Entanglement Entropy, Horizons and Holography

Nikolaos Tetradis

University of Athens

- We discuss the relation between entanglement entropy and the entropy in spaces that contain horizons.
- The divergent part of the entanglement entropy scales with the area of the entangling surface (Bombelli, Koul, Lee, Sorkin 1986, Srednicki 1993) .
- This feature suggests a connection with the entropy of the gravitational background when the entangling surface is identified with the horizon.

- In the framework of the AdS/CFT correspondence, we consider parametrizations for which the boundary metric takes the Rindler and static de Sitter form.
- We study the entanglement entropy for a CFT confined within a part of the AdS boundary delimited by an entangling surface $\mathcal{A}$.
- The entropy is proportional to the area of a minimal surface that starts from $\mathcal{A}$ and extends into the bulk (Ryu, Takayanagi 2006).
- We compute the entropy by indentifying the entangling surface $\mathcal{A}$ with the horizon of the boundary metric.

Plan

- Rindler entropy
- de Sitter entropy
- Generalizations
- Conclusions
- D. Giataganas, N. T. arXiv:1904.13119[hep-th], Phys. Lett. B 796, 88 (2019)

Rindler slicing of (d + 2)-dimensional AdS space

- Global coordinates:

$$\mathrm{ds}^2_{\mathrm{d}+2} = \frac{\mathrm{R}^2}{\cos^2\chi}\left[-\mathrm{d}\tau^2 + \mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega^2_{\mathrm{d}-1}\right)\right],$$

  $\mathrm{d} > 1$: $-\infty < \tau < \infty$, $0 \le \chi < \pi/2$, $-\pi/2 \le \theta \le \pi/2$.
  $\mathrm{d} = 1$: θ covers the full unit circle: $-3\pi/2 \le \theta \le \pi/2$.

- Fefferman-Graham coordinates with a Rindler boundary:

$$\mathrm{ds}^2_{\mathrm{d}+2} = \frac{\mathrm{R}^2}{\mathrm{z}^2}\left[\mathrm{dz}^2 - \mathrm{a}^2\mathrm{y}^2\mathrm{d}\eta^2 + \mathrm{dy}^2 + \mathrm{d}\vec{\mathrm{x}}_{\mathrm{d}-1}\right],$$

  with $-\infty < \eta < \infty$.
  $0 < \mathrm{y} < \infty$ covers the right (R) Rindler wedge
  $-\infty < \mathrm{y} < 0$ covers the left (L) wedge.

- All the coordinates in the above expressions are dimensionless, with R the only dimensionful parameter.

- Relation between the Euclidean Rindler and global coordinates for $\mathrm{d} = 1$:

$$
\begin{aligned}
\chi(\mathrm{z}, \eta_{\mathrm{E}}, \mathrm{y}) &= \tan^{-1}\left(\frac{1}{\mathrm{z}}\sqrt{\mathrm{y}^2\cos^2(\mathrm{a}\,\eta_{\mathrm{E}}) + \frac{1}{4}\left(\mathrm{y}^2+\mathrm{z}^2-1\right)^2}\right) \\
\tau_{\mathrm{E}}(\mathrm{z}, \eta_{\mathrm{E}}, \mathrm{y}) &= \tanh^{-1}\left(\frac{2\mathrm{y}\sin(\mathrm{a}\,\eta_{\mathrm{E}})}{\mathrm{y}^2+\mathrm{z}^2+1}\right) \\
\theta(\mathrm{z}, \eta_{\mathrm{E}}, \mathrm{y}) &= \tan^{-1}\left(\frac{\mathrm{y}^2+\mathrm{z}^2-1}{2\mathrm{y}\cos(\mathrm{a}\,\eta_{\mathrm{E}})}\right).
\end{aligned}
$$

