\mathbb{R} \times S^3$ [Arutyunov,Frolov,Staudacher] and recently $AdS_5 \times S^5$ [Beisert,Staudacher]

The dual gauge theory story

The dual gauge theory: Computation of Δ

- Scaling dimensions $\Delta(\lambda, N)$ from 2-pt function:

$$\langle \mathcal{O}_\alpha(x)\, \mathcal{O}_\beta(y) \rangle = \frac{\delta_{\alpha\beta}}{(x-y)^{2\,\Delta(\lambda,N)}}$$



- Consider pure scalar operators made of

$$Z = \Phi_1 + i\,\Phi_2 \quad \Leftrightarrow \quad J_1 = 1 \quad J_2 = 0 \qquad \text{“}SU(2)\text{ sector”}$$

$$W = \Phi_3 + i\,\Phi_4 \quad \Leftrightarrow \quad J_1 = 0 \quad J_2 = 1$$

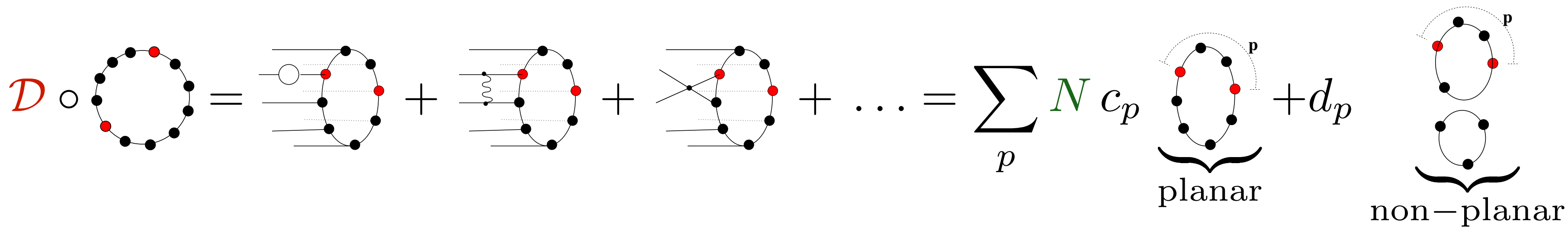
Operators: $\mathcal{O}_\alpha = \mathrm{Tr}(\text{word in } Z\ \&\ W)$ and products thereof

- Huge mixing problem for large J_1 and J_2: All $\mathrm{Tr}(Z^{J_1}\,W^{J_2}) + \text{perms}$ are degenerate.

The dilatation operator $\mathcal{D}$

- Introduction of the **dilatation operator** $\mathcal{D}$: $\boxed{\mathcal{D} \circ \mathcal{O}_\alpha = \Delta_\alpha \, \mathcal{O}_\alpha}$

  [Beisert,Kristjansen,Plefka,Staudacher]



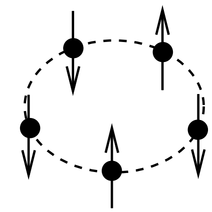
- Acts on operators at origin, perturbatively defined: $\mathcal{D} = J_1 + J_2 + \sum_{l=1}^{\infty} \lambda^l \, \mathcal{D}^{(l)}$

- Explicit form at one-loop: $\boxed{\mathcal{D}^{(1)} = -\frac{2}{N} \, \mathrm{Tr}\Big([Z, W] \, [\check{Z}, \check{W}] \Big)}$ $\qquad \check{Z}_{ij} := \frac{d}{dZ_{ji}}$

$\mathcal{D}^{(1)} \circ \mathrm{Tr}(Z \, A \, W \, B) = -\mathrm{Tr}(A) \, \mathrm{Tr}\Big([Z, W] \, B \Big) + (A \Leftrightarrow B)$ $\quad \Rightarrow$ Enhanced for $A = 1$

- Planar contribution: Nearest neighbor interactions

$$\mathcal{D}^{(1)}_{\text{planar}} \circ \mathrm{Tr}(\ldots Z\,W \ldots) = 2\Big(\mathrm{Tr}(\ldots Z\,W \ldots) - \mathrm{Tr}(\ldots W\,Z \ldots)\Big)$$

