

# Pure states, black hole formation and Choptuik scaling in the SYK model

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## Motivation :

- BH information loss usually associated with BH evaporation.
- Process of formation of a BH is a version of inf loss  
 $| \text{Pure state} \rangle \rightarrow \text{BH, mixed state, } T > 0$
- Gravitation collapse can be modelled by a quantum quench in AdS/CFT  
Under a sudden perturbation  
 $| \text{Pure state} \rangle \rightarrow | \text{Pure state} \rangle \text{ with thermal props.}$
- In AdS/CFT such a description is possible if we can solve the QFT on the boundary ..
- SYK model : (SYK, MS, MSY, Kourkoulou-Maldacena)
  - soluble at large  $N$  and strong coupling
  - explicit construction of 'thermal' state
- Dual to a model of 2-dim gravity  
(JT :  $\text{AdS}_4 \rightarrow S^2 \times \text{AdS}^2$ ; Polyakov gravity)  
(AP) (PN, GM, SRW)

## The SYK Model

QM of  $N$  real fermions:  $\Psi_a, a=1 \dots N$

$$\{\Psi_a, \Psi_b\} = \delta_{ab}$$

$$H = \sum_{1 \leq a < b < c < d \leq N} J_{abcd} \Psi_a \Psi_b \Psi_c \Psi_d$$

$$\langle J_{abcd} \rangle = 0, \quad \langle J_{abcd}^2 \rangle = 3! \frac{J^2}{N^3}$$

### Relevance / Interest for black hole physics

- (i) Simple microscopic degrees of freedom
- (ii) Soluble in large  $N$  and long wavelengths  
 $\omega \ll J$ ; can compute  $\lambda_L = \frac{2\pi}{\beta} + \text{corrections}$
- (iii) Easy to compute at finite  $N$  numerically
- (iv) There is a simple semi-classical gravity dual (JT gravity / Polyakov gravity)
- (v) SYK at finite temp. is dual to a geometry in 2-dim  $AdS_2$  spacetime

## Results:

- 1) We compute the evolution of a 'pure state' (that evaluates thermal averages of many observables), using a Hamiltonian that is tuned to the 'pure state'. By varying the strength of the Hamiltonian (piece wise in time) we can extract or pump energy into the system, via 2 simple quenches.
- 2) The second quench, pumps in energy into the system and enables an explicit time dependent evolution of a state that asymptotically can be identified with a BH (geom with a horizon) with  $T_{bh} \propto (\epsilon_{critical} - \epsilon_2)^{1/2}$  Choptuik scaling form.

## Large N mean field theory

Bilocal Variables :

$$G(z_1, z_2) = \frac{1}{N} \sum_{a=1}^N \langle \Psi_a(z_1) \Psi_a(z_2) \rangle$$

$$\Sigma(z_1, z_2) = \frac{1}{N} \sum_{a=1}^N \langle X_a(z_1) X_a(z_2) \rangle$$

$$X_a = \sum_{b,c,d} f_{abcd} \Psi_b \Psi_c \Psi_d$$

$$\text{Hamiltonian} \sim \sum_a \Psi_a X_a$$

Schwinger-Dyson eqns :

$$\Sigma(z_1, z_2) = J^2 G(z_1, z_2)^3, \quad G(\omega) = (i\omega - \Sigma(\omega))^{-1}$$

are derived from

$$S = -\frac{1}{2} \text{Pf}(\partial_z - \Sigma) + \int dz_1 dz_2 \left[ \Sigma G - \frac{J^2}{4} G^3 \right]$$

$$Z = \int \mathcal{D}G \mathcal{D}\Sigma e^{-NS}$$

Large  $N \Rightarrow$  quenched average = annealed average  
+  $O(N)$  Symmetry is manifest

## Solution in infra-red

$\omega \ll J$

SD eqns + action has an infinite dim symmetry :

$$z \rightarrow f(z) \quad (\text{Diff 1})$$

$$G^{[f]} = [f'(z_1) f'(z_2)]^{-4} G(z_1, z_2)$$

$$\Delta = \frac{1}{4}$$

$$\Sigma^{[f]} = [f'(z_1) f'(z_2)]^{-3\Delta} \Sigma(z_1, z_2)$$

$$\cdot \text{UV dim}[\Psi] = 0, \text{ IR dim}[\Psi] = \frac{1}{2}$$

Solution:

Diff 1  $\rightarrow$   $SL(2, \mathbb{R})$

$$G_{IR}(z) = \frac{b}{|z|^{2\Delta}} \text{sgn}(z), \quad b = \left(\frac{1}{4\pi J^2}\right)^{\frac{1}{4}}$$

$$\text{Finite temperature } T = \beta^{-1}: f(z) = \tan \frac{\pi i z}{\beta}$$

$$G_{IR}(z, \beta) = c \left[ \frac{\pi}{J\beta \sin \frac{\pi z}{\beta}} \right]^{\frac{1}{2}} \text{sgn}(z)$$



$$G_{IR}(t, \beta) = c \left[ \frac{\pi}{J\beta \sinh \frac{\pi t}{\beta}} \right]^{\frac{1}{2}} \quad \text{in real time}$$

## Low energy dynamics

Degrees of freedom  $\tau \rightarrow f(z) \in \frac{\text{Diff } \mathbb{R}^1}{\text{SL}(2, \mathbb{R})}$

Euclidean effective action

$$NS = -\frac{N\alpha_s}{J} \int dz \{f(z), z\} \quad (\text{easy to derive except for } \alpha_s)$$

Finite temp:  $f(z) = \tan \frac{i\pi g(z)}{\beta}$

$$NS = \frac{N\alpha_s}{2J} \int_0^\beta dz \left[ \left( \frac{g''}{g'} \right)^2 - \left( \frac{2\pi}{\beta} \right)^2 (g')^2 \right]$$

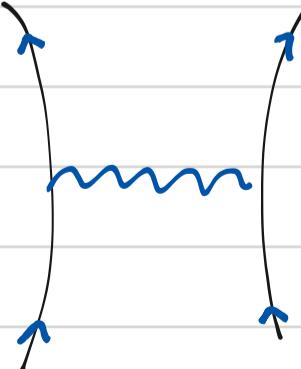
action on  
coadjoint  
orbit of  
diff's/ $\text{SL}(2)$

e.g.  $g(z) = z$  is a solution of EOM and

it corresponds to a space time with a horizon

and temp.  $\beta^{-1}$ .

