Mutual Information and Area Law at Finite Temperature

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Entanglement Entropy and Area Law

At free scalar field theory at its ground state, it is well known that *entanglement entropy obeys an area law*[1].
This property is a consequence of *locality* and the *symmetric property* of entanglement entropy, which enforces it to depend on the common characteristics of the considered subsystems, i.e. the entangling surface.
This property resembles the famous property of black hole entropy, feeding a scenario of emergent entropic gravitational interactions due to entanglement statistics.

[1] Entropy and area, Mark Srednicki, Phys.Rev.Lett. 71 (1993) 666-669

Entanglement Entropy at Mixed States

However, when mixed states are considered (e.g. thermal states), the entanglement entropy ceases having the symmetric property and receives contributions *proportional to the volume* of the considered subsystem, originating from the classical statistics of the *overall* system.
This is the very reason that entanglement entropy is *not* a good measure of entanglement for mixed states. These contributions have *nothing to do with the correlations of the considered subsystems*.

Mutual Information

For mixed states, the simplest measure of entanglement is *the mutual information*, which has the symmetric property by definition. It is defined as

$$I\left(A, A^C\right) := S_A + S_{A^C} - S_{A \cup A^C}.$$

So does it obey an area law in free scalar field theory at finite temperature?

The Harmonic Oscillator at Finite T

In coordinate representation, the energy eigenstates and the corresponding eigenvalues of the harmonic oscillator are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \sqrt[4]{\frac{\omega}{\pi}} e^{-\frac{\omega x^2}{2}} H_n\left(\sqrt{\omega} x\right), \quad E_n = \omega\left(n + \frac{1}{2}\right).$$

The density matrix describing a quantum harmonic oscillator at finite temperature T is given by

$$\rho(x, x') = \sum_{n=0}^{\infty} 2 \sinh \frac{\omega}{2T} e^{-\frac{\omega}{T}\left(n+\frac{1}{2}\right)} \frac{1}{2^n n!} \sqrt{\frac{\omega}{\pi}} \times e^{-\frac{\omega\left(x^2+x'^2\right)}{2}} H_n\left(\sqrt{\omega} x\right) H_n\left(\sqrt{\omega} x'\right).$$

The Harmonic Oscillator at Finite T

As a consequence of Mehler's formula,

$$\sum_{n=0}^{\infty} \frac{H_n(x)\, H_n(y)}{n!} \left(\frac{w}{2}\right)^n = \frac{1}{\sqrt{1-w^2}} e^{\frac{2xyw-\left(x^2+y^2\right)w^2}{1-w^2}},$$

the density matrix ρ assumes the form

$$\rho\left(x, x'\right) = \sqrt{\frac{\omega}{\pi}(a+b)} e^{-\frac{a\left(x^2+x'^2\right)}{2}} e^{-bxx'},$$

where $a \equiv \omega \coth \frac{\omega}{T}$ and $b \equiv -\omega \operatorname{csch} \frac{\omega}{T}$.
So it is still Gaussian, and, thus, *Srednicki's calculation can be performed as at the ground state*.

The Density Matrix

We first consider two coupled oscillators,

$$H = \frac{1}{2}\left(p^2 + p_C^2 + k_0(x^2 + x_C^2) + k_1(x - x_C)^2\right).$$

The density matrix trivially reads

$$\rho\left(x_+, x_+{}', x_-, x_-{}'\right) = \frac{\sqrt{(a_+ + b_+)(a_- + b_-)}}{\pi} \times e^{-\frac{a_+\left(x_+{}^2 + x_+{}'^2\right) + a_-\left(x_-{}^2 + x_-{}'^2\right)}{2}} e^{-b_+ x_+ x_+{}'} e^{-b_- x_- x_-{}'},$$

where $x_\pm$ are the normal mode coordinates $x_\pm = \frac{x \pm x_C}{\sqrt{2}}$, and $a_\pm \equiv \omega_\pm \coth\frac{\omega_\pm}{T}$, $b_\pm \equiv -\omega_\pm \mathrm{csch}\frac{\omega_\pm}{T}$, where $\omega_+^2 = k_0$, $\omega_-^2 = k_0 + 2k_1$

The Reduced Density Matrix

We trace out the x_C oscillator to obtain the reduced density matrix

$$\rho\left(x,x'\right)=\int dx_C\rho\left(x,x',x_C,x_C\right)=\sqrt{\frac{\gamma-\beta}{\pi}}e^{-\frac{\gamma\left(x^2+x'^2\right)}{2}}e^{\beta xx'},$$

where

$$\gamma-\beta=2\frac{\left(a_+ + b_+\right)\left(a_- + b_-\right)}{a_+ + a_- + b_+ + b_-},\quad \gamma+\beta=\frac{1}{2}\left(a_+ + a_- - b_+ - b_-\right).$$

The eigenfunctions read

$$f_n\left(x\right)=H_n\left(\sqrt{\alpha}x\right)e^{-\frac{\alpha x^2}{2}},\quad \text{where}\quad \alpha\equiv\sqrt{\gamma^2-\beta^2}.$$

The Spectrum of the Reduced Density Matrix

The respective eigenvalues are

$$p_n = \left(1 - \frac{\beta}{\gamma + \alpha}\right)\left(\frac{\beta}{\gamma + \alpha}\right)^n \equiv (1 - \xi)\,\xi^n,$$

where

$$\xi \equiv \frac{\beta}{\gamma + \alpha} = \frac{\sqrt{\frac{\gamma+\beta}{\gamma-\beta}} - 1}{\sqrt{\frac{\gamma+\beta}{\gamma-\beta}} + 1}.$$

Entanglement Entropy and Mutual Information

The entanglement entropy equals to

$$S_{EE} = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln\xi.$$

In the case of the two coupled oscillators, it holds that $S_{A^C} = S_A$, due to the symmetry of the system. Thus, the mutual information is given by,

$$I\left(A : A^C\right) = 2S_A - S_{\text{th}},$$

where S_{th} it the thermal entropy of the overall system.