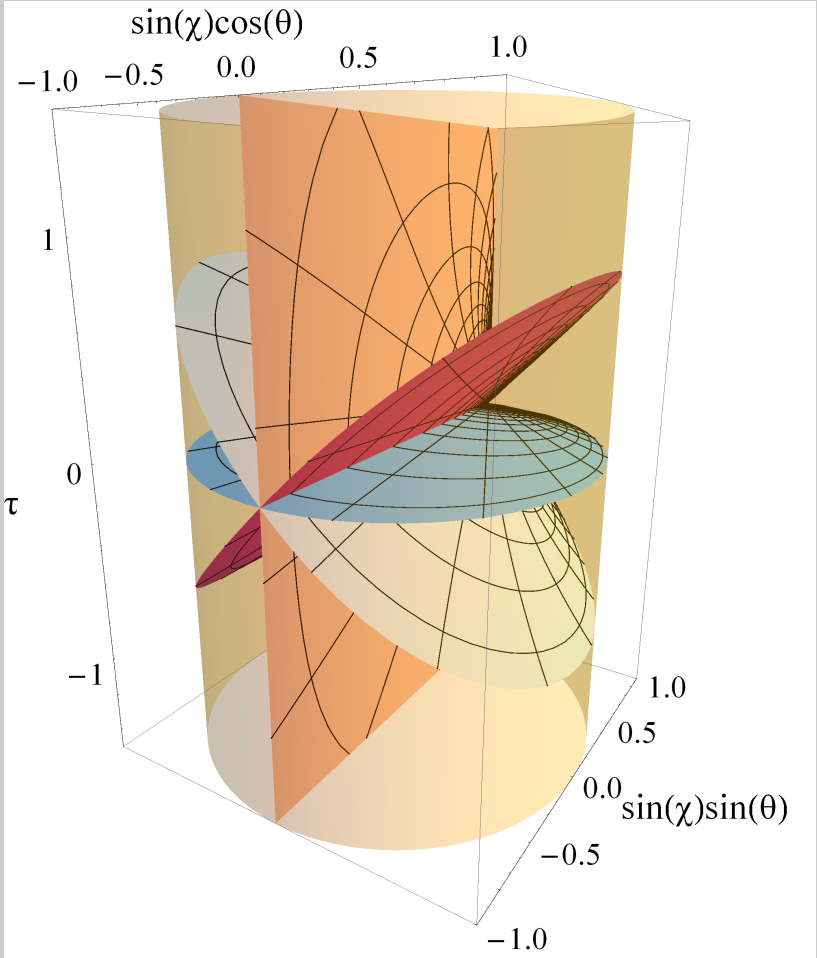

Figure: The slicing of the Euclidean AdS_3 cylinder for a Rindler boundary with $a = 1$.

- For $\eta = 0$ (or Minkowski time $t = 0$), the coordinate $y = x > 0$ covers the positive x-axis of Minkowski space, and $y = x < 0$ the negative x-axis.
- The two regions are separated by the horizon at $y = 0$.
- There is entanglement between the states localized in the right wedge ($y > 0$) and the states in the left wedge ($y < 0$).
- Calculate the entanglement entropy through holography.
- Consider a strip with width ℓ along the y-axis and infinite length along the perpendicular directions. The minimal surface extends into the bulk up to a turning point at

$$z_* = \frac{\Gamma\left(1/2d\right)}{2\sqrt{\pi}\,\Gamma\left(d + 1/2d\right)}\,\ell.$$

- The holographic entanglement entropy is obtained by dividing the area of the minimal surface by $4G_{d+2}$:

$$S_A = \frac{2R(R^{d-1}L^{d-1})}{4G_{d+2}} \left(\frac{1}{(d-1)\epsilon^{d-1}} + \frac{\sqrt{\pi}}{2d} \frac{\Gamma\left(\frac{1-d}{2d}\right)}{\Gamma\left(\frac{1}{2d}\right)} \frac{1}{z_*^{d-1}} \right).$$

- ϵ is a cutoff imposed on z near the boundary.
- L is the large length of the directions perpendicular to the strip, and $R^{d-1}L^{d-1}$ the corresponding volume.
- For $d = 1$, one must substitute $1/((d-1)\epsilon^{d-1})$ with $\log(1/\epsilon)$. The term in the parenthesis becomes $\log(\ell/\epsilon)$.