One important implication of the Schwarzian:



$$\Rightarrow \langle \{ \psi_a(t), \psi_b(0) \}^2 \rangle \propto \frac{J}{H} e^{+\frac{2\pi i t}{\beta}}$$

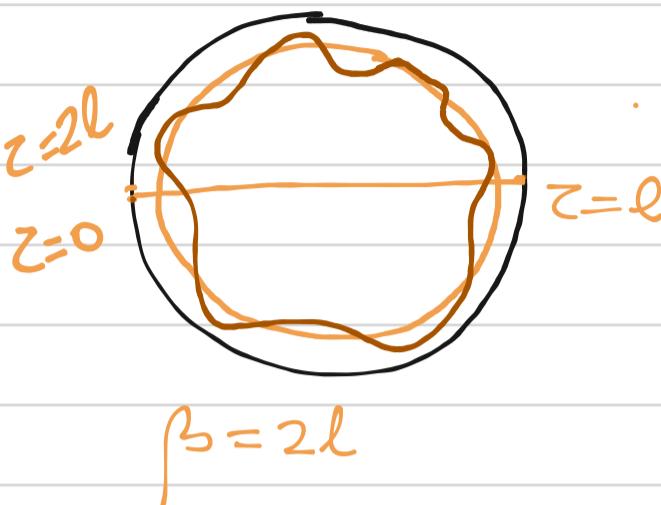
## Dual gravity theory:

JT - Dim. reduction / Polyakov gravity

of near extremal BH

(Virasoro  
Co-adjoint  
orbit ...)

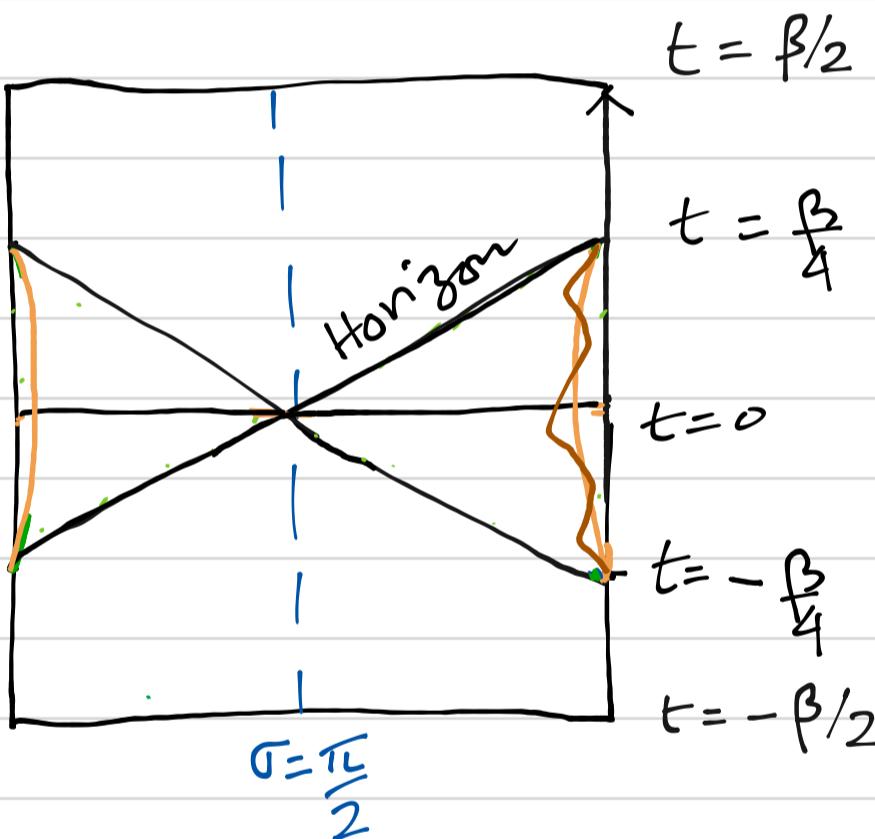
Euclidean :  
Signature



Poincaré disc

Minkowski:  
metric

Eternal  
Black Hole  
in  $AdS_2$



$$z = \delta \ll 1$$

$$\rightarrow \sigma \quad 0 \leq \sigma \leq \pi$$

- Degrees of freedom of both gravity theories  
 $f(t) \in \text{diff } 1 / \text{SL}(2, \mathbb{R})$

- Gravitational action  $\Rightarrow S(f) = \frac{\Phi}{G_2} \int d\tau \{f, \tau\}$  solution  $\downarrow$   
 (Euclidean)

- $ds^2 = -\frac{dt^2}{z^2} \left(1 + \{f, t\} \frac{z^2}{2}\right)^2 + \frac{dz^2}{z^2}$   $\left\{ \begin{array}{l} f(t) = \tanh \frac{\pi}{\beta} t \\ \{f, t\} = -\left(\frac{2\pi}{\beta}\right)^2 \end{array} \right.$   
 (Minkowski)

## SYK micro-states

$$\{\Psi_a, \Psi_b\} = 2\delta_{ab} \Rightarrow \Psi_a = \frac{\gamma_a}{\sqrt{2}} \quad (\gamma\text{-matrices})$$

$$\hat{S}_k = 2i \Psi_{2k-1} \Psi_{2k}, \quad k=1, 2, \dots, \frac{N}{2}$$

$$\hat{S}_k^2 = 1 \Rightarrow \text{eigenvalues } s_k = \pm 1$$

Hence Hilbert space  $\mathcal{H}$  is spanned by

the complete set  $\{|s_1, s_2, \dots, s_N\rangle\}$

$$\text{and } \dim \mathcal{H} = 2^{N/2}$$

$$\text{Define } |B_s\rangle = |s_1, s_2, \dots, s_N\rangle$$

These are high energy states :

$$\langle B_s | H_{\text{SYK}} | B_s \rangle \sim 0 \quad \text{and}$$

$$\langle GND | H_{\text{SYK}} | GND \rangle \sim - NJ$$

Low energy projection :

$$|B_s(l)\rangle = \bar{e}^{-lH_{\text{SYK}}} |B_s\rangle \text{ for large } l = \frac{\beta}{2}.$$

## Properties of $|B_s(l)\rangle$

$$1) \sum_{\{S_k\}} \langle B_s(l) | B_s(l) \rangle = \text{Tr } e^{-2lH}$$

$$2) \sum_{\{S_k\}} \langle B_s(l) | \prod_i O_i | B_s(l) \rangle = \text{Tr} (e^{-2lH} \prod_i O_i)$$

$$3) \text{Flip Symmetry: } \Psi_{2k} \rightarrow -\Psi_{2k}, \Psi_{2k-1} \rightarrow \Psi_{2k-1}$$