- Spin chain picture: $\mathrm{Tr}(ZZWZW) \mathrel{\hat{=}} |{\downarrow\downarrow\uparrow\downarrow\uparrow}\rangle \quad \mathrel{\hat{=}}$ 

$$\boxed{\mathcal{D}^{(1)}_{\text{planar}} = 2\sum_{l=1}^{L}(1 - P_{l,l+1}) = \sum_{l=1}^{L}(-\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} + 1)}$$

Is spin=1/2 Heisenberg chain! [Minahan,Zarembo]

- Ground state: $|{\downarrow\downarrow \ldots \downarrow}\rangle \mathrel{\hat{=}} \mathrm{Tr}(Z^J)$ with $\Delta = 0 \Leftrightarrow$ rotating point particle sol.

- String excitations $\alpha^\dagger_n$: "magnons": $|m\rangle = |\underbrace{\uparrow \downarrow \ldots \downarrow \uparrow}_{m} \downarrow\rangle\rangle \mathrel{\hat{=}} \mathrm{Tr}(WZ^mWZ^{J-m})$

Integrability

- Heisenberg spin chain is integrable: Existence of L commuting charges Q_n: $\boxed{[Q_m, Q_n] = 0}$ $\forall (m,n)$!

$$Q_2 = \sum_{l=1}^{L} \vec{\sigma}_l \cdot \vec{\sigma}_{l+1} = \mathcal{D}^{(1)}_{\text{planar}}, \quad Q_3 = \sum_{l=1}^{L} (\vec{\sigma}_l \times \vec{\sigma}_{l+1}) \cdot \vec{\sigma}_{l+2}, \quad \ldots$$

- Spectrum follows from **Bethe equations**!

  ⋆ Integrability proven up to 3 loop order in $\mathcal{N} = 4$ SYM [Beisert; Eden, Sokatchev, Stanev]

  ⋆ In dim. reduced toy model up to 4 loops [Fischbacher,Klose,Plefka]

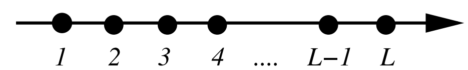
- Fascinating possibility: Does integrability and BMN scaling (emergence of $\lambda' = \frac{\lambda}{J^2}$ eff. coupling) fix the dilatation operator $\mathcal{D}$ to all loop orders?

  ⇒ Exact spectrum of planar $\mathcal{N} = 4$ Super Yang-Mills theory!

The coordinate Bethe-Ansatz

- How to diagonalize $\widehat{\mathcal{D}}$? Open up the trace (no cyclicity)

$$\mathrm{Tr}(WZZW\ldots WZ) \longrightarrow |WZZW\ldots WZ\rangle$$



- Consider two-magnon states $|\psi\rangle = \sum_{1\leq x_1<x_2\leq L} \psi(x_1,x_2)\,|\ldots \underset{x\,\hat{=}\,x_1}{Z\,W\,Z}\ldots \underset{x\,\hat{=}\,x_2}{Z\,W\,Z}\ldots\rangle$

- One-loop Schrödinger eq. $\boxed{\sum_{i=1}^{L}(1-P_{i,i+1})\,|\psi\rangle = E_2\,|\psi\rangle}$ in "position space":

$x_2 > x_1 + 1$ $\qquad E_2\,\psi(x_1,x_2) = 2\,\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1+1,x_2)$

$\ldots ZWZ\ldots ZWZ\ldots$ $\qquad 2\,\psi(x_1,x_2) - \psi(x_1,x_2-1) - \psi(x_1,x_2+1)$

$x_2 = x_1 + 1$ $\qquad E_2\,\psi(x_1,x_2) = 2\,\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1,x_2+1)$

$\ldots ZWWZ\ldots$

$$x_2 > x_1 + 1 \qquad E_2\, \psi(x_1, x_2) = 2\,\psi(x_1, x_2) - \psi(x_1 - 1, x_2) - \psi(x_1 + 1, x_2)$$
$$2\,\psi(x_1, x_2) - \psi(x_1, x_2 - 1) - \psi(x_1, x_2 + 1) \quad (1)$$

$$x_2 = x_1 + 1 \qquad E_2\, \psi(x_1, x_2) = 2\,\psi(x_1, x_2) - \psi(x_1 - 1, x_2) - \psi(x_1, x_2 + 1) \quad (2)$$

- Solved by Bethe's ansatz (1931):