- Take the limit in which the strip covers the whole positive axis.
- The entropy arises from the entanglement of the right wedge with the left wedge of Rindler space.
- As $z_* \to \infty$, only the first term survives. This term is strongly dependent on the cutoff and independent of the strip width ℓ.
- Define the effective Newton's constant as (Hawking, Maldacena, Strominger 2001)

$$G_{d+1} = (d-1)\epsilon^{d-1}\frac{G_{d+2}}{R},$$

with $(d-1)\epsilon^{d-1}$ replaced by $1/\log(1/\epsilon)$ for $d = 1$.

- A natural definition within an effective theory with a cutoff, such as the RS model (Randall, Sundrum 1999), after correcting for a factor of 2 from the two copies of AdS space.
- Alternatively, consider the regulated theory in holographic renormalization (de Haro, Solodukhin, Skenderis 2001).

- The strip extends from the horizon at $y = 0$ to a value y_m for which the limit $y_m \to \infty$ is taken.
- For any finite value of y_m the strip is entangled not only with the left wedge, but also with the (infinite domain) beyond y_m.
- As the space is essentially flat, the two contributions to the entropy are expected to be equal.
- If one is interested in the entanglement with the left wedge only, the limit $y_m \to \infty$ must be accompagnied by a division by 2 of the computed entanglement entropy.
- There is only one horizon on the boundary.
- The final result for the Rindler entropy is

$$S_R = \frac{R^{d-1} L^{d-1}}{4 G_{d+1}},$$

in agreement with Laflamme 1987.

dS slicing of (d + 2)-dimensional AdS space

- Fefferman-Graham coordinates with a static dS boundary:

$$\begin{aligned} ds^2_{d+2} &= \frac{R^2}{z^2}\left[dz^2 + \left(1 - \frac{1}{4}H^2 z^2\right)^2 \times \right. \\ &\left. \left(-(1 - H^2\rho^2)dt^2 + \frac{d\rho^2}{1 - H^2\rho^2} + \rho^2\, d\Omega^2_{d-1}\right)\right]. \end{aligned}$$

  $d > 1$: $0 \leq \rho \leq 1/H$ covers the static patch. There are two such patches, with $\rho = 0$ for the "North" and "South pole".
  $d = 1$: each static patch is covered by $-1/H \leq \rho \leq 1/H$.

- All the coordinates are dimensionless, with R the only dimensionful parameter. The physical Hubble scale is H/R.

- Relation between the Euclidean de Sitter and global coordinates for $\mathrm{d} = 1$:

$$
\begin{aligned}
\chi(\mathrm{z}, \mathrm{t_E}, \rho) &= \tan^{-1}\left(\frac{1-\frac{1}{4}\mathrm{H}^2\mathrm{z}^2}{\mathrm{Hz}}\sqrt{\cos^2(\mathrm{Ht_E}) + \mathrm{H}^2\rho^2\sin^2(\mathrm{Ht_E})}\right) \\
\tau_{\mathrm{E}}(\mathrm{z}, \mathrm{t_E}, \rho) &= \tanh^{-1}\left(\frac{1-\frac{1}{4}\mathrm{H}^2\mathrm{z}^2}{1+\frac{1}{4}\mathrm{H}^2\mathrm{z}^2}\sin(\mathrm{Ht_E})\sqrt{1-\mathrm{H}^2\rho^2}\right) \\
\theta(\mathrm{z}, \mathrm{t_E}, \rho) &= \tan^{-1}\left(\frac{\mathrm{H}\rho}{\sqrt{1-\mathrm{H}^2\rho^2}\,\cos(\mathrm{Ht_E})}\right).
\end{aligned}
$$