(Subgroup of  $O(N)$ )

4) For flip invariant operators

$$\sum_{\{S_k\}} \langle B_s(l) | O_1 \dots O_n | B_s(l) \rangle = 2^{N/2} \langle B_{S_0}(l) | O_1 \dots O_n | B_{S_0}(l) \rangle$$

$\Rightarrow$

$$\bullet G_\beta(t, t') = \langle B_s(l) | \underbrace{\Psi_a(t) \Psi_a(t')}_{\text{Thermal}} | B_s(l) \rangle = \frac{c_\alpha}{\left[ \frac{\pi}{\beta} \sinh \frac{\pi}{\beta}(t-t') \right]^{1/2}}$$

$$\bullet \langle B_s(l) | \underbrace{s_k \Psi_{2k-1}(t) \Psi_{2k}(t')}_{\text{Non-thermal}} | B_s(l) \rangle$$

$$= -2i G_\beta(t, -il) G_\beta(t', -il)$$

- expressed as product of thermal correlators

- same for all sets  $\{S_1, S_2, \dots, S_{N/2}\}$

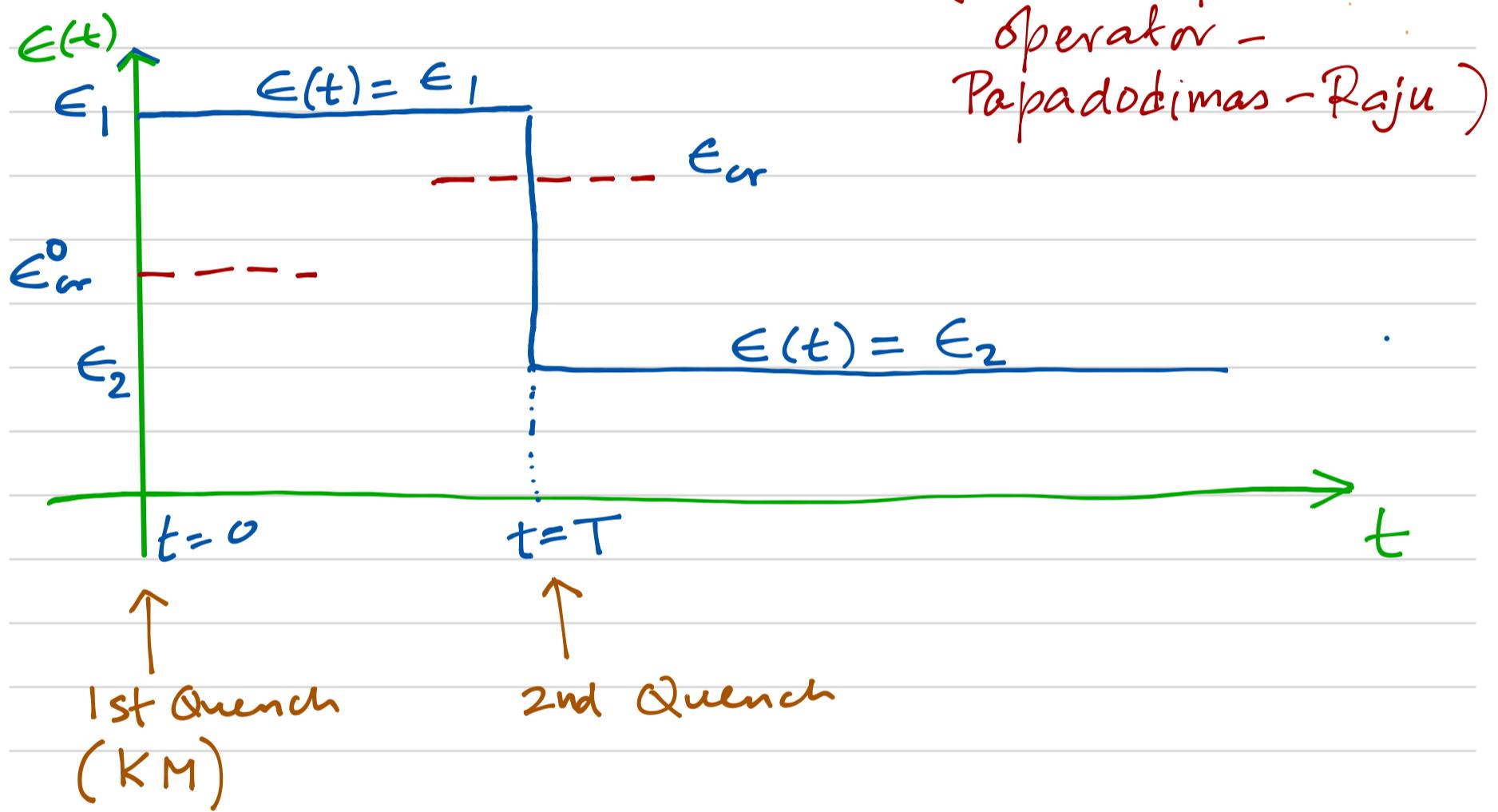
## State dependent Operator added to $H_{\text{SYK}}$

$$H = H_0 + \epsilon(t) H_M, \quad H_0 = H_{\text{SYK}}, \quad t \geq 0$$

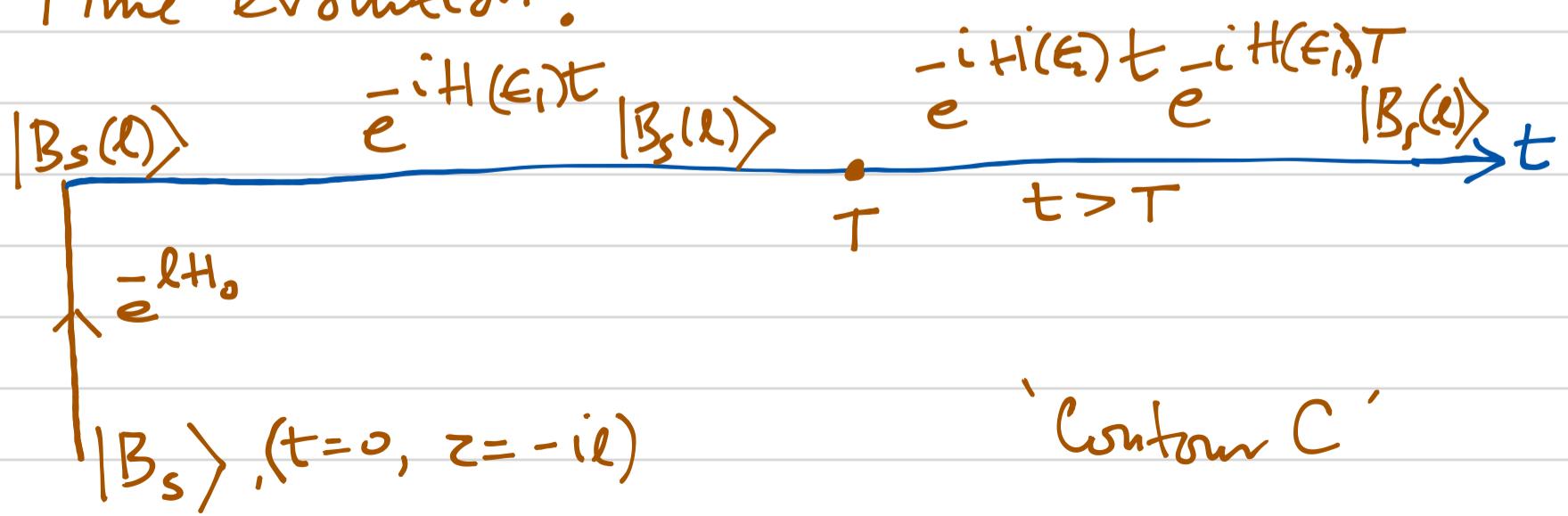
$$H_M = -i \sum_{k=1}^{N/2} \delta_k \Psi_{2k-1} \Psi_{2k}$$

