Rational Conformal Field Theory and the Holomorphic Modular Bootstrap

Sunil Mukhi

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Based on:

"Towards a classification of two-character rational conformal field theories",
A. Ramesh Chandra and Sunil Mukhi,
JHEP 1904 (2019) 153, arXiv:1810.09472.

"Curiosities above $c = 24$",
A. Ramesh Chandra and Sunil Mukhi,
SciPost 6 (2019), 053, arXiv:1812.05109.

And previous work:

"On 2d conformal field theories with two characters",
Harsha Hampapura and Sunil Mukhi,
JHEP 1601 (2016) 005, arXiv: 1510.04478.

"Cosets of meromorphic CFTs and modular differential equations",
Matthias Gaberdiel, Harsha Hampapura and Sunil Mukhi,
JHEP 1604 (2016) 156, arXiv: 1602.01022.

And older work:

"On the classification of rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Phys. Lett. B213 (1988) 303.

"Reconstruction of CFT from modular geometry on the torus",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B318 (1989) 483.

Outline

- 2d CFTs play multiple roles in Physics and Mathematics:

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- Spectrum has following structure:

$$\begin{aligned} \text{primaries } \phi_i, &\quad \text{dimensions } (h_i, \bar{h}_i) \\ \text{secondaries } \mathcal{W}_{-n,-\bar{n}}\, \phi_i, &\quad \text{dimensions } (h_i + n, \bar{h}_i + \bar{n}) \end{aligned}$$

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- $\mathcal{W}_{-n,-\bar{n}}$ stands for arbitrary products of creation modes of the spin-1, spin-2, spin-3 $\cdots$ chiral algebra.

- Defining $q = e^{2\pi i\tau}$, the partition function:

$$Z(\tau, \bar{\tau}) = \operatorname{tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}$$

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  counts the total number of states, both primary and secondary.

- For consistency, the partition function must be modular invariant:

$$Z(\gamma\tau, \gamma\bar{\tau}) = Z(\tau, \bar{\tau})$$

  where:

$$\gamma\tau \equiv \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL(2,Z)}$$

- Rational Conformal Field Theory (RCFT): partition function takes the form:

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- In [Mathur-Mukhi-Sen, 1988] a programme was initiated to classify all RCFT's with small n using modular invariance.

- The character $\chi_i(\tau)$ for a primary ϕ_i is defined as:

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- General form:

$$\chi_i(q) = q^{-\frac{c}{24}+h_i}(a_0^i + a_1^i q + a_2^i q^2 + \cdots)$$

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- For $Z(\tau,\bar{\tau})$ to be modular-invariant, the characters must be vector-valued modular functions:

$$\chi_i\left(\gamma\tau\right) = \sum_{j=0}^{p-1} M_{ij}(\gamma)\chi_j(\tau), \quad \gamma \in \mathrm{SL(2,Z)}$$

with $M^\dagger M = 1$.

- From [Belavin-Polyakov-Zamolodchikov (1984)] and [Knizhnik-Zamolodchikov (1984)], we know many examples of RCFT's.

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- This approach has yielded numerous RCFT and taught us a lot.
- However as we will see, it explores a very small corner of the space of RCFT.

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Classification of RCFT Characters

- Our scheme to classify "admissible" RCFT characters $\chi_i(\tau)$ uses three key features:
  - Holomorphy
  - Modular covariance.
  - Integrality of the q-series.
- An RCFT with n characters is characterised by a central charge c and $n-1$ conformal dimensions h_i.
- The Riemann-Roch theorem tells us that the Wronskian Index or W-Index:

$$\ell \equiv \frac{nc}{4} - 6\sum_{i=1}^{n-1} h_i + \frac{n(n-1)}{2}$$

is a non-negative integer $\neq 1$.

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- For example [Hampapura-Mukhi 2016] the $c = 1$ compact free boson at radius $\sqrt{2p}$ has $\ell = 0$ for all p, but its Z_2 orbifold has:

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- What is the "physical" meaning of the W-index?