An Effective Oscillator

The reduced density matrix is identical to the density matrix of a single harmonic oscillator, with eigenfrequency.

$$\omega_{\text{eff}} = \alpha \quad \text{at temperature} \quad T_{\text{eff}} = -\frac{\alpha}{\ln \xi}.$$

There is no experiment that can be performed to the one of the two coupled oscillators that can distinguish it from this single *effective* harmonic oscillator.

An Effective Oscillator - High T

At high temperatures,

$$\omega_{\text{eff}} = \sqrt{\frac{2\omega_+^2\omega_-^2}{\omega_+^2+\omega_-^2}}\left[1+\frac{1}{48}\frac{\left(\omega_+^2-\omega_-^2\right)^2}{\omega_+^2+\omega_-^2}\frac{1}{T^2}+\mathcal{O}\left(\frac{1}{T^4}\right)\right],$$

$$T_{\text{eff}} = T\left[1+\frac{1}{24}\frac{\left(\omega_+^2-\omega_-^2\right)^2}{\omega_+^2+\omega_-^2}\frac{1}{T^2}+\mathcal{O}\left(\frac{1}{T^4}\right)\right].$$

At infinite temperature, ω_{eff} tends to the finite value,

$$\omega_{\text{eff}}^{\infty} = \sqrt{\frac{2\omega_+^2\omega_-^2}{\omega_+^2+\omega_-^2}},$$

whereas T_{eff} is dominated by the physical temperature of the composite system.

Entanglement Entropy and Mutual Information High T

At high temperatures, the entanglement entropy and mutual information read

$$S_{EE} = \frac{1}{2}\ln\frac{(k_0+k_1)\,T^2}{k_0\,(k_0+2k_1)} + 1 + \frac{k_0+k_1}{24T^2} + \frac{3k_0^4+12k_0^3k_1+20k_0^2k_1^2+16k_0k_1^3+9k_1^4}{2880(k_0+k_1)^2T^4} + \mathcal{O}\left(\frac{1}{T^6}\right)$$

and

$$I\left(A:A^C\right) = \frac{1}{2}\ln\frac{(k_0+k_1)^2}{k_0\,(k_0+2k_1)} + \frac{k_1^2\,(k_0-k_1)\,(k_0+3k_1)}{1440(k_0+k_1)^2T^4} + \mathcal{O}\left(\frac{1}{T^6}\right),$$

respectively.

Entanglement Entropy and Mutual Information Low T

At small temperatures, the entanglement entropy tends to the zero temperature result, plus exponentially suppressed corrections

$$S_{EE} = S^0_{EE} + \frac{\omega_- + \omega_+}{4T^0_{\text{eff}}}\left(e^{-\frac{\omega_-}{T}} + e^{-\frac{\omega_+}{T}}\right) + \cdots .$$

Similarly, the mutual information is equal to

$$I\left(A : A^C\right) = 2S^0_A + \left(\frac{\omega_- + \omega_+}{2T^0_{\text{eff}}} - \frac{\omega_-}{T} - 1\right) e^{-\frac{\omega_-}{T}} + \left(\frac{\omega_- + \omega_+}{2T^0_{\text{eff}}} - \frac{\omega_+}{T} - 1\right) e^{-\frac{\omega_+}{T}} + \cdots .$$