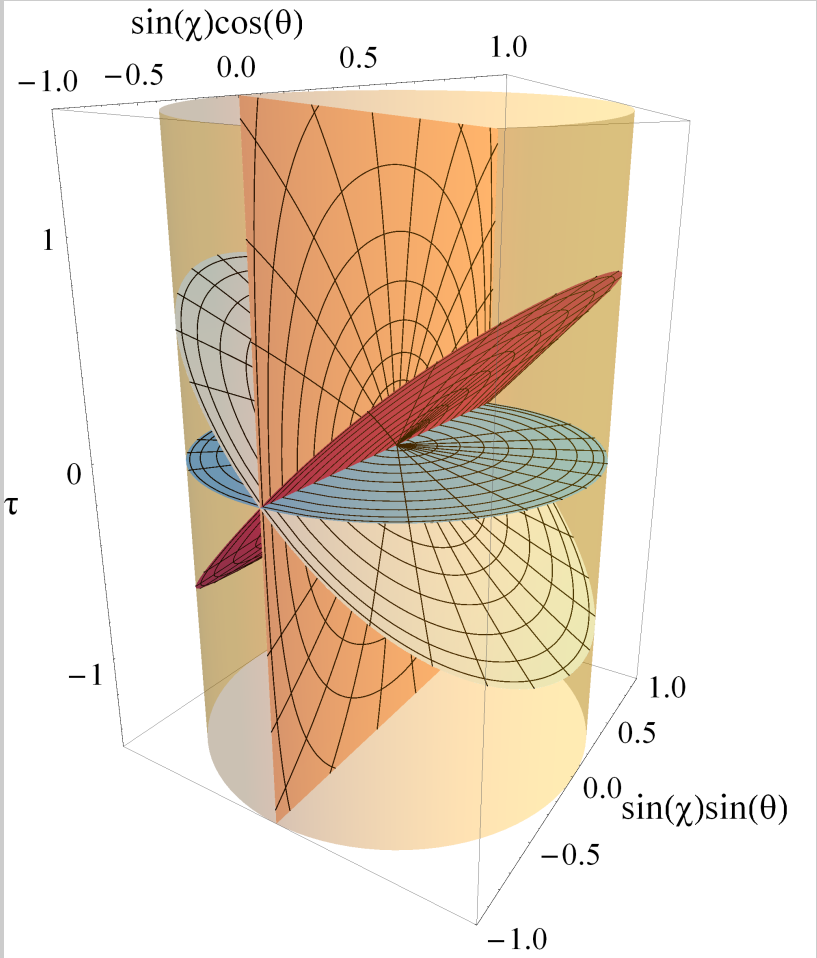

Figure: The slicing of the Euclidean AdS_3 cylinder for a static de Sitter boundary with $H = 1$.

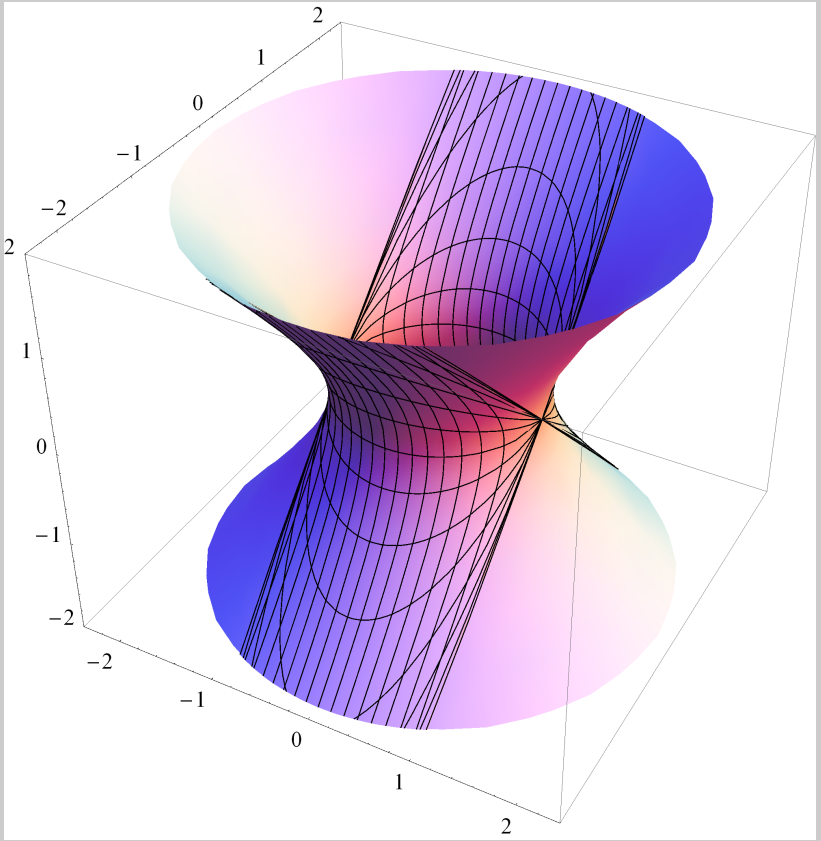

Figure: Static dS_2.