where  $\{\delta_k\}$  are the same as those in  $|B_s(\ell)\rangle$

(state dependent operator - Papadodimas-Raju)



Time evolution:



## Effective action from overlap:

$$A_s(t, \ell) = \langle B_s(\ell) | U_o^+(t) | U(t) | B_s(\ell) \rangle$$

$t \gg 1$

$$H = H_{SYK} + \hat{\epsilon} H_m$$

$$U(t) = \bar{e}^{iHt} = \bar{e}^{-iH_0 t} T\left(\bar{e}^{i \int_0^t H_I(t') dt'}\right) = U_o(t) U_I(t)$$

$$\Rightarrow A_s(t, \ell) = \int dG \partial \Sigma e^{i S_{SYK}(\Sigma, G) + \Delta S}$$

$$\Delta S = (-i \epsilon N J) \int_C dt' G^2(t', -i\ell)$$

(we have used the large  $N$  limit)

$$G(t', -i\ell) = \left\langle \sum_{a=1}^N \psi^a(t') \psi^a(-i\ell) \right\rangle$$

In the SYK model

$$\bar{G}(t, -i\ell) = \bar{G}(t + i\ell, 0) = \frac{c_\Delta}{\left[ \frac{i\beta}{\pi} \coth \frac{\pi}{\beta} t \right]^{\frac{1}{2}}}$$

Low energy effective action:  $t \rightarrow \varphi(t)$

$$\Delta S \propto \epsilon J N \int dt \frac{|\varphi'(t)|^{2\Delta}}{\left[ \frac{\beta J}{\pi} \coth \frac{\pi}{\beta} \varphi(t) \right]^{4\Delta}} \propto \epsilon J N \int dt (f')^{2\Delta}$$

$$f(t) = \frac{\pi}{J^2 \beta} \tanh \frac{\pi}{\beta} \varphi$$

Collecting all formulas :

$$A_s(t, \beta) = \int_{\mathcal{D}G \otimes \Sigma} e^{i S(G, \Sigma)}$$

$$S(G, \Sigma) = S_{SYK}(G, \Sigma) - \epsilon N J \int_0^t dt' G^2(t', -il)$$

(only large  $N$  used)

$\beta J \gg 1$ ,  $\epsilon \ll 1$ , saddle point  $\bar{G}$  + low energy

$$S(f) = -\frac{N\alpha}{J} \int_C dt \left[ \{f(t), t\} - \frac{J^2}{2} \epsilon(t) (f')^{2\Delta} \right]$$

( $KM$ )

$$2\Delta = \frac{1}{2}$$

Change variables  $\dot{f} = e^\phi$ .

$$S(f, \lambda, \phi) = \frac{N\alpha}{J} \int_C \left[ \frac{\dot{\phi}^2}{2} + \frac{J}{2} \lambda(t) (e^\phi - f) + \frac{J^2}{2} \epsilon(t) e^{2\Delta\phi} \right]$$

## Equations of motion:

$$1) \ddot{\phi} = J^2 \epsilon(t) \Delta e^{2\Delta\phi} + J \frac{\lambda(t)}{2} e^{\phi(t)} \equiv -V'(\phi)$$