Mutual Information as Function of the Temperature

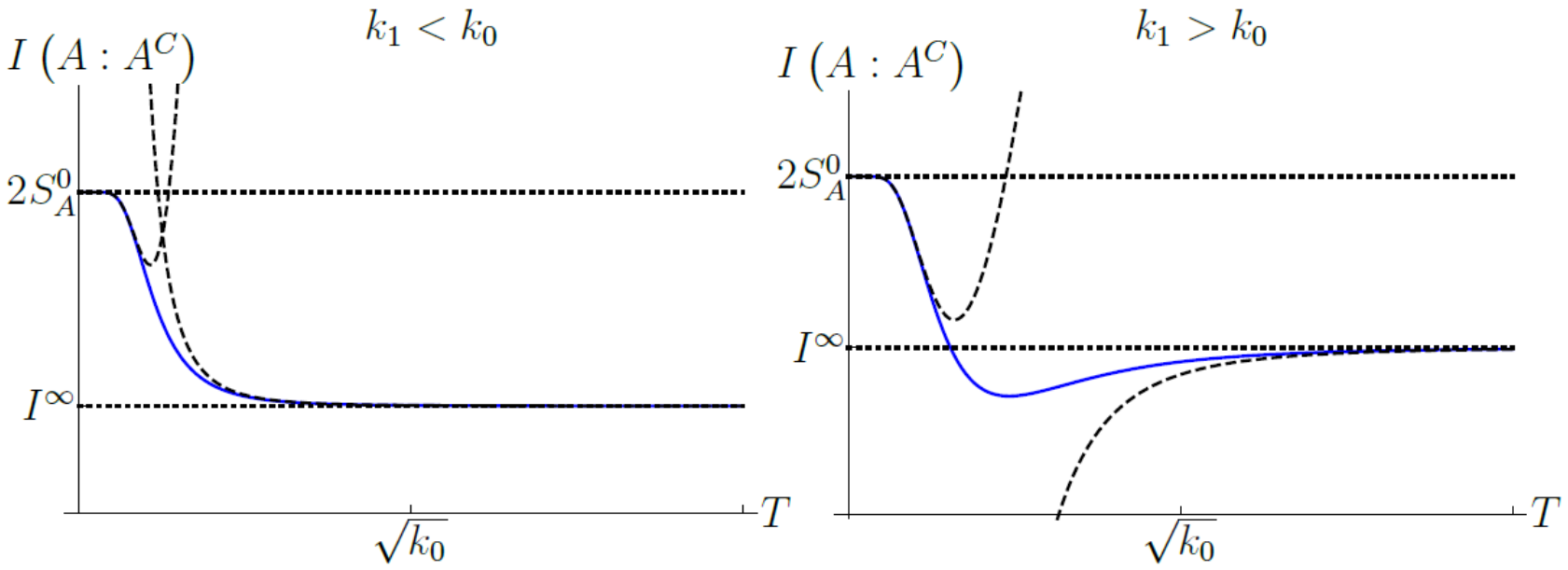


Figure 1 – The mutual information as function of the temperature. The dashed lines are the small and large temperature expansions of the mutual information, whereas the dotted lines are the asymptotic values for $T \to 0$ and $T \to \infty$.

The Mutual Information at Infinite Temperature

The mutual information does not vanish at infinite temperature, but it tends to the value

$$I^{\infty} = \frac{1}{2} \ln \frac{(k_0 + k_1)^2}{k_0 (k_0 + 2k_1)} = 2 \ln \frac{\omega_{\text{eff}}^0}{\omega_{\text{eff}}^{\infty}}.$$

This does not happen at qubit systems. The difference is due to the dimensionality of the Hilbert space.
The mutual information captures both classical and quantum correlations. What is the origin of the remnant?

The Mutual Information at Infinite Temperature

In the classical system, the probability of finding the particle at position x is inverse proportional to the magnitude of its velocity

$$p_E(x) = \frac{\omega}{\pi\sqrt{2E - \omega^2 x^2}}.$$

At finite temperature T the spatial probability distribution is

$$p_{\text{can}}(x; \omega, T) = \int_{\frac{1}{2}\omega^2 x^2}^{\infty} p(E)\, p_E(x)\, dE = \frac{\omega}{\sqrt{2\pi T}} e^{-\frac{\omega^2 x^2}{2T}},$$

where $p(E)$ is the Boltzmann factor.

The Mutual Information at Infinite Temperature

For the two coupled oscillators, working with the normal modes, trivially

$$p\left(x_1, x_2; T\right) = \frac{\omega_+ \omega_-}{2\pi T} e^{-\frac{\omega_+^2 \left(x_1 + x_2\right)^2 + \omega_-^2 \left(x_1 - x_2\right)^2}{4T}}.$$

The probability distribution for the one of the two coupled oscillators can be calculated integrating out the other coordinate. Simple algebra yields

$$p\left(x_1; T\right) = \int p\left(x_1, x_2; T\right) dx_2 = \frac{\omega_{\text{eff}}^{\infty}}{\sqrt{2\pi T}} e^{-\frac{\left(\omega_{\text{eff}}^{\infty}\right)^2 x_1^2}{2T}}.$$

The Mutual Information at Infinite Temperature

It is now straightforward to find the classical version of the "entanglement" entropy

$$S_A^{\mathrm{cl}} = S_{A^C}^{\mathrm{cl}} = -\int p\left(x_1; T\right) \ln p\left(x_1; T\right) dx_1 = \frac{1}{2}\left(1 - \ln \frac{\left(\omega_{\mathrm{eff}}^{\infty}\right)^2}{2\pi T}\right)$$

It follows that the classical mutual information is equal to

$$I^{\mathrm{cl}}\left(A : A^C\right) = \ln \frac{\left(\omega_{\mathrm{eff}}^{0}\right)^2}{\left(\omega_{\mathrm{eff}}^{\infty}\right)^2} = \ln \frac{\omega_+^2 + \omega_-^2}{2\omega_+\omega_-} = I^{\infty}.$$

This *does not depend* on the temperature[2] and it is *equal to the asymptotic value of the quantum mutual information at infinite temperature.*

[2] M. Cramer, et.al., An Entanglement-area law for general bosonic harmonic lattice systems, Phys. Rev. A73, 012309 (2006)