- For the entanglement entropy, we consider the interior of a spherical entangling surface on the boundary. (For $d = 1$ we consider a line segment between the two horizons at $\rho = \pm 1/H$.)
- The minimal surface in the bulk can be determined through the minimization of the area

$$A = R^d S^{d-1} \int d\rho\, \rho^{d-1} \frac{\left(1 - \frac{1}{4}H^2 z^2\right)^{d-1}}{z^d} \sqrt{\frac{\left(1 - \frac{1}{4}H^2 z^2\right)^2}{1 - H^2\rho^2} + \left(\frac{dz(\rho)}{d\rho}\right)^2},$$

with S^{d-1} the volume of the $(d-1)$-dimensional unit sphere.

- Through the definitions $\sigma = \sin^{-1}(H\rho)$, $w = 2\tanh^{-1}(Hz/2)$, the above expression becomes

$$A = R^d S^{d-1} \int d\sigma \frac{\sin^{d-1}(\sigma)}{\sinh^d(w)} \sqrt{1 + \left(\frac{dw(\sigma)}{d\sigma}\right)^2}.$$

- Minimization of the area results in the differential equation

$$\tan(\sigma)\tanh(\mathrm{w})\,\mathrm{w}''+(\mathrm{d}-1)\tanh(\mathrm{w})\left((\mathrm{w}')^3+\mathrm{w}'\right)+\mathrm{d}\tan(\sigma)\left((\mathrm{w}')^2+1\right)=0$$

- The solution is

$$\mathrm{w}(\sigma)=2\tanh^{-1}\left(\sqrt{\frac{2+\cos(2\sigma)-4\cos(\sigma)\cos(\sigma_0)+\cos(2\sigma_0)}{\cos(2\sigma)-\cos(2\sigma_0)}}\right)$$

- For $\mathrm{H}\to 0$ and small σ, w we recover the known equation for the minimal surface in the case of a Minkowski boundary. Its solution is $\mathrm{z}(\rho)=\sqrt{\rho_0^2-\rho^2}$, with ρ_0 the radius of the entangling surface on the boundary.