- The tensor product of two distinct RCFT's having n, n' characters and W-indices ℓ, ℓ' respectively, is an RCFT with $\tilde{n} = nn'$ characters and W-index:

$$\tilde{\ell} = \frac{1}{2}nn'(n-1)(n'-1) + n'\ell + n\ell'$$

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- What are all the irreducible (non-tensor-product) RCFT with n characters and arbitrary W-index $\ell \geq 0$?
- No a priori assumption is made about the chiral algebra.

- For $n = 1$ (one character) the partition function is:

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$$j(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

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- Non-negative integer q-series puts strong restrictions on the function. Allowed examples:

$c = 8$:	$\chi = j^{\frac{1}{3}}$	E_8 (unique)
$c = 16$:	$\chi = j^{\frac{2}{3}}$	$E_8 \times E_8,\ \mathrm{Spin}_{32}/\mathrm{Z}_2$
$c = 24$:	$\chi = j + \mathcal{N}$	free boson, Niemeier lattice
$c = 32$:	$\chi = j^{\frac{1}{3}}(j + \mathcal{N})$	free boson, even unimodular 32d lattice

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- The W-index for these theories is just:

$$\ell = \frac{c}{4}$$

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- It was argued in [Schellekens (1992)] that there are altogether 71 such theories. Their characters are all of the form $j + \mathcal{N}$ with just 30 distinct values of $\mathcal{N}$.
- For all other values of $\mathcal{N}$ there seem to be no consistent CFT ("modular swampland").

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- Practical only for small values of both n and ℓ.
- Hence we mainly focused on $n = 2, 3$.

- For two characters, one can prove that the W-index is even: $\ell = 0, 2, 4, 6, \cdots$.

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- For $\ell = 0, 2, 4$ there are finite many solutions to the bootstrap.

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m_1	c	h	Identification
1	$\frac{2}{5}$	$\frac{1}{5}$	$c=-\frac{22}{5}$ minimal model $(c \leftrightarrow c-24h)$
3	1	$\frac{1}{4}$	$k=1$ SU(2) WZW model
8	2	$\frac{1}{3}$	$k=1$ SU(3) WZW model
14	$\frac{14}{5}$	$\frac{2}{5}$	$k=1$ G_2 WZW model
28	4	$\frac{1}{2}$	$k=1$ SO(8) WZW model
52	$\frac{26}{5}$	$\frac{3}{5}$	$k=1$ F_4 WZW model
78	6	$\frac{2}{3}$	$k=1$ E_6 WZW model
133	7	$\frac{3}{4}$	$k=1$ E_7 WZW model
190	$\frac{38}{5}$	$\frac{4}{5}$	?
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- Most of these theories are known, but they occur in different minimal series for their respective Kac-Moody algebras. Here they occurred all together for the first time.

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C. R. Acad. Sci. Paris, t. 322, Série I, p. 321-326, 1996
Algèbres de Lie/*Lie Algebra*

La série exceptionnelle de groupes de Lie

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Résumé. Numérologie des groupes exceptionnels et une interprétation conjecturale.

The exceptional series of Lie groups

Abstract. *Numerology of exceptional Lie groups and a conjectural explanation.*

Soit G^0 le groupe déployé adjoint de l'un des types suivant : A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8. On fixe un épinglage de G^0. On note G le groupe des automorphismes de G^0. Pour Γ le groupe des

- The case of $\ell = 2$ [Naculich (1989), Hampapura-Mukhi (2015)].

a_1^0	c	h
410	$\frac{82}{5}$	$\frac{6}{5}$
323	17	$\frac{5}{4}$
234	18	$\frac{4}{3}$
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Unlike the $\ell = 0$ list, these are not readily identifiable as CFT. Their status remained unclear for decades.

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- We found the $\ell = 2$ theories are novel cosets of the meromorphic $c = 24$ theories of [Schellekens (1992)] by a corresponding $\ell = 0$ theory.