$\downarrow$ S-matrix

$$\boxed{\psi(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S(p_2, p_1)\, e^{i(p_2 x_1 + p_1 x_2)}}$$

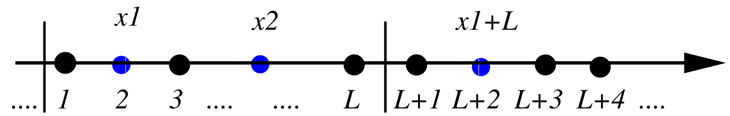
- Then (1) is solved for any $S(p_2, p_1)$ with $\boxed{E_2 = \sum_{k=1}^{M} 4\, \sin^2(\frac{p_k}{2})}$

N.B. $2 - e^{-ip} - e^{ip} = 4 \sin^2 \frac{p}{2}$

- (2) determines S-matrix: $\boxed{S(p_2, p_1) = \frac{\varphi(p_1) - \varphi(p_2) + i}{\varphi(p_1) - \varphi(p_2) - i}}$ with $\varphi(p) = \frac{1}{2} \cot(\frac{p}{2})$

Bethe-equations: Follow from periodicity

- Demand $\psi(x_1, x_2) = \psi(x_2, x_1 + L)$



$$\Rightarrow \quad \boxed{e^{ip_1 L} = S(p_1, p_2) \quad \text{and} \quad e^{ip_2 L} = S(p_2, p_1)}$$

solve for p_1 & p_2 $\Rightarrow$ $E_2(p_1, p_2) = \sum_{k=1}^{2} 4 \sin^2 \frac{p_k}{2}$ spectrum!

- Big leap ($\Leftrightarrow$ factorized scattering from integrability): M-**body problem**
  Total phase acquired by one magnon cycling around the chain:

$$\boxed{e^{ip_k L} = \prod_{i=1, i \neq k}^{M} S(p_k, p_i)} \qquad k = 1, \ldots, M$$

Scatters off all other magnons exactly once

- Energy additive:

$$E_2(p_1,\dots,p_M) = \sum_{k=1}^{M} 4\sin^2\tfrac{p_k}{2}$$

- Cyclicity of trace condition: $\sum_{k=1}^{M} p_k = 0 \Leftrightarrow$ vanishing total momentum

- Example: Two magnons: $p := p_1 = -p_2$

$$\begin{aligned} e^{ipL} &= \frac{\cot\frac{p}{2}+i}{\cot\frac{p}{2}-i} = e^{ip} \quad \Rightarrow \quad e^{ip(L-1)} = 1 \quad \Rightarrow \quad p = \frac{2\pi\, n}{L-1} \\ E_2 &= 8\sin^2\left(\frac{\pi\, n}{L-1}\right) \quad \overset{L\to\infty}{\longrightarrow} \quad 8\pi^2\,\frac{n^2}{L^2}\,, \end{aligned}$$

Recall $\Delta_1 = \frac{\lambda}{8\pi^2}\,E_2 \to n^2\,\lambda/L^2$
Agrees with plane-wave string spectrum $E_{\mathrm{light-cone}} = 2\sqrt{1+n^2\,\lambda/J^2}$

Thermodynamic limit of the spin chain

- To make contact with spinning strings: Take M (number of magnons) and L (length of chain) $\to \infty$

- Reformulate Bethe eqs. via roots $u_k = \frac{1}{2} \cot \frac{p_k}{2}$:

$$\left(\frac{u_i + i/2}{u_i - i/2}\right)^L = \prod_{k \neq i}^{M} \frac{u_i - u_k + i}{u_i - u_k - i}, \qquad \prod_{i=1}^{M} \frac{u_i + i/2}{u_i - i/2} = 1\,.$$

↑Total momentum $= 0$

The energy then is $\quad Q_2 = \sum_{i=1}^{M} \frac{1}{{u_i}^2 + \frac{1}{4}}\,.$

- Satisfy total momentum constraint by choice $(u_i, -u_i, u_i^*, -u_i^*)$.