$$2) \dot{\lambda} = 0 \Rightarrow \lambda = \text{constant}$$

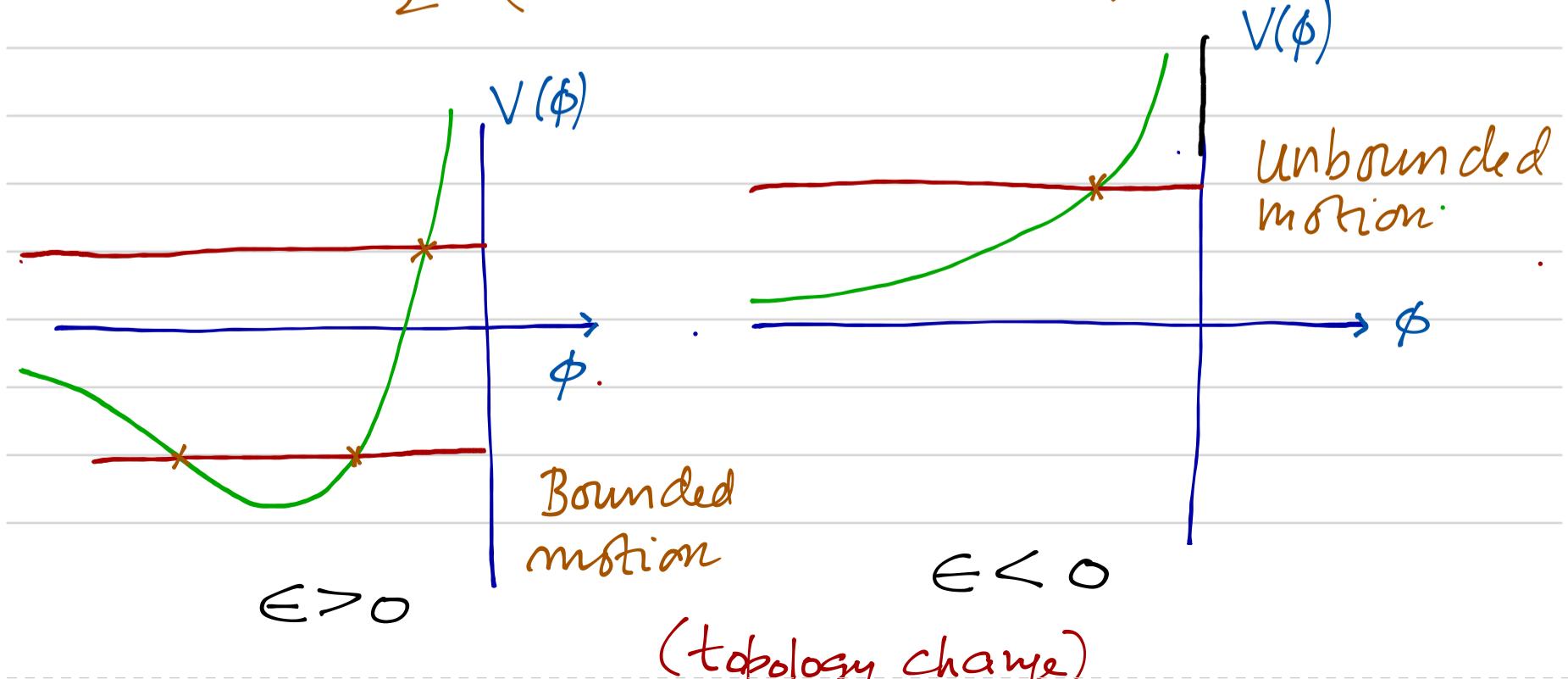
$$3) \dot{f} = e^\phi$$

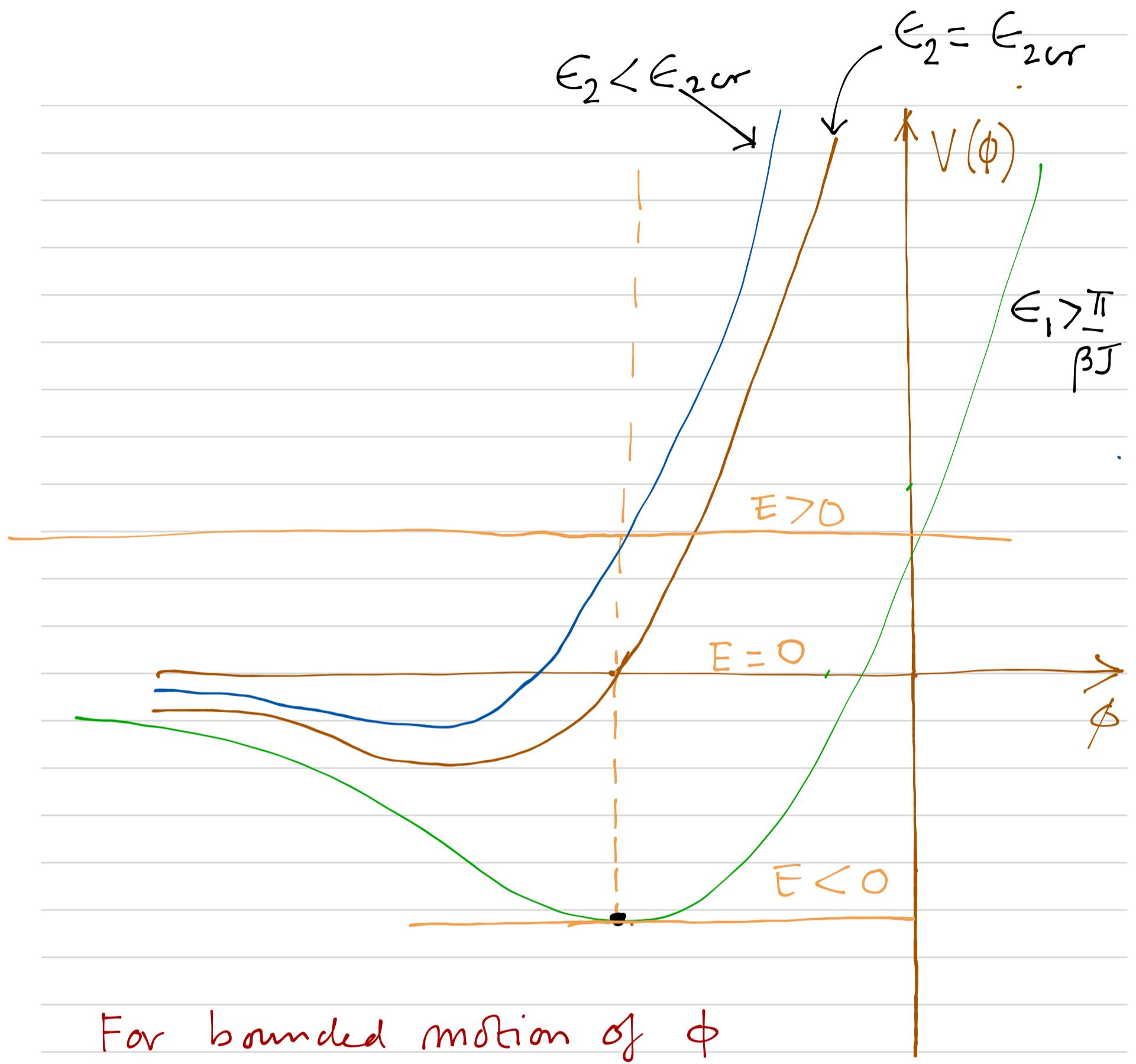
In imaginary time  $(-i\ell, 0)$ ,  $\epsilon = 0$

$$\Rightarrow \lambda = -J \quad (\text{using } f_E(z) = \frac{\pi}{\beta J^2} \tan \frac{\pi}{\beta} z)$$

$$\Rightarrow S = \frac{N\alpha}{J} \int_{t \geq 0} dt \left[ \frac{\dot{\phi}^2}{2} - V(\phi) \right]$$

$$V(\phi) = \frac{J^2}{2} \left( e^\phi - \epsilon(t) e^{2\Delta\phi} \right)$$





For bounded motion of  $\phi$

$$\dot{\phi} = e^\phi \neq 0$$

For unbounded motion of  $\phi$ ,  $\dot{\phi} = e^\phi \rightarrow 0$

as  $\phi \rightarrow -\infty$

Solutions:

i)  $0 \leq t \leq T$ ,  $\epsilon_1 > \frac{\pi}{\beta J}$ ,  $\dot{\phi}(0) = 0$ ,  $e^{\phi(0)} = \left(\frac{\pi}{\beta J}\right)^2$

$$\bar{\epsilon}_0 = \frac{\pi}{\beta J}, \quad \bar{\epsilon}_1 = \epsilon_1 - \frac{\pi}{\beta J}$$

matching with  
Euclidean solution

$$\dot{f}(t) = e^{\phi(t)} = \frac{\bar{\epsilon}_0^2}{\left[ \left( 1 - \frac{\epsilon_1 \bar{\epsilon}_1^{-1}}{2} \right) \cos \left( \frac{\sqrt{\bar{\epsilon}_0 \bar{\epsilon}_1}}{2} t \right) + \epsilon_1 \bar{\epsilon}_1 \right]^2}$$