The Density Matrix

We consider a system an N coupled harmonic oscillators

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j,$$

where K symmetric, positive definite.
It is a matter of trivial algebra to show that the density matrix at a thermal state is

$$\rho\left(\mathbf{x},\mathbf{x}'\right) = \sqrt{\frac{\det\left(a+b\right)}{\pi^N}} e^{-\frac{\mathbf{x}^T a\mathbf{x}+\mathbf{x}'^T a\mathbf{x}'}{2}} e^{-\mathbf{x}^T b\mathbf{x}'}.$$

where

$$a = \sqrt{K}\coth\frac{\sqrt{K}}{T}, \quad b = -\sqrt{K}\operatorname{csch}\frac{\sqrt{K}}{T}.$$

Block Notation

We consider as subsystem A the last $N-n$ oscillators. We use the block form notation

$$\mathbf{x} = \begin{pmatrix} x^C \\ x \end{pmatrix}, \quad \text{where} \quad x^C = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_N \end{pmatrix}.$$

We also write any symmetric matrix M in block form, as

$$M = \begin{pmatrix} M_A & M_B \\ M_B^T & M_C \end{pmatrix},$$

where M_A is an $n \times n$ matrix, M_C is an $(N-n) \times (N-n)$ matrix and finally M_B is an $n \times (N-n)$ matrix. The indices A, B and C will always indicate the corresponding blocks of such matrices.

Reduced Density Matrix

We trace out the first n degrees of freedom to find the reduced density matrix for subsystem A.

$$\rho\left(x, x'\right) = \int dx^C \rho\left(\left\{x, x^C\right\}, \left\{x', x^C\right\}\right)$$
$$= \sqrt{\frac{\det\left(\gamma - \beta\right)}{\pi^{N-n}}} e^{-\frac{x^T \gamma x + x'^T \gamma x'}{2}} e^{x^T \beta x'},$$

where

$$\gamma = a_C - \frac{1}{2}\left(a_B^T + b_B^T\right)\left(a_A + b_A\right)^{-1}\left(a_B + b_B\right),$$
$$\beta = -b_C + \frac{1}{2}\left(a_B^T + b_B^T\right)\left(a_A + b_A\right)^{-1}\left(a_B + b_B\right).$$

Entanglement Entropy

Similarly to the ground state case the spectrum reads

$$p_{n_{n+1},\ldots,n_N} = \prod_{i=n+1}^{N} (1-\xi_i)\,\xi_i^{n_i}, \quad n_i \in \mathbb{Z},$$

where the quantities ξ_i are given by

$$\xi_i = \frac{\beta_{Di}}{1+\sqrt{1-\beta_{Di}^2}}$$

and β_{Di} the eigenvalues of the matrix $\gamma^{-1}\beta$. It follows that the entanglement entropy is given by

$$S = \sum_{j=n+1}^{N} \left(-\ln\left(1-\xi_j\right) - \frac{\xi_j}{1-\xi_j} \ln \xi_j \right).$$

High Temperature Expansion

The high temperature expansions of the mutual information is

$$I\left(A:A^C\right) = -\frac{1}{2}\ln\det\left[I-(K_A)^{-1}K_B(K_C)^{-1}K_B^T\right] + \frac{0}{T^2} - \frac{1}{720\,T^4}\left\{\mathrm{Tr}\left[\left(K_B^T(K_A)^{-1}K_B\right)^2\right] + \mathrm{Tr}\left[\left(K_B(K_C)^{-1}K_B^T\right)^2\right] - \frac{1}{2}\mathrm{Tr}\left(K_B^T K_B\right)\right\}.$$

Notice that the $1/T^2$ term *always vanishes*.

Discretization of the Degrees of Freedom

We consider free massive scalar field theory

$$H = \frac{1}{2}\int d^3x \left[\pi^2(\vec{x}) + \left|\vec{\nabla}\varphi(\vec{x})\right|^2 + m^2\varphi^2(\vec{x})\right].$$

We expand in spherical harmonic modes and then introduce a lattice of N spherical shells with radii $r = na$ in order to discretize the degrees of freedom[3]. We arrive at

$$H = \frac{1}{2a}\sum_{\ell,m}\sum_{j=1}^{N}\left[\pi^2_{\ell m,j} + \left(j+\frac{1}{2}\right)^2\left(\frac{\varphi_{\ell m,j+1}}{j+1} - \frac{\varphi_{\ell m,j}}{j}\right)^2 + \left(\frac{\ell(\ell+1)}{j^2} + m^2a^2\right)\varphi^2_{\ell m,j}\right]$$