- Take $\log$ of Bethe eqs.: $(n_i \in \mathbb{Z})$

$$L\,\ln\left(\frac{u_i+i/2}{u_i-i/2}\right)=\sum_{k=1\,(k\neq i)}^{M}\ln\left(\frac{u_i-u_k+i}{u_i-u_k-i}\right)-2\,\pi\,i\,n_i\,,$$

In thermodynamic limit: $L\to\infty$ then $p_i\sim 1/L$ hence Bethe roots $u_i\sim L$

$$\frac{1}{u_i}=2\pi\,n_j+\frac{2}{L}\sum_{k=1\,(k\neq i)}^{M}\frac{1}{u_j-u_k}\,.$$

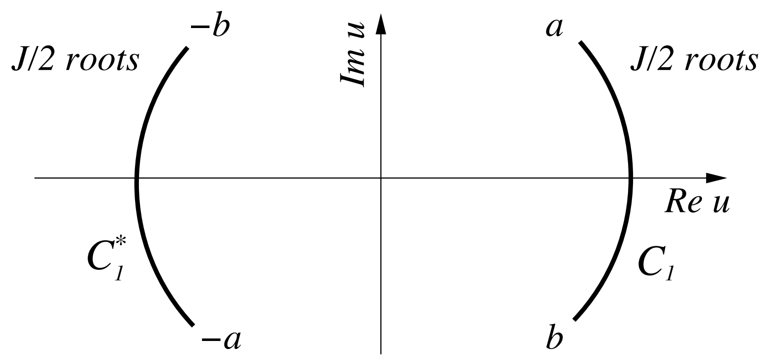
- For $M\to\infty$: Introduce Bethe root density: $\rho(u):=\frac{1}{M}\sum_{j=1}^{M}\delta(u-u_j)$

$$\fint_C dv\,\frac{\rho(v)\,u}{v-u}=-\frac{1}{2\,\alpha}+\frac{\pi\,n_{C(u)}\,u}{\alpha}\qquad\text{where}\quad u\in C\quad\text{and}\quad \alpha:=\frac{M}{L}\,.$$

Has to solve this singular integral eq. for $\rho(u)$ Energy: $Q_2=M\,\int_C\frac{\rho(u)}{u^2}$

Bethe strings

- Roots u condense on smooth contours "Bethe strings". Root distribution dual to folded spinning string: [Beisert,Minahan,Staudacher,Zarembo]



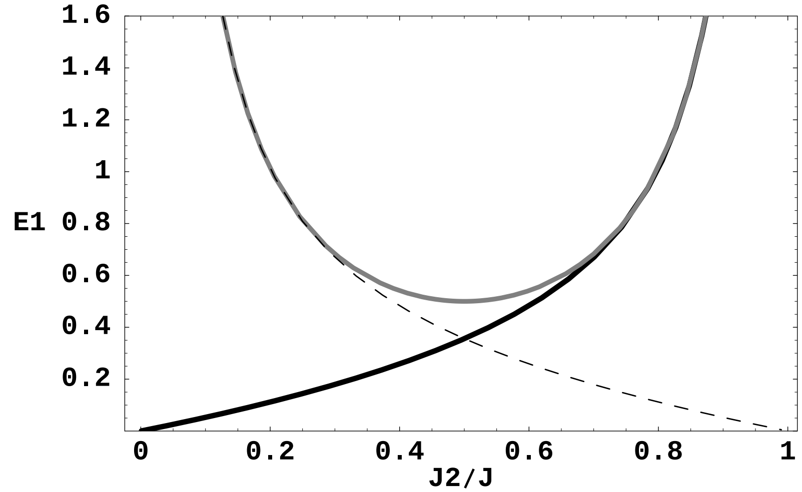
- Upshot:

$$E_2 = \frac{1}{2\pi^2} K(q) \Big[2\, E(q) - (2-q)\, K(q) \Big] \quad \text{with} \quad \frac{J_2}{J} = \frac{1}{2} - \frac{1}{2\sqrt{1-q}} \frac{E(q)}{K(q)}$$

Agrees with string result! Needs to relate $q_0^{\text{string}} = -\frac{(1-\sqrt{1-q})^2}{4\sqrt{1-q}}$

- Bethe integral eqs. can be similarly solved for $\rho(u)$ dual to circular spinning string and agrees. [Beisert,Frolov,Staudacher,Tseytlin]

- Complete functions $E_2(J_2/J)$ matched!



- Same is true for all higher charges $q_k(J_2/J)$ of spin chain and string. [Arutyunov,Staudacher]

Higher loops: Is integrability stable?