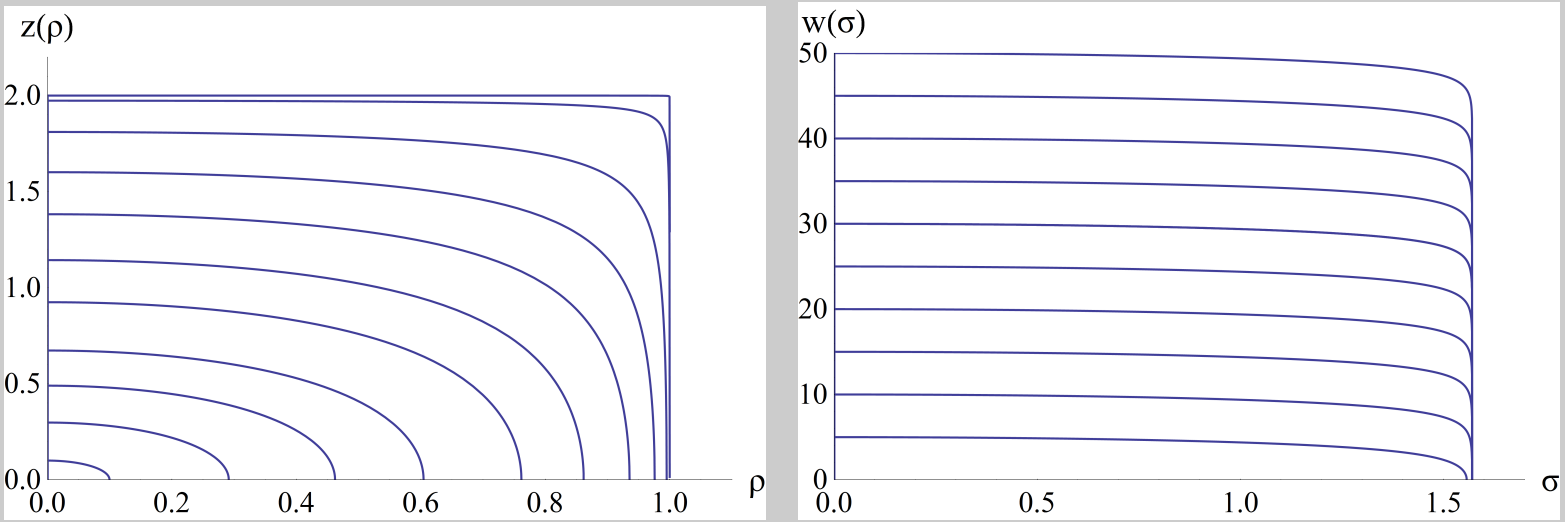

Figure: Minimal surfaces for a de Sitter boundary with H = 1.

- The area of the minimal surface is dominated by the region near the boundary. Cutting off at $z = \epsilon$, the leading contribution is

$$\mathrm{Area}(\gamma_A) = R^d S^{d-1} \int_{H\epsilon} \frac{dw}{w^d} = \frac{R^d S^{d-1}}{(d-1) H^{d-1} \epsilon^{d-1}}.$$

- The entropy becomes

$$S_{dS} = \frac{R^d S^{d-1}}{4 G_{d+2} (d-1) H^{d-1} \epsilon^{d-1}} = \frac{S^{d-1}}{4 G_{d+1}} \left(\frac{R}{H}\right)^{d-1},$$

  which reproduces the entropy of Gibbons, Hawking 1977.
- The above expressions are valid for $d = 1$ as well, with $1/((d-1)\epsilon^{d-1})$ replaced by $\log(1/\epsilon)$.
- No division by 2 is necessary. For $d = 1$ there are two horizons on the boundary.

- Is the identification with the entanglement entropy also valid for more general bulk gravitational theories?
- Consdider Gauss-Bonnet bulk gravity, which is dual to a CFT with a more general class of central charges. (Buchel, Escobedo, Myers, Paulos, Sinha, Smolkin 2010).
- The theory admits an AdS bulk solution of the form

$$ds^2_{d+2} = \frac{R^2}{z^2}\left[\frac{dz^2}{f} - dt^2 + d\vec{x}_d\right],$$

where $f = (1+\sqrt{1-4\lambda})/(2\lambda)$, with λ the GB coupling.

- The AdS radius is equal to $\tilde{R} = R/\sqrt{f}$.
- The effective Newton's constant is (Myers, Pourhasan, Smolkin 2013)

$$\tilde{G}_{d+1} = (d-1)\epsilon^{d-1}\frac{G_{d+2}}{(1+2\lambda f)\tilde{R}}.$$

- The holographic calculation of the entanglement entropy requires a modified bulk functional (Hung, Myers, Smolkin 2011, Camps 2014), which generalizes the Wald entropy (Wald 1993).
- For an entangling surface with a strip geometry on a flat boundary, the leading contribution is (Myers, Singh 2012)

$$\tilde{S}_A = \frac{(1+2\lambda f)\tilde{R}^d L^{d-1}}{2(d-1)\epsilon^{d-1} G_{d+2}},$$

- The entropy for a Rindler boundary is

$$\tilde{S}_R = \frac{\tilde{R}^{d-1} L^{d-1}}{4\tilde{G}_{d+1}}.$$

- The divergences of the total area of the minimal surface are included in the integral

$$A = R^d S^{d-1} I(\epsilon) = R^d S^{d-1} \int_{H\epsilon} \frac{dw}{\sinh^d(w)}.$$

- For $d = 3$ we have

$$I(\epsilon) = \frac{1}{2H^2\epsilon^2} + \frac{1}{2}\log(H\epsilon) + \mathcal{O}(\epsilon^2).$$