[3] Entropy and area, Mark Srednicki, Phys.Rev.Lett. 71 (1993) 666-669

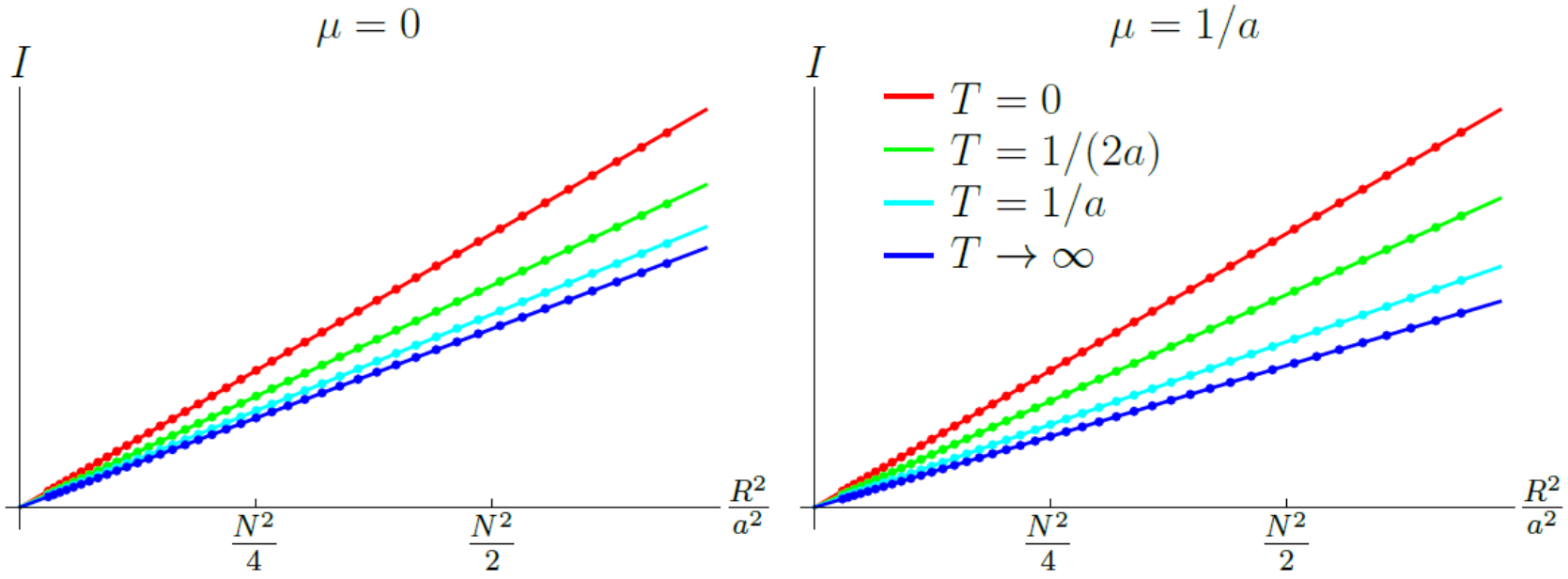
Entanglement Entropy

This procedure has introduced a UV cutoff $1/a$ and an IR cutoff $1/(Na)$.
Different ℓ, m pairs do not mix and m does not appear at all.
Thus, the entanglement entropy and mutual information can be calculated as

$$S_{\mathrm{EE}}/I\,(N,n) = \sum_{\ell=0}^{\infty} (2\ell+1)\, S_\ell/I_\ell\,(N,n)$$

where S_ℓ and I_ℓ can be calculated as in the coupled oscillators case.

Area Law


$\mu = 0$
I
$\frac{N^2}{4}$
$\frac{N^2}{2}$
$\frac{R^2}{a^2}$
$\mu = 1/a$
I
$T = 0$
$T = 1/(2a)$
$T = 1/a$
$T \to \infty$
$\frac{N^2}{4}$
$\frac{N^2}{2}$
$\frac{R^2}{a^2}$

Hopping Expansion

First, we study systems of coupled oscillators, where only adjacent degrees of freedom are coupled, i.e. coupling matrices K of the form

$$K_{ij} = k_i \delta_{ij} + \left(l_i \delta_{i,j+1} + l_j \delta_{i+1,j} \right).$$

Assuming that $k_i \ll l_i$, one may calculate the entanglement entropy and the mutual information perturbatively [4]. In other words, one may define

$$K_{ij} \equiv \frac{1}{\varepsilon} K_{ij}^{(0)} + K_{ij}^{(1)},$$

where $K_{ij}^{(0)} = \varepsilon k_i \delta_{ij}$ and $K_{ij}^{(1)} = l_i \delta_{i,j+1} + l_j \delta_{i+1,j}$ and perform an expansion in ε (or equivalently in l/k).

[4] An Inverse Mass Expansion for Entanglement Entropy in Free Massive Scalar Field Theory, D. Katsinis, G. P., Eur.Phys.J. C78 (2018) no.4, 282

Definitions

We define the functions

$$
\begin{aligned}
f_1(x) &= \sqrt{x}\coth\sqrt{x}, \quad \text{so that} \quad a = f_1(K), \\
f_2(x) &= -\sqrt{x}\operatorname{csch}\sqrt{x}, \quad \text{so that} \quad b = f_2(K), \\
f_3(x) &= f_1(x) + f_2(x) = \sqrt{x}\tanh\left(\sqrt{x}/2\right), \\
f_4(x) &= -f_2(x)/f_1(x) = \operatorname{sech}\sqrt{x}.
\end{aligned}
$$

The Matrix $\gamma^{-1}\beta$

Expanding the matrix $\gamma^{-1}\beta$ in ε,

$$\gamma^{-1}\beta = \left(\gamma^{-1}\beta\right)^{(0)} + \varepsilon\left(\gamma^{-1}\beta\right)^{(1)} + \varepsilon^2\left(\gamma^{-1}\beta\right)^{(2)} + \mathcal{O}\left(\varepsilon^3\right),$$

one can show that the zeroth and first order terms are given by

$$\left(\gamma^{-1}\beta\right)_i^{0(0)} = f_4\left(\frac{k_{n+i}}{T^2}\right)$$

and

$$\left(\gamma^{-1}\beta\right)_i^{\pm 1(1)} = \frac{l_{n+i}}{k_{n+i} - k_{n+i+1}}\left(f_4\left(\frac{k_{n+i}}{T^2}\right) - f_4\left(\frac{k_{n+i+1}}{T^2}\right)\right)$$

and all other elements are vanishing. The second order result is given by a little more complicated expressions.