- Recall: $\mathcal{D} = J_1 + J_2 + \sum_{l=1}^{\infty} \lambda^l \, \mathcal{D}^{(l)}$

$$\widehat{\mathcal{D}}_{1-\text{loop}} = \sum_{i=1}^{L} (1 - P_{i,i+1}) = \sum_{i=1}^{L} 1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}$$

Heisenberg $\text{XXX}_{1/2}$ model!

$$\widehat{\mathcal{D}}_{2-\text{loop}} = \sum_{i=1}^{L} -\vec{\sigma}_l \cdot \vec{\sigma}_{l+2} + 4\,\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} - 3 \cdot 1$$

[Beisert,Kristjansen,Staudacher]

$$\widehat{\mathcal{D}}_{3-\text{loop}} = \sum_{i=1}^{L} -\vec{\sigma}_l \cdot \vec{\sigma}_{l+3} + (\vec{\sigma}_l \cdot \vec{\sigma}_{l+2})\,(\vec{\sigma}_{l+1} \cdot \vec{\sigma}_{l+3}) - (\vec{\sigma}_l \cdot \vec{\sigma}_{l+3})\,(\vec{\sigma}_{l+1} \cdot \vec{\sigma}_{l+2})$$
$$+ 10\,\vec{\sigma}_l \cdot \vec{\sigma}_{l+2} - 29\,\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} + 20 \cdot 1$$

[Beisert]

k-loop piece: $(k+1)$ nearest neighbor interactions!

- Integrable structure is stable, i.e. higher loop deformations of Q_n may be constructed.

- Are facing an **integrable long range spin chain** of novel type, structure of infinite tower of conserved higher charges

$$Q_2 = \sum_{i=1}^{L} \Big[1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1} + \lambda \left(-\vec{\sigma}_l \cdot \vec{\sigma}_{l+2} + 4\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} - 3 \cdot 1 \right) + \lambda^2 \, \widehat{\mathcal{D}}_{3-\mathrm{loop}} + \ldots \Big]$$

$$Q_3 = \sum_{i=1}^{L} \Big[\left(\vec{\sigma}_i \times \vec{\sigma}_{i+1} \right) \cdot \vec{\sigma}_{i+2} + \lambda \left(\text{parity odd range 4 interaction} \right) + \ldots \Big]$$

$$Q_4 = \sum_{i=1}^{L} \Big[\left(\text{parity even range 4 interaction} \right) + \lambda \left(\text{parity even range 5 interaction} \right) \ldots \Big]$$

with $\boxed{[Q_k, Q_l] = 0 \quad \forall k.}$ $\quad Q_k = \sum_{l=0}^{\infty} \lambda^l \, Q_k^l \sim$ (parity $(-1)^k$ range k+l interaction)

⋆ diagonalized through asymptotic Bethe-Ansatz $\Rightarrow$ to be discussed

⋆ Underlying “algebraic” structure, like Yang-Baxter equation, unknown

- Of novel type, not encountered in statistical mechanics literatur yet (??):

$$\mathcal{H} = \sum_{i<j}^{L} h(|i-j|)\,\vec{\sigma}_i \cdot \vec{\sigma}_j \qquad h(s) = \begin{cases} \frac{1}{\sin^2(\frac{\pi\, s}{L})} & \text{Haldane-Shastry} \\ \frac{1}{\sinh^2(\kappa\, s)} & \text{Inozemtsev } (L \to \infty) \\ \mathcal{P}_{L, i\pi/\kappa} & \text{Weierstrass p-fct}(L \text{ finite}) \end{cases}$$

Here also have $(\vec{\sigma}_i \cdot \vec{\sigma}_j)(\vec{\sigma}_k \cdot \vec{\sigma}_l)$ and higher type interactions!

