Loop quantum gravity: an outside view

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with Hermann Nicolai and Marija Zamaklar

Why Loop Quantum Gravity?

- Assumption: **Einstein gravity in 4d can be quantised**.
- Perturbative quantisation leads to ∞ counterterms,

$$\Gamma^{(2)}_{\text{div}} = \frac{1}{\epsilon}\frac{209}{2880}\frac{1}{(16\pi^2)^2}\int \mathrm{d}^4x\,\sqrt{g}\,C_{\mu\nu\rho\sigma}C^{\rho\sigma\lambda\tau}C_{\lambda\tau}{}^{\mu\nu}\,,$$

- Perhaps only the series in Newton's constant is bad **?**
  - Non-trivial UV fixed point (asymptotic safety) [Weinberg]
  - Exact renormalisation group [Reuter]
  - Euclidean path integrals [Gibbons, Hawking]
  - Dynamical triangulations [Ambjørn, Loll]
- However, the nonperturbative canonical quantisation of gravity has, so far, failed. $\longrightarrow$ LQG

Outline of this talk

★ **What makes Loop Quantum Gravity different ?**

How do the calculations work ?

What has been achieved ?

What are the problems ?

. . . with emphasis on the physics, not the maths

Loop variables and spin networks

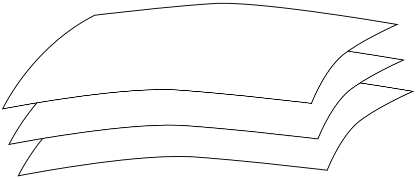
Quantisation → unusual Hilbert spaces

Quantised area and volume

Implementing constraints → physical Hilbert space

Applications and key problems

Canonical quantisation of gravity



$$E_\mu{}^A = \begin{pmatrix} N & N^a \\ 0 & e_m{}^a \end{pmatrix}$$

"coordinates": dreibein $e_m{}^a$ on the hypersurface

"momenta": extrinsic curvature K_{ab} of the surface

$E_a{}^\mu E_b{}^\nu \nabla_\mu E_{\nu 0}$

$$\Pi_a{}^m := \frac{\delta \mathcal{S}}{\delta \partial_t e_m{}^a} = \frac{1}{2} e e_b{}^m (K_{ab} - \delta_{ab} K) \,,$$

Quantisation: $(e_a{}^m, \Pi_a{}^m) \rightarrow \left(e_m{}^a, \frac{\delta}{\delta e_m{}^a}\right) \,, \qquad \Psi(e_m{}^a) \,.$

Two major technical problems

1 Einstein gravity has nonlinear constraints.

$$H = \partial_t e_a{}^m \Pi_m{}^a - \mathcal{L} = N H_0 + N_a H_a \,,$$

Hamiltonian constraint

$$H_0 \equiv e^{-1}\left(\Pi_{ab}\Pi_{ab} - \tfrac{1}{2}\Pi^2\right) - eR^{(3)}$$

diff constraint

$$H_a \equiv D_m \Pi_a{}^m$$

Hard to solve once quantised.

(Lorentz constraint

$$L_{ab} \equiv e_{m[a}\Pi_{b]}{}^m)$$

2 Resulting functional differential equations ill-defined.

$$H_0(\mathbf{x})\Psi[e] = 0\,, \qquad H_a(\mathbf{x})\Psi[e] = 0$$

this contains

$$\frac{\delta}{\delta e_m{}^a(\mathbf{x})}\frac{\delta}{\delta e_n{}^b(\mathbf{x})}$$

[Wheeler & DeWitt]

Considerable improvement: Ashtekar variables

Perform a transformation on phase-space,

$$\left(e_m{}^a\,,\;\Pi_a{}^m\right) \longrightarrow \left(\tilde{E}_a{}^m, A_{ma}\right)$$

inverse densitised 3-bein: $\tilde{E}_a{}^m := e\, e_a{}^m$

Ashtekar connection: $A_{ma} := -\frac{1}{2}\epsilon_{abc}\omega_{mbc} + \gamma\; e^{-1}(\Pi_{ma} - \frac{1}{2}e_{ma}\Pi)$