The Hierarchy of $\gamma^{-1}\beta$

- We want to specify the eigenvalues of the matrix $\gamma^{-1}\beta$ in the ε expansion. The problem is more difficult than the zero temperature problem.
- In the ground state case, the elements of the matrix $\gamma^{-1}\beta$ obey an hierarchy in both its directions, i.e. the leading contribution to the element $\left(\gamma^{-1}\beta\right)_{ij}$ are of order $i+j$.
- However, in the case of finite temperature, the thermal contributions have changed the structure of the matrix $\gamma^{-1}\beta$. The leading contribution to the element $\left(\gamma^{-1}\beta\right)_{ij}$ is of order $i-j$.

The Hierarchy of $\gamma^{-1}\beta$

- The appropriate perturbative approach depends on the ratio of the non-diagonal elements to the *differences* of the diagonal ones.
- When this ratio is small non-degenerate perturbation theory applies
- When this ratio is large degenerate perturbation theory applies

Non-Degenerate Perturbation Theory

The contributions to the entanglement entropy from the eigenvalues are identical to those of the thermal entropy, apart from the two eigenvalues corresponding to the eigenvectors localized at the boundary of the two subsystems at first order in ε. It turns out that

$$I = \frac{l_n^2}{4T^2\left(k_n - k_{n+1}\right)}\left(\frac{1}{f_3\left(\frac{k_{n+1}}{T^2}\right)} - \frac{1}{f_3\left(\frac{k_n}{T^2}\right)}\right) + \mathcal{O}\left(\varepsilon^3\right).$$

Expanding for large temperatures the above result yields

$$I = \frac{l_n^2}{2k_n k_{n+1}} + \frac{l_n^2}{1440T^4} + \mathcal{O}\left(\frac{1}{T^6}\right),$$

which is identical to the ε expansion of the high temperature expansion of the generic oscillatory system.

Degenerate Perturbation Theory

We will focus on a subclass of this kind of problems that emerges from the discretization of $1+1$ dimensional field theory, namely the case where the matrix K is of the form

$$k_i = k, \quad l_i = l.$$

Degenerate Perturbation Theory

A similar cancellation between the contributions from all eigenvalues, occurs in this case too.

$$I = \frac{l^2}{16k^{\frac{3}{2}}T^2}\operatorname{csch}^2\frac{\sqrt{k}}{2T}\left(\sqrt{k} + T\sinh\frac{\sqrt{k}}{T}\right) + \mathcal{O}\left(l^3\right).$$

The above formula is exactly the smooth limit of the latter as $k_i \to k$ and $l_i \to l$, i.e.

$$I = -\frac{l^2}{4T^2}\frac{d}{dk}\left(\frac{1}{f_3\left(\frac{k}{T^2}\right)}\right) + \mathcal{O}\left(l^3\right).$$

The non-degenerate and degenerate perturbative calculations resulted in different results for the entanglement entropy, but in the *same result for the mutual information*.

Low Temperature Expansion

The $1/m$ expansion fails at low temperatures, but unlike the generic oscillatory system, a low temperature expansion can be obtained.