Recall $\widehat{\mathcal{D}}_{3-\mathrm{loop}}$:

$$\widehat{\mathcal{D}}_{3-\mathrm{loop}} = \sum_{i=1}^{L} \Big[-\vec{\sigma}_l \cdot \vec{\sigma}_{l+3} + (\vec{\sigma}_l \cdot \vec{\sigma}_{l+2})\,(\vec{\sigma}_{l+1} \cdot \vec{\sigma}_{l+3}) - (\vec{\sigma}_l \cdot \vec{\sigma}_{l+3})\,(\vec{\sigma}_{l+1} \cdot \vec{\sigma}_{l+2})$$

$$+ 10\,\vec{\sigma}_l \cdot \vec{\sigma}_{l+2} - 29\,\vec{\sigma}_l \cdot \vec{\sigma}_{l+1} + 20 \cdot 1 \Big]$$

Higher Loop Bethe-ansatz

- **Long-range interactions:** Ansatz needs to be modified [Staudacher; Fischbacher,Klose,Plefka]

$$\psi(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)}\, f(x_2 - x_1, p_1, p_2) + S(p_2, p_1)\, e^{i(p_2 x_1 + p_1 x_2)}\, f(L - x_2 + x_1, p_1, p_2)$$

with "Fudge functions": $f(x, p_1, p_2) = 1 + \sum_{n=0}^{\infty} \lambda^{n+|x|}\, f_n(x, p_1, p_2) \xrightarrow{x \gg 1} 1$

- Yields perturbative S-matrix: $S(p_1, p_2) = S_0(p_1, p_2) + \sum_{n=1}^{\infty} \lambda^n\, S_n(p_1, p_2)$

and unmodified Bethe equations: $\boxed{e^{ip_k L} = \prod_{i=1, i\neq k}^{M} S(p_k, p_i)}$

- But charges are corrected $q_r(p) = q_{r,0} + \sum_{n=1}^{\infty} \lambda^r\, q_{r,n}(p)$

- Seems very likely that exact spectrum of $\mathcal{N} = 4$ SYM will be expressed in this language $\Rightarrow$ What is the all loop form of $S(p_1, p_2)$??

- Remarkable asymptotic all loop conjecture [Beisert,Dippel,Staudacher]

$$S(p_1, p_2) = \frac{\varphi(p_1) - \varphi(p_2) + i}{\varphi(p_1) - \varphi(p_2) - i} \qquad \varphi(p) = \tfrac{1}{2} \cot(\tfrac{p}{2}) \sqrt{1 + \lambda \sin^2(\tfrac{p}{2})}$$

$$q_2(p) = \frac{1}{\lambda}(\sqrt{1 + 8\,\lambda \sin^2(\tfrac{p}{2})} - 1)$$

(But explicit form of Hamiltonian unknown!)

- Reproduces all known gauge theory data:
  - ⋆ Dilop firmly known in $SU(2)$ sector to 3 loops
  - ⋆ Dilop up to 5 loops assuming integrability + BMN scaling

- Restriction: length of chain $>$ loop order (wrapping issue):
$\Rightarrow$ Prediction up to $l - 1$-loops for operator of length l

Discrepancies at three loops

- Fact: For folded, spinning string two-loop energies agree, **but**: [Serban,Staudacher]

$$E_3^{\text{string theory}}\left(\frac{J_1}{J_1+J_2}\right) \neq E_3^{\text{gauge theory}}\left(\frac{J_1}{J_1+J_2}\right)$$

same is true for quartic fluctuations about point particle solution [Callan,Schwarz,Swanson,et. al.]

- Resolution:

(a) AdS/CFT correspondence is wrong (not favored)

(b) BMN argument too "naive": Non-commuting order of limits problem: [Beisert,Dippel,Staudacher]

⋆ String theory: First $J \to \infty$ then expansion in λ'.
⋆ Gauge theory: First expansion in λ then $J \to \infty$.

- But then also perturbative BMN scaling (existence of the effective GT loop-counting parameter λ') not guaranteed!

  As a matter of fact we recently observed a breakdown of BMN scaling in a dim. reduced "toy model" (plane wave matrix theory) at 4-loops $E_{4\text{loop}} \sim \frac{\lambda^4}{J^7}$ [Fischbacher,Klose,Plefka]

Further developments

- Larger sectors of the gauge theory
  - ⋆ $SO(6)$ @ 1-loop [Minahan,Zarembo]
  - ⋆ $SU(2,2|4)$ @ 1-loop $\Rightarrow$ super spin chain [Beisert,Staudacher]
  - ⋆ $SU(2|3)$ to 3-loops [Beisert]

- Recently assymptotic all-loop conjecture generalized to full theory ($SU(2,2|4)$) [Beisert,Staudacher]

- Leigh-Strassler deformations of $\mathcal{N}=4$ SYM and corresponding Lunin-Maldacena background: All structures seem to lift.