Barbero-Immirzi parameter

$$\{A_m{}^a(\mathbf{x})\,,\;\tilde{E}^n_b(\mathbf{y})\} = \gamma\;\delta^n_m\delta^a_b\delta^{(3)}(\mathbf{x},\mathbf{y})\,,$$

The Hamiltonian constraint becomes very simple when $\gamma = \pm i$,

$$-\gamma^2\left(\Pi_{ab}\Pi_{ab} - \frac{1}{2}\Pi^2\right) - e^2 R^{(3)} \longrightarrow \epsilon^{abc}\tilde{E}_a{}^m\tilde{E}_b{}^n F_{mnc}$$

The Barbero-Immirzi parameter

inverse densitised 3-bein: $\tilde{E}_a{}^m := e\, e_a{}^m$

Ashtekar connection: $A_{ma} := -\frac{1}{2}\epsilon_{abc}\omega_{mbc} + \gamma\; e^{-1}(\Pi_{ma} - \frac{1}{2}e_{ma}\Pi)$

$$H[N] = \int_\Sigma \mathrm{d}^3x\, N \frac{\tilde{E}^m_a \tilde{E}^n_b}{\sqrt{\det \tilde{E}}}\left(\epsilon^{abc}F_{mnc} - \frac{1}{2}(1+\gamma^2)K_{[m}{}^a K_{n]}{}^b\right).$$

- Old literature: take $\gamma = \pm i$.
  Simple Hamiltonian, but complex phase space.

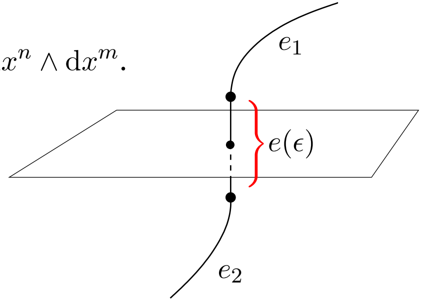
- New literature: take γ real

Loop variables

su(2) valued connection

- **holonomy:** $h_e[A] = \mathcal{P}\exp\int_e A_m^a \tau_a \, \mathrm{d}x^m,$
- **flux:** $F_S[\tilde{E}, f] = \int_S f_a \, \epsilon_{mnp} \tilde{E}^{ma} \mathrm{d}x^n \wedge \mathrm{d}x^m.$

inverse densitised dreibein



$$\left\{ (h_e)_{\alpha\beta},\, F_S \right\}$$

$$= \lim_{\epsilon \to 0} \left[\left(h^\epsilon_{e_1}\right)_{\alpha\gamma} \left\{ \int_{e(\epsilon)} A^a(\mathbf{x}) \tau^a_{\gamma\delta} \, , \, \int_S f_b(\mathbf{y}) \tilde{E}^b(\mathbf{y}) \right\} \left(h^\epsilon_{e_2}\right)_{\delta\beta} \right]$$

$$= \pm \gamma \, f_a(P) \left(h_{e_1} \tau^a \, h_{e_2}\right)_{\alpha\beta} .$$

Wave functions . . .

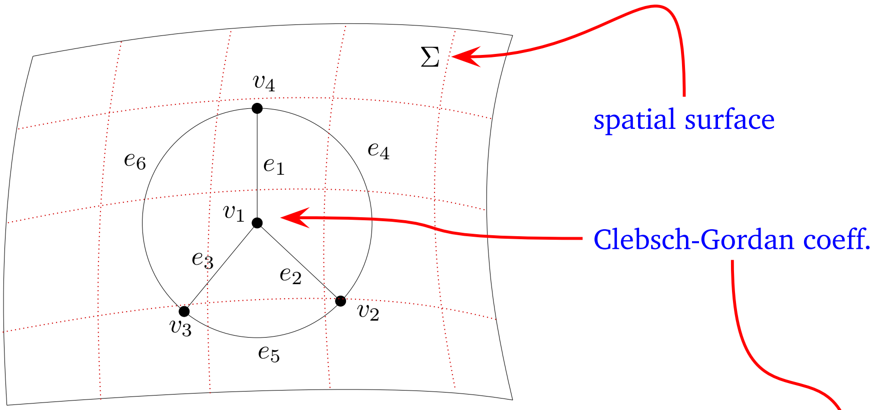
over all dreibein functions: $\Psi\left[e_m{}^a(\mathbf{x})\right]$

over all Ashtekar connections: $\Psi\left[A_m{}^a(\mathbf{x})\right]$

over the connections on a curve $\mathcal{C}$: $\Psi\left[\mathcal{P}\int_{\mathcal{C}} A_m{}^a(\mathbf{x})\right]$

$$\Psi\left[e_m{}^a(\mathbf{x})\right] \longleftrightarrow \Psi\left[h_{\mathcal{C}_1}[A]\right], \quad \Psi\left[h_{\mathcal{C}_2}[A]\right], \ldots$$