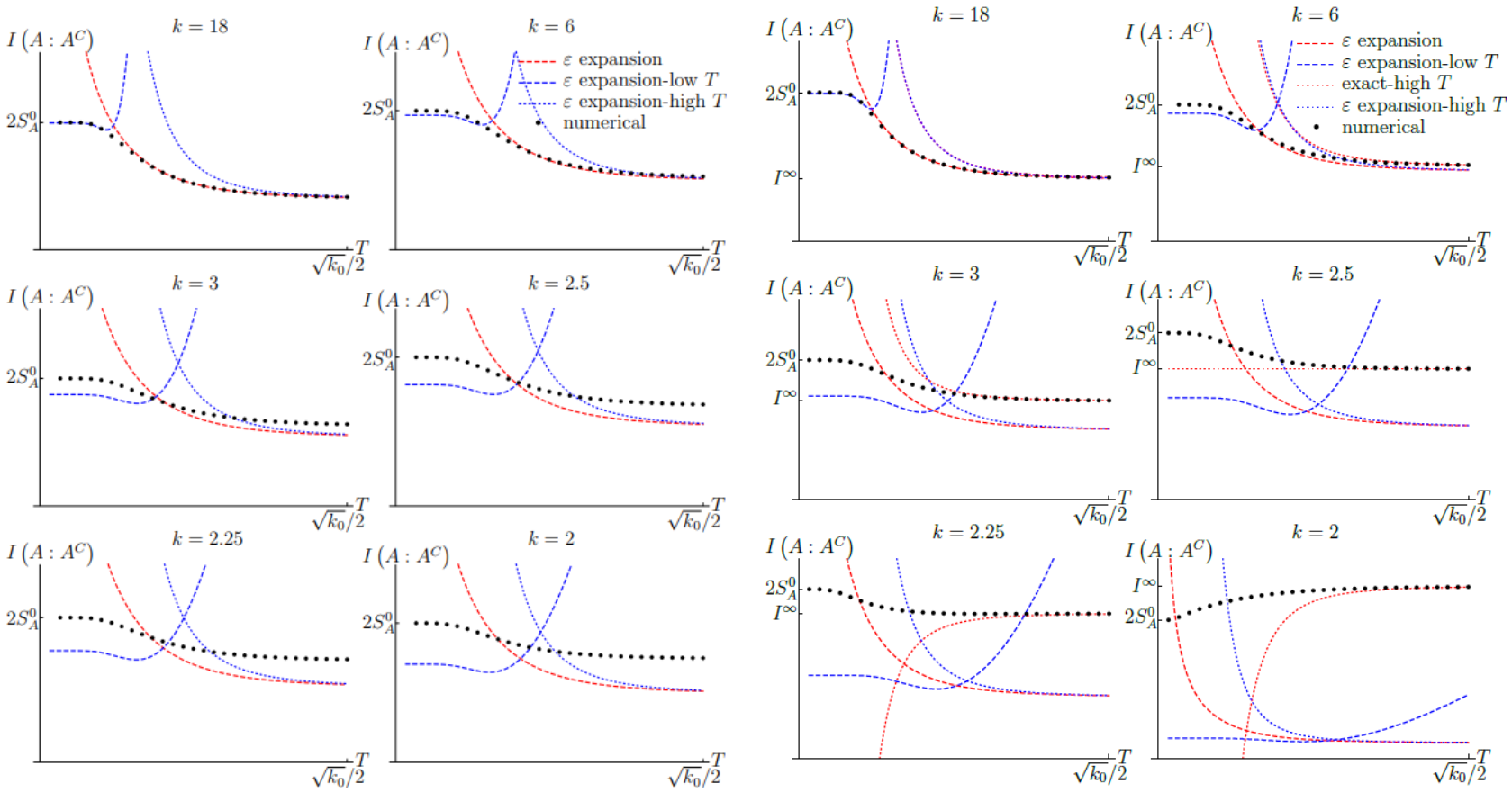
$$I = -\log\left(\frac{\beta_n^{(0)}}{2}\right)\left(1+2\beta_n^{(0)}\right) + (n \to n+1)$$
$$+\left[-\log\left(\frac{\beta_n^{(0)}}{2}\right)\left(1+\beta_n^{(0)}\right) - \left(1+\frac{\sqrt{k_n}}{T}\left(1+\frac{k_n^{(2)}}{2k_n^{(0)}}+\mathcal{O}\left(l^3\right)\right)\right)\right]$$
$$\times \exp\left[-\frac{\sqrt{k_n}}{T}\left(1+\frac{k_n^{(2)}}{2k_n^{(0)}}+\mathcal{O}\left(l^3\right)\right)\right] + (n \to n+1) + \ldots,$$

Two Indicative Examples

We consider coupling matrices of the form

$$K_{\mathrm{Non}} = \begin{pmatrix} k & l & 0 & 0 & \cdots \\ l & 2k & l & 0 & \cdots \\ 0 & l & k & l & \cdots \\ 0 & 0 & l & 2k & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix} \qquad K_{\mathrm{Deg}} = \begin{pmatrix} k & l & 0 & 0 & \cdots \\ l & k & l & 0 & \cdots \\ 0 & l & k & l & \cdots \\ 0 & 0 & l & k & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Plots


$k = 18$
$k = 6$
$k = 3$
$k = 2.5$
$k = 2.25$
$k = 2$
$I(A:A^C)$
$2S_A^0$
I^∞
T
$\sqrt{k_0}/2$
ε expansion
ε expansion-low T
ε expansion-high T
exact-high T
numerical

Free scalar QFT

The above apply to scalar QFT, as the discretized Hamiltonian contains coupling only between adjacent neighbours. We assume that the entangling sphere lies in the middle between the n-th and $(n+1)$-th site of the spherical lattice. We define

$$n_R := n + \frac{1}{2},$$

then the radius of the entangling sphere will be

$$R = n_R a.$$

However, we still have to perform the sum in all ℓ sectors.

Euler-MacLaurin formula

We will sum over ℓ using the Euler-MacLaurin formula

$$\sum_{n=a}^{b} f(n) = \int_a^b dx f(x) + \frac{f(a)+f(b)}{2} + \sum_{k-1}^{\infty} \frac{B_{2k}}{(2k)!} \left[\left. \frac{d^{2k-1} f(x)}{dx^{2k-1}} \right|_{x=b} - \left. \frac{d^{2k-1} f(x)}{dx^{2k-1}} \right|_{x=a} \right],$$

where the coefficients B_k are the Bernoulli numbers.

Large R expansion

It can be shown that the dominant for large R area law term emerges from the integral term solely. Thus, we approximate

$$I \simeq \int_0^\infty d\ell\,(2\ell+1)\,I_\ell\,(N, n, \ell\,(\ell+1)).$$

We are interested in the behaviour of this integral for large R. This behaviour cannot be isolated trivially, since n_R appears in the integrand in the form of the fraction $\ell(\ell+1)/n_R^2$ and ℓ takes arbitrarily large values within the integration range. This can be bypassed performing the change of variables $\ell(\ell+1)/n_R^2 = y$.

$$I \simeq n_R^2 \int_0^\infty dy\, I_\ell \left(N, n_R - \frac{1}{2}, y n_R^2\right),$$

which can be expanded for large n_R.