Periodic orbit,  $\dot{f}(t) \neq 0 \quad t \in [0, T]$

If  $\epsilon_1 = 2 \frac{\pi}{\beta J}$ ,  $1 - \frac{\epsilon_1 \bar{\epsilon}_1^{-1}}{2} = 0$

and  $\dot{f}(t) = \text{constant}$

particle sits at the bottom of the potential

$$E = V(\phi) < 0$$

$$2) \quad T \leq t < \infty \quad \epsilon(t) = \epsilon_2$$

$$\dot{f}(t) = e^{\phi(t)} = \frac{\bar{\epsilon}_1^2}{\left[ \left( 1 + \frac{\epsilon_2 \bar{\epsilon}_2^{-1}}{2} \right) \cosh \left( \frac{J(t-T)}{2} \sqrt{\bar{\epsilon}_1 \bar{\epsilon}_2} \right) - \frac{\epsilon_2 \bar{\epsilon}_2}{2} \right]^2}$$

$$\bar{\epsilon}_2 = \bar{\epsilon}_1 - \epsilon_2$$

$$\text{at } t=T, \quad \phi(T-\delta) = \phi(T+\delta), \quad \delta \rightarrow 0$$

$$\dot{\phi}(T-\delta) = \dot{\phi}(T+\delta), \quad \delta \rightarrow 0$$

$$\text{but} \quad \ddot{\phi}(T-\delta) \neq \ddot{\phi}(T+\delta) \quad \text{acc discontinuous}$$

$$\Delta \ddot{\phi} = (\epsilon_1 - \epsilon_2) e^{\phi(T)/2}$$

$$t \rightarrow \infty, \quad \dot{f} = e^{\phi} \rightarrow \exp \left[ - J t \sqrt{\bar{\epsilon}_1 (\bar{\epsilon}_1 - \bar{\epsilon}_2)} \right]$$

$$(f_{bh} = \frac{\pi}{\beta J^2} \tanh \frac{\pi t}{\beta}, \quad \dot{f}_{bh} = \left( \frac{\pi}{\beta J} \right)^2 \operatorname{Sech}^2 \left( \frac{\pi t}{\beta} \right))$$

$$\dot{f}_{bh} \rightarrow \left( \frac{\pi}{\beta J} \right)^2 e^{-\frac{2\pi t}{\beta}}$$

$$\Rightarrow \frac{2\pi}{\beta} = J \sqrt{\bar{\epsilon}_1 (\bar{\epsilon}_1 - \bar{\epsilon}_2)} \quad \Rightarrow$$

$$T_{bh} = \frac{J}{2\pi} \sqrt{\left( \bar{\epsilon}_1 - \frac{\pi}{\beta J} \right) \left( \bar{\epsilon}_1 - \bar{\epsilon}_2 - \frac{\pi}{\beta J} \right)}$$

$$T_{bh} = C \sqrt{\underbrace{(\epsilon_1 - \epsilon_2)}_{\Delta \epsilon} - \underbrace{(\epsilon_1 - \epsilon_{cr})}_{\Delta \epsilon_{cr}}}$$

3)  $\epsilon = 0$  solution is a power law

$$e^{\phi(t)} = \left[ \frac{\pi/4\beta J}{1 + \left( \frac{\pi}{4\beta J} \right)^2 \frac{J^2 t^2}{4}} \right]^2$$

The key point in the above analysis

is that by tuning the Hamiltonian

there are solutions for which

$f(t)$  never vanishes, and solutions

for which  $f(t) = 0$ .

They correspond to geometries without

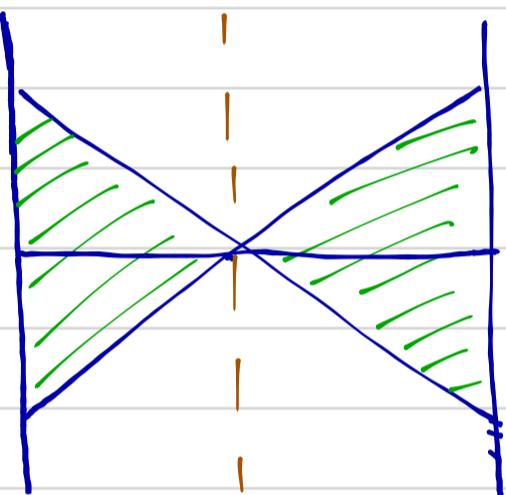
horizons and with horizons respectively.

## Bulk geometry and $|B_s(\lambda)\rangle$

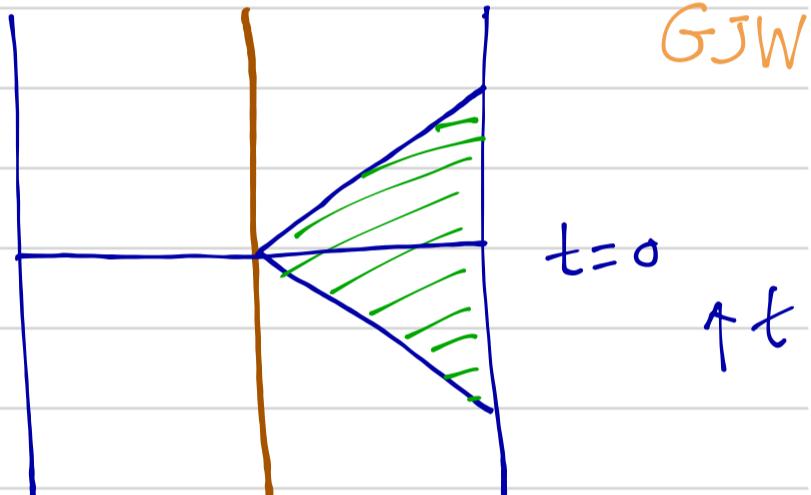
The following can be derived :

$$\langle TFD_\beta | P_{SL} | O_R | TFD_\beta \rangle \propto \langle B_s(\lambda) | O_R | B_s(\lambda) \rangle$$

where  $P_{SL} = |B_s\rangle \langle B_s| \left( \sum_k S_k^L S_k^R \xrightarrow{P_L} \sum_k S_k^L S_k^R \right)$