Spin network wave functions



$$\Psi[A]_{\text{graph, spins, intertwiners}} =$$

$$\Big(\rho_{j_1}(h_{e_1}[A])\Big)_{\alpha_1\beta_1}\Big(\rho_{j_2}(h_{e_2}[A])\Big)_{\alpha_2\beta_2}\Big(\rho_{j_3}(h_{e_3}[A])\Big)_{\alpha_3\beta_3}C^{j_1\,j_2\,j_3}_{\beta_1\,\beta_2\,\beta_3}\ldots$$

Ψ only feels A on curves: like δ-function basis.

So far . . .

★ **We have not done anything yet !**

We have just introduced fancy new variables !

What is the difference with "old" canonical quantisation ?

✓ Loop variables and spin networks

Quantisation → unusual Hilbert spaces

Quantised area and volume

Implementing constraints → physical Hilbert space

Applications and key problems

They key ingredient: the inner product

Wave functions labelled by a graph, spins, Clebsch-Gordan coefficients,

$$\Psi_{\Gamma,\{j\},\{C\}}\left[h_e[A]\,,\ldots\right]$$

LQG uses an inner product which leads to uncountable basis,

$$\left\langle \Psi_{\Gamma,\{j\},\{C\}} \,\middle|\, \Psi'_{\Gamma',\{j'\},\{C'\}} \right\rangle =$$

$$\int \prod_{e_i\in\Gamma} \mathrm{d}h_{e_i}\; \delta_{\Gamma,\Gamma'}\; \bar{\psi}_{\Gamma,\{j\},\{C\}}\Big(h_{e_1},\ldots\Big)\; \psi'_{\Gamma',\{j'\},\{C'\}}\Big(h_{e_1},\ldots\Big)$$

only nonzero if graphs equal

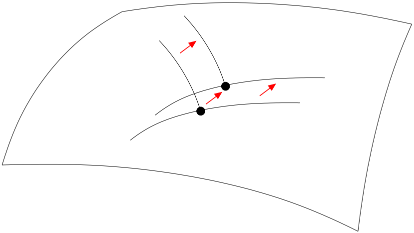
Loss of weak continuity

$$\left\langle \Psi_{\Gamma,\{j\},\{C\}} \,\middle|\, \Psi'_{\Gamma',\{j'\},\{C'\}} \right\rangle =$$

$$\int \prod_{e_i \in \Gamma} \mathrm{d}h_{e_i} \; \delta_{\Gamma,\Gamma'} \; \bar{\psi}_{\Gamma,\{j\},\{C\}}\Big(h_{e_1},\ldots\Big)\, \psi'_{\Gamma',\{j'\},\{C'\}}\Big(h_{e_1},\ldots\Big)$$

only nonzero if graphs equal

Consequence: loss of weak continuity (and the Stone-Von Neumann theorem)



continuous deformation in Σ

discontinous deformation in $\mathcal{H}$

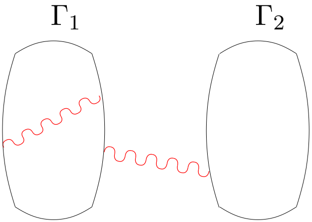
Compare with Yang-Mills in Fock quantisation

One-point functions:

$$\langle\, \Psi_\Gamma[A]\,\rangle = \langle\, \mathbf{1} \,|\, \Psi_\Gamma[A]\,\rangle = 0\,,$$

$$\langle\, W_\Gamma[A]\,\rangle = \left\langle\, \exp\left[ie \oint_\Gamma A_\mu \mathrm{d}x^\mu\right]\right\rangle = \exp\left[-\frac{ie}{2}\oint_\Gamma \oint_\Gamma \mathrm{d}x^\mu \mathrm{d}x^\nu\, \Delta_{\mu\nu}(x-y)\right]$$