The non-planar sector and string interactions

The non-planar sector ($g_S \neq 0$)

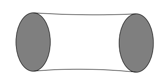
- Non-planar piece of $\mathcal{D}^{(1)} = \mathrm{Tr}[Z,W][\check{Z},\check{W}]$: Induces string splitting

$$\mathcal{D}^{(1)} \circ \mathrm{Tr}(Z\,A\,W\,B) = -\mathrm{Tr}(A)\,\mathrm{Tr}\Big([Z,W]\,B\Big) + (A \Leftrightarrow B)$$

- The full one-loop "Hamiltonian" for two-magnons: $\boxed{\mathcal{D}^{(1)} = \mathcal{D}^{(1)}_{\mathrm{planar}} + \mathcal{D}^{(1)}_{\mathrm{Int}}}$

  [Beisert,Kristjansen,Plefka,Staudacher]

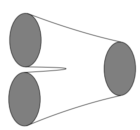
$$\mathcal{D}^{(1)}_{\mathrm{planar}}\,|x;1\rangle \;=\; -\lambda'\,\partial_x^2\,|x;1\rangle$$



"free string"

$$\mathcal{D}^{(1)}_{\mathrm{Int}}|x;1\rangle \;=\; \lambda'\,g_2\int_0^x dr\,\partial_x\,|x-r;1-r\rangle\otimes|r\rangle$$

"string splitting"



$$-\lambda'\,g_2\int_0^{1-x} dr\,\partial_x\,|x;1-r\rangle\otimes|r\rangle$$

Plane-wave **string field theory** from **gauge theory**! $\quad g_2 := \frac{J^2}{N}$

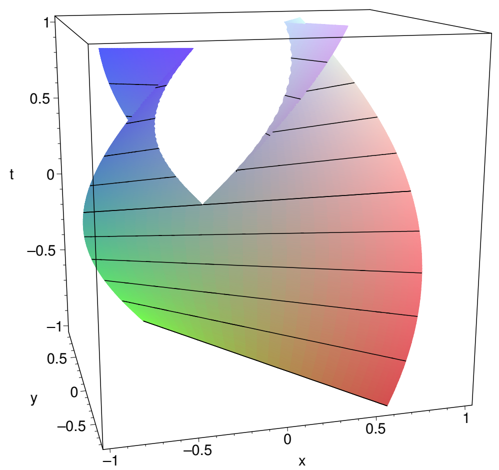
[Kristjansen,Plefka,Semenoff,Staudacher
Constable,Freedman,Minwalla et al]

Spinning string decay [Peeters,Plefka,Zamaklar]

- Non-planar sector in gauge theory for $J_1, J_2 \to \infty$?
  Again emergence of the effective genus-counting parameter $g_2 = \frac{(J_1+J_2)^2}{N}$

- Gauge theory side: Integrability lost in non-planar sector, but splitting vertex known.
  Problem: Enormous complexity of the Bethe-wavefunction! Determination of decay rates has not been achieved . . .

- String side: "Cutting" (change of boundary conditions) of the string world-sheet:



Yields prediction for dominant contribution to the decay amplitude in the limit $J_1, J_2 \to \infty$:

From continuity:

Energies $E^{(I/II)}$ of decay products known.

- Higher charges Q_n are conserved in semiclassical situation!

  $\Rightarrow$ Genrerally not the case, as non-planar effects destroy the integrable structure.

Summary & outlook

- "Snapshot" at the $AdS_5 \times S^5$ string spectrum through $J \to \infty$ limit: Analytic function in $\lambda' := \frac{\lambda}{J^2}$!

- Corresponds to novel (non 't Hooftian) large N limit of the gauge theory: $N, J \to \infty$ with $\frac{J^2}{N}$ fixed $\Rightarrow$ 2 new effect. parameters λ' (loops) and g_2 (genus).

  ⋆ Planar sector $\Leftrightarrow$ integrable (long-range) spin chain
  ⋆ Non-planar sector: string interactions

- Integrability of (planar) Super Yang-Mills $\Rightarrow$ key to the exact all loop spectrum?

  Parallel development: Integrable structure of the (class.) $AdS_5 \times S^5$ string

- Where is the spin chain in quantum string theory (on $AdS_5 \times S^5$)?