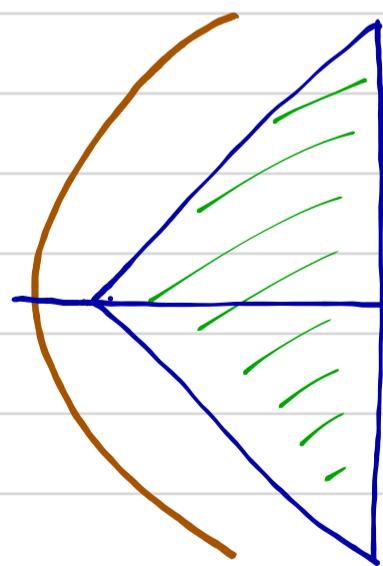
$Z_2$



$t=0$

$\uparrow t$

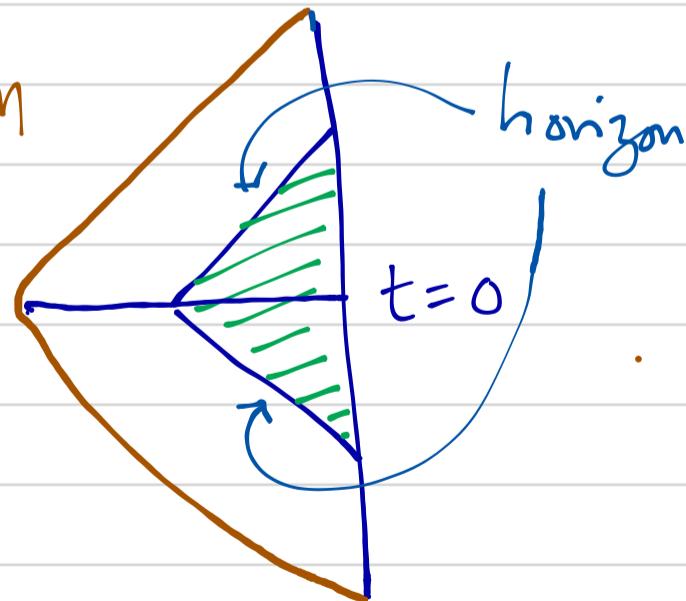
← 'End of the world brane'  
· particle of mass  $M$



$M \rightarrow 2M$



$t=0$



$t=0$

$\uparrow t$

Except for a very high energy particle behind the horizon, geom same as before (Karakostas - Maldacena)

## $f(t)$ dynamics + horizons

Bulk geometry entirely specified by  $f(t)$

i.e. the curve

$$(t, z) \rightarrow (t^{[f]}(t, z), z^{[f]}(t, z))$$

Near the boundary,  $z \approx 0$

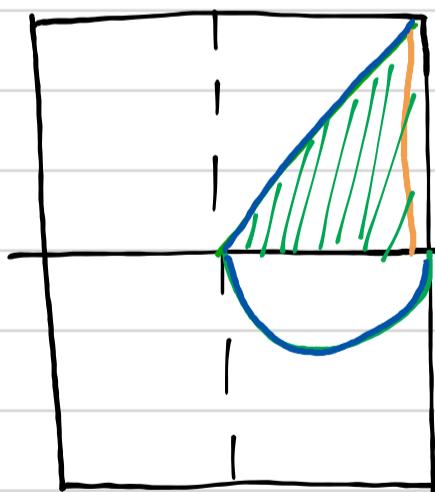
$$t^{[f]}(t, z) \approx f(t), \quad z^{[f]}(t, z) \sim z \dot{f}(t)$$

If  $\dot{f}(t) \neq 0$ ,  $z^{[f]}(t, z) \neq 0$

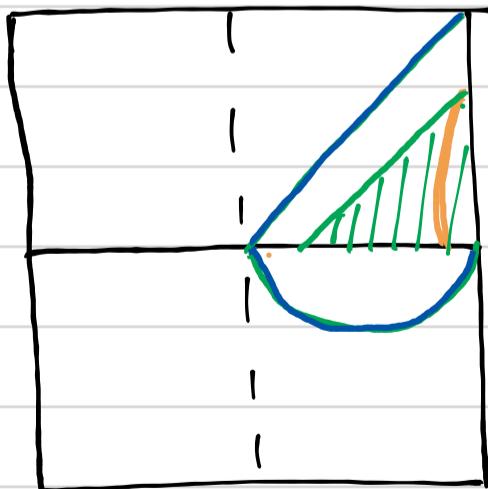
and hence the entire bulk is visible from

the boundary. If  $\dot{f}(t) = 0$ ,  $z^{[f]} = 0$

and a horizon forms.



$t=0$



$t=0$

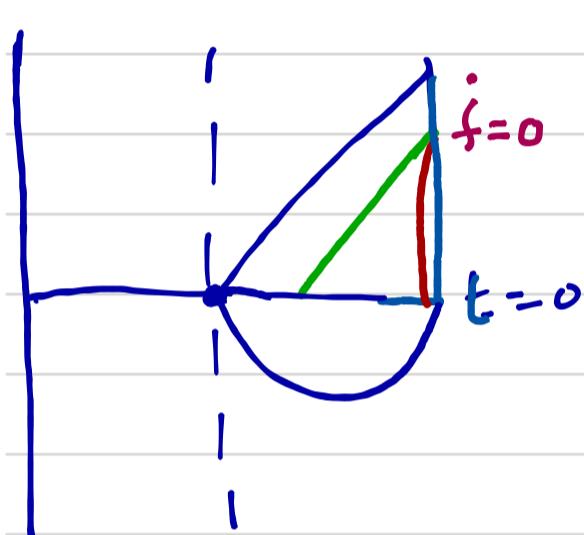
Green line is a horizon.