Correlators of Wilson loops:



$$\langle\, \Psi_{\Gamma_1,\{C\}}[A] \,|\, \Psi_{\Gamma_2,\{C\}}[A]\,\rangle = \delta_{\Gamma_1,\Gamma_2}\,,$$

$$\langle\, W_{\Gamma_1}[A] \,|\, W_{\Gamma_2}[A]\rangle = \exp\left[-\frac{ie}{2}\oint_{\Gamma_1} \mathrm{d}x^\mu \oint_{\Gamma_2} \mathrm{d}y^\nu\, \Delta_{\mu\nu}(x-y)\right].$$

So far . . .

★ **LQG uses a weird Hilbert space (non-separable)**

What are the consequences**?**

How do operators behave on this Hilbert space**?**

✓ Loop variables and spin networks

✓ Quantisation $\longrightarrow$ unusual Hilbert spaces

Quantised area and volume

Implementing constraints $\longrightarrow$ physical Hilbert space

Applications and key problems

Area operator

Given a spin network wave function Ψ
and given a two-dimensional surface in Σ
what is the expectation value of the area **?**

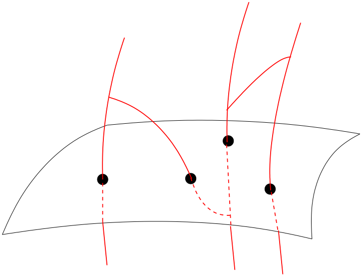
$$\mathrm{d}F_a := \epsilon_{abc}\, e_m{}^a e_n{}^b\, \mathrm{d}x^m \wedge \mathrm{d}x^n$$

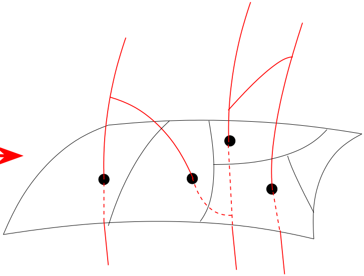
$$A_S = \int_S \sqrt{\mathrm{d}F^a \cdot \mathrm{d}F^a} = \lim_{N\to\infty} \sum_{I=1}^{N} \sqrt{F^a_{S_I}\, F^a_{S_I}}$$

$$\hat{F}^a_{S_I}\, \hat{F}^a_{S_I} \quad \big(h_e\big)_{\alpha\beta} \sim \big(h_{e_1}\, \tau_a\, \tau^a\, h_{e_2}\big)_{\alpha\beta}$$

$$= \big(j_e(j_e+1)\big)\, \big(h_e\big)_{\alpha\beta}$$

spectrum: $\quad 8\pi l_P^2 \hbar\, \gamma \sum_e \sqrt{j_e(j_e+1)}$

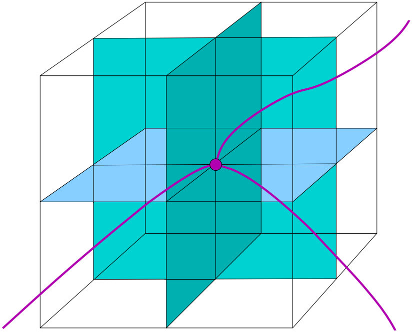




Volume operator

$$\hat{V}(\Omega) = \lim_{N\to\infty} \sum_{I=1}^{N} \hat{V}(\Omega_I) = \lim_{N\to\infty} \sum_{I=1}^{N} \sqrt{\left|\hat{q}(S_I^1, S_I^2, S_I^3)\right|}\,,$$

$$\hat{q}(S_I^1, S_I^2, S_I^3) := \frac{1}{3!}\epsilon_{abc}\hat{F}^a_{S_I^1}\,\hat{F}^b_{S_I^2}\,\hat{F}^c_{S_I^3}\,.$$



- vanishes on 3-valent vertices
- limiting procedure subtle
- surface orientation needs averaging

- spectrum of $q = V^2$ unknown

$\Downarrow$

action of $\hat{V}$ unknown

So far . . .