Area Law

The coupling matrix is

$$K_{ij} = \frac{1}{a}\left[\left(2 + \frac{l(l+1)}{i^2} + \mu^2 a^2\right)\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}\right].$$

The corresponding mutual information reads

$$I = n_R^2 \frac{\coth\left[\frac{1}{2aT}\sqrt{2 + a^2\mu^2}\right]}{4aT\sqrt{2 + a^2\mu^2}} + \mathcal{O}(n_R).$$

This formula has the high temperature expansion

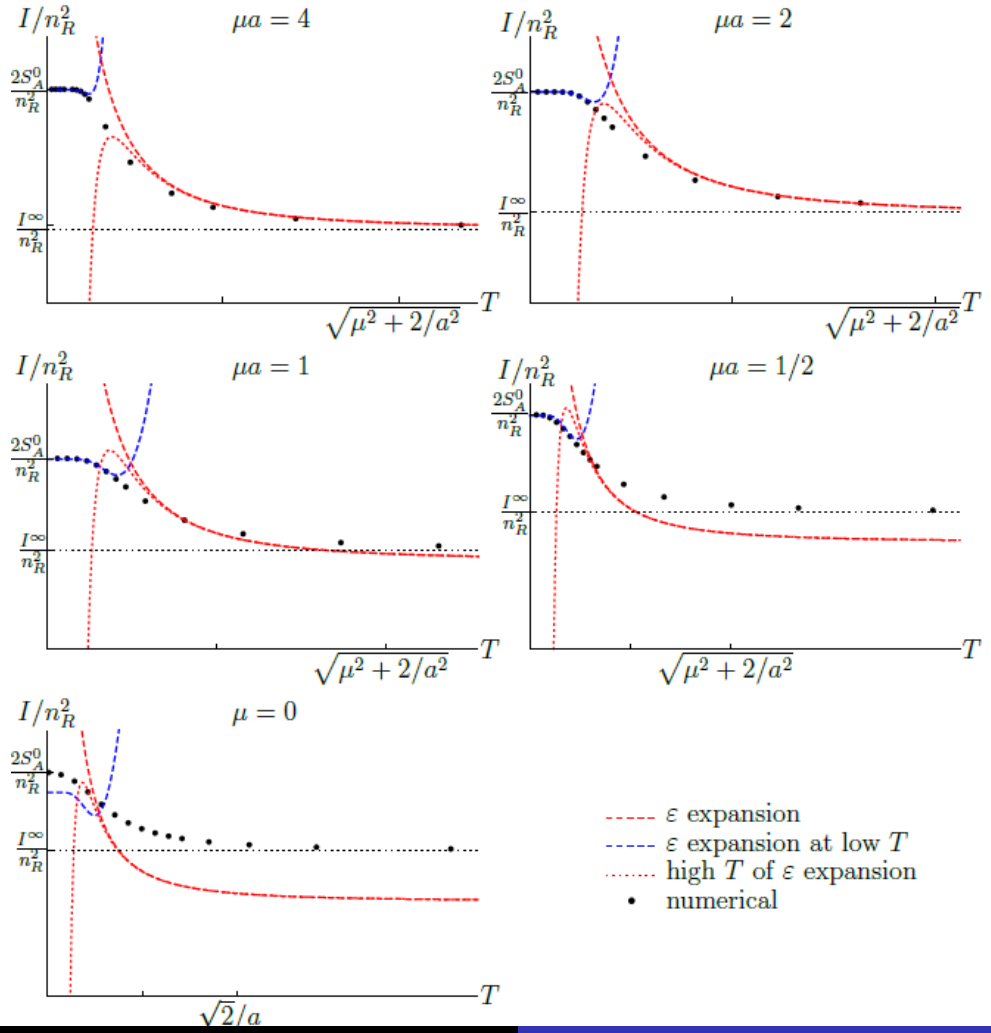
$$I = n_R^2\left(\frac{1}{2(2 + a^2\mu^2)} + \frac{1}{24a^2T^2} - \frac{2 + a^2\mu^2}{1440a^4T^4} + \mathcal{O}\left(\frac{1}{T^6}\right)\right) + \mathcal{O}(n_R).$$

Area Law - Low T

At low temperatures, the integral cannot be performed analytically, but one may use a saddle point approximation. The mutual information is

$$I \simeq I_{T=0} + 2n_R^2\sqrt{2\pi T}\sqrt[4]{\frac{3\left(2+\mu^2a^2\right)}{2}}$$
$$\times\left[2\log\left(4\left(2+\mu^2a^2\right)\right)-1-\frac{\sqrt{2+\mu^2a^2}}{T}\right]\exp\left[-\frac{\sqrt{2+\mu^2a^2}}{T}\right].$$

Coefficient of Area Law as function of T



This behaviour of the area law coefficient is the analogue of the freezing of the degrees of freedom in the context of entanglement.

Summary

- We managed to show an area law behaviour for the mutual information at finite temperature.
- We managed to calculate the coefficient of the area law term in an $1/m$ expansion.
- as well as high and low temperature expansions
- This approach technically has the advantage of the specification of the whole reduced density matrix as an intermediate result.

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Thank you for your attention!