## Summary

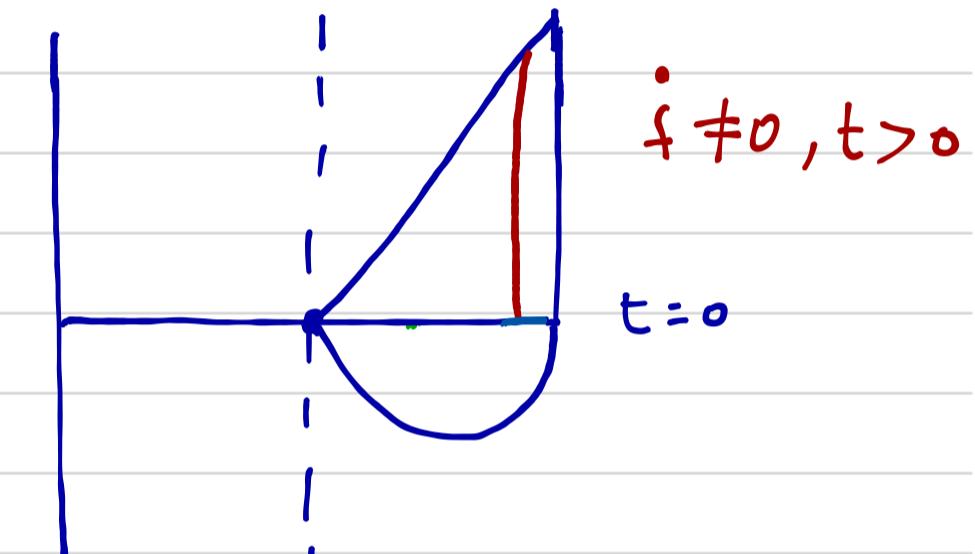
Given a solution for  $\Phi(t)$ , there is an associated time dependent geometry.

$$ds^2 = -\frac{dt^2}{z^2} \left(1 - \frac{z^2}{2} \left[\frac{J^2}{4} e^{\frac{\Phi(t)}{2}} - E\right]\right)^2 + \frac{dz^2}{z^2}$$


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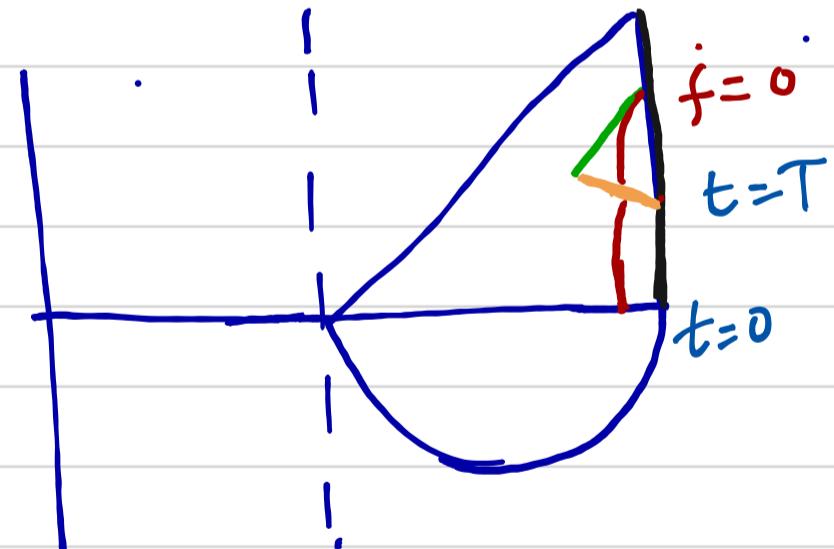


$$H = H_{SYK}$$



$$H = H_{SYK} + \epsilon_1 H_M \quad t \geq 0$$

$$\epsilon_1 > \frac{\pi}{\beta J}$$

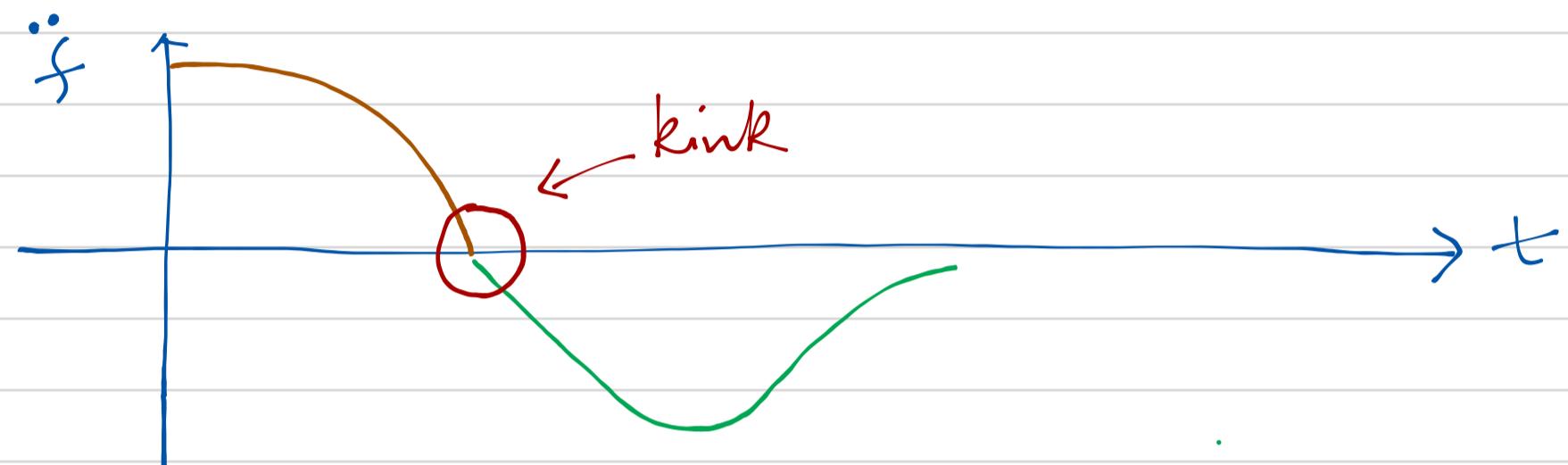


$$H = \begin{cases} H_{SYK} + \epsilon_1 H_M, & 0 \leq t \leq T \\ H_{SYK} + \epsilon_2 H_M, & 0 \leq T < \infty \\ \epsilon_1 > \frac{\pi}{\beta J}, \quad \epsilon_2 < \epsilon_M \end{cases}$$

## Concluding remarks

1. 2-pt functions at the crossover (at  $t=\bar{t}$ )

reflect the discontinuity of  $\phi$  and  $f$



2. The growth of the OTOC 4-pt. function

$$t \ll \bar{t}, \quad \sim \exp[2\pi Jt] \leftarrow (\text{Roberts, Stanford, Streicher})$$

$$t \gg \bar{t}, \quad \sim \exp\left[\frac{2\pi}{\beta_{bh}} t\right]$$

3. Bulk description incomplete because we

do not know the bulk matter coupling to

$$\in J \int dt (f')^{2\alpha}$$

Dilaton eqn can be solved only near the boundary

## 4. Hawking radiation and evaporation