★ **Funny Hilbert space → quantised area & volume**

Still need to implement the constraints **!**

This is usually "the hard part" **!**

✓ **Loop variables and spin networks**

✓ **Quantisation → unusual Hilbert spaces**

✓ **Quantised area and volume**

Implementing constraints → physical Hilbert space

Applications and key problems

Constraint 1: Gauss constraint

Gauss' constraint expresses the fact that the Ashtekar connection is a bit like an SU(2) gauge field. In Maxwell:

$$\Pi^0 = \frac{\delta L}{\delta \dot{A}_0} \approx 0$$

Gauss' law
$(\Pi^m = F^{0m})$

$$\{\Pi^0, H\}_{\text{PB}} = 0 \quad \longrightarrow \quad \partial_m \Pi^m \approx 0$$

In gravity it's just like in gauge theory,

inverse densitised dreibein
$(\tilde{E}_a{}^m = e\, e_a{}^m)$

$$D_m \tilde{E}_a{}^m \approx 0$$

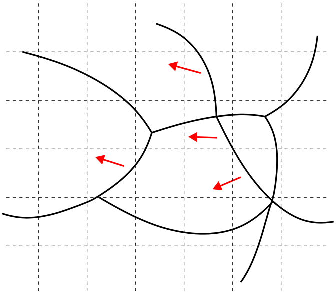
wrt. Ashtekar connection

Gauss' constraint in absence of sources implies vanishing charge.

This is easy to implement: just make sure all SU(2) indices are contracted.

Constraint 2: Diffeomorphism constraint

Spin-network states not automatically diffeomorphism invariant.



Diffeomorphism-invariant states are infinite sums of spin networks.

$$|\Psi\rangle_{\text{diff}} = \sum_{\text{diffeomorphisms } g} g\,|\Psi\rangle$$

$$_{\text{diff}}\langle\Psi_1|\Psi_2\rangle_{\text{diff}} = \sum_{\text{diffeos } g_1} \sum_{\text{diffeos } g_2} \Big(\langle\Psi_1|g_1^\dagger\Big)\Big(g_2|\Psi_2\rangle\Big)$$

reduces to single term

Constraint 3: Hamiltonian constraint

- The commutator of two Hamiltonians does not close in the strict Lie-Poisson algebra sense:

$$\Big\{H[N],\ H[M]\Big\} = \int \mathrm{d}^3x \, \underbrace{(N\partial_r M - M\partial_r N)\,\frac{\tilde{E}_a{}^m \tilde{E}^{an}}{\det \tilde{E}}}_{\text{field-dependent structure function}} \; \overbrace{\tilde{E}_a{}^r\, F^a_{st}}^{\text{diffeomorphism generator}}$$

- No choice: implement Hamiltonian constraint as operator equation.

Rewriting tricks

Barbero-Immirzi parameter

$$H[N] = \int_\Sigma \mathrm{d}^3x\, N \frac{\tilde{E}^m_a \tilde{E}^n_b}{\sqrt{\det \tilde{E}}} \left(\epsilon^{abc} F_{mnc} - \frac{1}{2}(1+\gamma^2) K_{[m}{}^a K_{n]}{}^b \right) .$$

need to implement these in terms
of holonomies and fluxes

Rewriting tricks

$$H[N] = \int_\Sigma \mathrm{d}^3x\, N \frac{\tilde{E}^m_a \tilde{E}^n_b}{\sqrt{\det \tilde{E}}} \left(\epsilon^{abc} F_{mnc} - \frac{1}{2}(1+\gamma^2) K_{[m}{}^a K_{n]}{}^b \right) .$$

dreibein

volume

$$e_m{}^a(\mathbf{x}) = \epsilon_{mnp} \epsilon^{abc} \tilde{E}^{-1/2} \tilde{E}^n_b \tilde{E}^p_c(\mathbf{x}) = \frac{1}{4\gamma} \left\{ A_m{}^a(\mathbf{x}), V \right\}$$

extrinsic curvature

$$K_m{}^a(\mathbf{x}) = \frac{1}{\gamma} \{ A_m{}^a(\mathbf{x}) \,,\, \int_\Sigma \mathrm{d}^3x\, K_m{}^a \tilde{E}_a{}^m \}$$