- The dS entropy in four dimensions is proportional to the area of the horizon, with a coefficient that contains a logarithmic correction:

$$S_{dS} = \frac{A}{4G_4}\left(1 + H^2\epsilon^2 \log H\epsilon\right).$$

- What is the origin of the correction? It must be attributed to higher-curvature terms in the effective gravitational action.
- Use known results from holographic renormalization (de Haro, Solodukhin, Skenderis 2001). The bulk metric of a five-dimensional asymptotically AdS space is written as

$$\begin{aligned} ds^2 &= \frac{R^2}{z^2}\left(dz^2 + g_{ij}(x,z)dx^i dx^j\right) \\ g(x,z) &= g_{(0)} + z^2 g_{(2)} + z^4 g_{(4)} + h_{(4)} z^4 \log z^2 + \mathcal{O}(z^5). \end{aligned}$$

- A solution is then obtained order by order.
- The on-shell gravitational action is regulated by restricting the bulk integral to the region $z > \epsilon$.
- The divergent terms are subtracted through the introduction of appropriate counterterms. An effective action is obtained in terms of the induced metric γ_{ij} on the surface at $z = \epsilon$.

- The holographic entanglement entropy displays similar divergences for $\epsilon \to 0$.
- In our approach the entropy is not renormalized, but we incorporate the ϵ-dependence in the effective couplings.
- We employ the regulated form of the effective action before the subtraction of divergences.
- The induced metric γ_{ij} includes a factor ϵ^{-2} relative to g_{ij}. We redefine γ_{ij} by extracting this factor: $\gamma_{ij} \to \epsilon^2 \gamma_{ij}$. In this way γ, $g_{(0)}$, $g_{(2)}$ etc are all of the same order.

- The regulated action is

$$S = \frac{1}{16\pi G_5}\int d^4x\sqrt{-\gamma}\left[-\frac{6}{R\epsilon^4}+\frac{R}{2\epsilon^2}\mathcal{R}-\frac{R^3}{4}\log\epsilon\left(\mathcal{R}_{ij}\mathcal{R}^{ij}-\frac{1}{3}\mathcal{R}^2\right)\right].$$

- The first term is a cosmological constant, which must be (partially) cancelled by vacuum energy on the surface $z=\epsilon$.
- The second term is the Einstein term if the effective Newton's constant G_4 is defined as before.
- The third term is responsible for the holographic conformal anomaly.
- The structure of the effective action is the same for the RS model, up to a rescaling of z (Myers, Pourhasan, Smolkin 2013).

- The effective action supports a dS solution. The entropy must take into account the presence of the third term.
- The Wald entropy gives the horizon entropy in theories with higher curvature interactions (Wald 1993):

$$S_{\rm Wald} = \frac{A}{4G_4} - \frac{R^3}{32G_5} \log \epsilon \int_{\mathcal{A}} d^2y \sqrt{h} \left(2\mathcal{R}^{ij}\gamma^{\perp}_{ij} - \frac{4}{3}\mathcal{R} \right).$$

  The integration is over the horizon, with induced metric h, and $\gamma^{\perp}$ the metric in the transverse space.
- For a dS background we obtain

$$S_{\rm Wald} = \frac{A}{4G_4} \left(1 + H^2\epsilon^2 \log \epsilon\right).$$

- The correction to the dS entropy agrees with the singular part of the correction provided by the holographic calculation.

- Within a construction that implements a physical UV cutoff, such as the RS model (Randall, Sundrum 1999), the effective Newton's constant in the Einstein action arises through the integration of the bulk degrees of freedom.
- In the general context of the AdS/CFT correspondence the bulk degrees of freedom correspond to matter fields of the dual theory.
- The picture is consistent with the expectation that the entropy associated with gravitational horizons can be understood as entanglement entropy if Newton's constant is induced by quantum fluctuations of matter fields (Jacobson 1994).
- However, the validity could be more general. According to Dvali 2008, in a theory with N^2 species there is a UV cutoff $\Lambda^2 \sim N^2/G_4$. Keeping Λ fixed, one concludes that $G_4 \sim N^2$.