Ashtekar curvature

$$= \frac{1}{\gamma} \left\{ A_m{}^a(\mathbf{x}) \,,\, \frac{1}{\gamma^{3/2}} \int \left\{ \frac{\tilde{E}_a{}^m \tilde{E}_b{}^n}{\sqrt{\tilde{E}}} \epsilon^{abc} F_{mnc} \,,\, V \right\} \right\} .$$

Action of the Hamiltonian constraint

- Let's look at the first term, which classically is

$$H_1 = \int_\Sigma \mathrm{d}^3x \, N \frac{\tilde{E}^m_a \tilde{E}^n_b}{\sqrt{\det \tilde{E}}} \left(\epsilon^{abc} F_{mnc} \right) .$$

plaquette volume

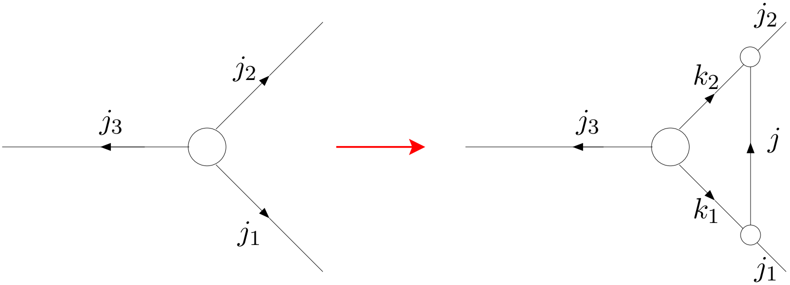
$$\hat{H}_1 = \sum_\alpha N(v_\alpha)\, \epsilon^{mnp} \, \mathrm{Tr} \left(\left(h_{\partial P_{mn}(\epsilon)} - h^{-1}_{\partial P_{mn}(\epsilon)} \right) h_p^{-1} \left[h_p, \hat{V} \right] \right)$$

act on each node

In pictures

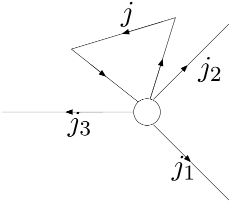
$$\hat{H}_1 = \sum_{\alpha} N(v_\alpha)\,\epsilon^{mnp}\,\mathrm{Tr}\left(\left(h_{\partial P_{mn}(\epsilon)} - h^{-1}_{\partial P_{mn}(\epsilon)}\right) h_p^{-1}\,\left[h_p, \hat{V}\right]\right)$$

The action is a bit tricky, however, the net effect is



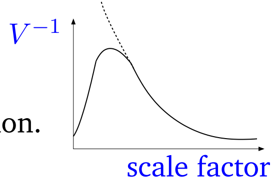
Ambiguous:

- Spin j of the trace.
- Orientation of the plaquette
- Operator ordering.
- Action of V itself not known.
- Ambiguities in $\hat{V}$.



Claimed achievements

- Loop quantum cosmology



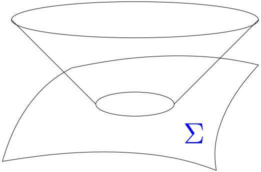
Not derived from the full LQG formalism.

LQG methods to quantise a mini-superspace action.

Inverse volume operator spectrum bounded.

Recent work: this is not the case in the full theory.

- Black hole entropy



Find a way to select kinematical states satisfying the "isolated horizon" conditions.

The total number of states with a given $\langle\text{area}\rangle$ can then be counted.

$$S(A) \sim A$$

Barbero-Immirzi parameter enters.

So far . . .

★ **Many ambiguities, and regularisation dependence**

Are there any consistency checks ?

Is the constraint algebra satisfied ?

Do we see long-range correlations ?

- ✓ Loop variables and spin networks
- ✓ Quantisation $\longrightarrow$ unusual Hilbert spaces
- ✓ Quantised area and volume
- ✓ Implementing constraints $\longrightarrow$ physical Hilbert space
- Applications and key problems

Requirement 1: off-shell closure

Classically:

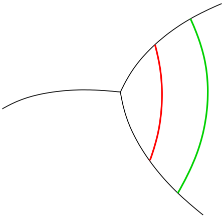
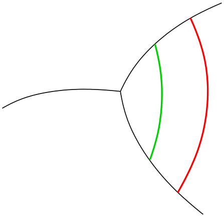
$$\Big\{H[M]\,,\,H[N]\Big\}_{\mathrm{PB}} = \int_\Sigma \mathrm{d}^3x\,(M\partial_m N - N\partial_m M)g^{mn}D_n\,.$$

Quantum theory:

$$\Big[\hat{H}[M]\,,\,\hat{H}[N]\Big]\Big|\psi\Big\rangle = \int_\Sigma \mathrm{d}^3x\,(M\partial_m N - N\partial_m M)g^{mn}\hat{D}_n\Big|\psi\Big\rangle\,.$$

but $D_n\big|\psi\big\rangle$ does not exist, only $\exp\big(\alpha D_n\big)\big|\psi\big\rangle$ **!**

Only a check on $\big|\psi\big\rangle$ such that $\exp\big(\alpha D_n\big)\big|\psi\big\rangle = 0$ done **!**

$[H[N],H[N']]\,|\psi\rangle =$  $-$  $= \; 0$

The importance of closure

Compare string theory:

$$(T_{++} + T_{--})|\psi\rangle \approx 0 \quad \longrightarrow \quad \mathcal{C} \times (T_{++} + T_{--})|\psi\rangle \approx 0$$

any Casimir
for instance $\mathcal{C} = (J^2 - \text{number})$

but keep $\quad (T_{++} - T_{--})|\psi\rangle \approx 0$

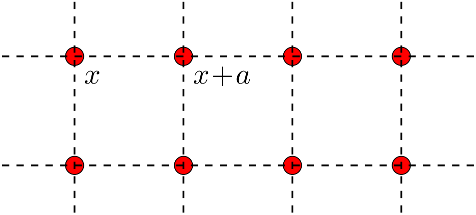
Incorrect mass spectrum !

Why incorrect ? Because the constraint algebra is not implemented !

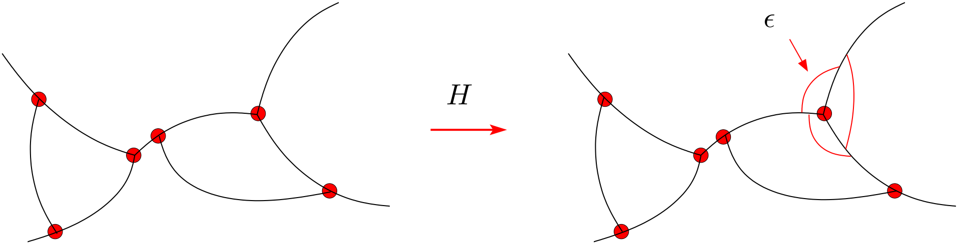
Requirement 2: long-range correlations

- In lattice gauge theory, neighbouring sites talk:

$$\nabla\phi \longrightarrow \frac{1}{a}\left(\phi_{x+a}-\phi_x\right)$$



- In loop quantum gravity, the operators act only at one node,



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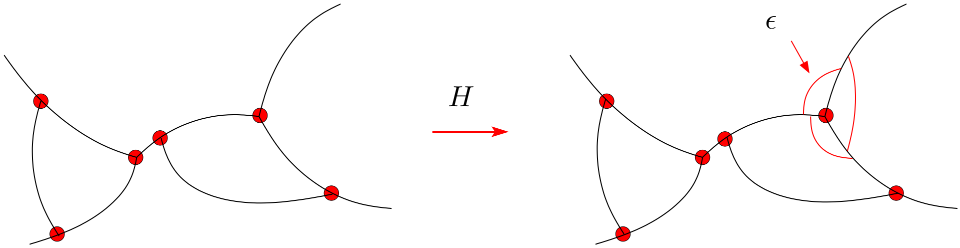

- Can there be long-range correlations?
  Waves, propagating gravitons, classical solutions?

- Loop quantum gravity not "gravity put on a coordinate lattice"
  (lattice quantum gravity).

- Spin foams try to repair this problem.
  Still suffer from ambiguities of the Hamiltonian approach.

Summary

★ **The good**

Four-dimensional.

No new degrees of freedom or symmetries.

“Discretisation” at the Hilbert-space level.

★ **The bad**

Constraint algebra not verified (if possible at all).

Are long-range correlations possible?

Not a single physical, interpretable state known.

No approximation methods around simple state.

Not discussed: matter couplings, anomalies.