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- The above cosets are different from the standard coset construction:

$$\mathcal{C} = \frac{\mathrm{WZW}_1}{\mathrm{WZW}_2}$$

	$\ell = 0$				$\ell = 2$				
No.	c	h	m_1	Algebra	$\tilde{c}$	$\tilde{h}$	$\tilde{m}_1$	$m_1 + \tilde{m}_1$	Example KM algebra
1	1	$\frac{1}{4}$	3	$A_{1,1}$	23	$\frac{7}{4}$	69	72	$(A_{1,1})^{23}$
2	2	$\frac{1}{3}$	8	$A_{2,1}$	22	$\frac{5}{3}$	88	96	$(A_{5,2})^2 C_{2,1} A_{2,1}$
3	$\frac{14}{5}$	$\frac{2}{5}$	14	$G_{2,1}$	$\frac{106}{5}$	$\frac{8}{5}$	106	120	$E_{6,3}(G_{2,1})^2$
4	4	$\frac{1}{2}$	28	$D_{4,1}$	20	$\frac{3}{2}$	140	168	$(D_{4,1})^5$
5	$\frac{26}{5}$	$\frac{3}{5}$	52	$F_{4,1}$	$\frac{94}{5}$	$\frac{7}{5}$	188	240	$E_{7,2} B_{5,1}$
6	6	$\frac{2}{3}$	78	$E_{6,1}$	18	$\frac{4}{3}$	234	312	$A_{11,1} D_{7,1}$
7	7	$\frac{3}{4}$	133	$E_{7,1}$	17	$\frac{5}{4}$	323	456	$D_{10,1} E_{7,1}$

Table: Characters with $\ell = 0$ and $\ell = 2$.

- Notice that $c + \tilde{c} = 24$ and $h + \tilde{h} = 2$ in every line.

	$\ell = 0$				$\ell = 2$				
No.	c	h	m_1	Algebra	$\tilde{c}$	$\tilde{h}$	$\tilde{m}_1$	$m_1 + \tilde{m}_1$	Example KM algebra
1	1	$\frac{1}{4}$	3	$A_{1,1}$	23	$\frac{7}{4}$	69	72	$(A_{1,1})^{23}$
2	2	$\frac{1}{3}$	8	$A_{2,1}$	22	$\frac{5}{3}$	88	96	$(A_{5,2})^2 C_{2,1} A_{2,1}$
3	$\frac{14}{5}$	$\frac{2}{5}$	14	$G_{2,1}$	$\frac{106}{5}$	$\frac{8}{5}$	106	120	$E_{6,3}(G_{2,1})^2$
4	4	$\frac{1}{2}$	28	$D_{4,1}$	20	$\frac{3}{2}$	140	168	$(D_{4,1})^5$
5	$\frac{26}{5}$	$\frac{3}{5}$	52	$F_{4,1}$	$\frac{94}{5}$	$\frac{7}{5}$	188	240	$E_{7,2} B_{5,1}$
6	6	$\frac{2}{3}$	78	$E_{6,1}$	18	$\frac{4}{3}$	234	312	$A_{11,1} D_{7,1}$
7	7	$\frac{3}{4}$	133	$E_{7,1}$	17	$\frac{5}{4}$	323	456	$D_{10,1} E_{7,1}$

Table: Characters with $\ell = 0$ and $\ell = 2$.

- Notice that $c + \tilde{c} = 24$ and $h + \tilde{h} = 2$ in every line.
- The $\ell = 2$ theories are simple, yet correspond to complicated invariants of direct sums of Kac-Moody algebras. Unlike the $\ell = 0$ theories, they were not previously known.

- An aside: one can take 3- and 4-character WZW models and use the novel coset construction to find new CFT's. None of them was previously known, and most of them are (almost) perfect metals!

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	$\mathcal{D}$					$\mathcal{C}$					
No.	c	h_1	h_2	m_1	Algebra	$\tilde{c}$	$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{m}_1$	$m_1+\tilde{m}_1$	Schellekens No.
1	$\frac{3}{2}$	$\frac{3}{16}$	$\frac{1}{2}$	3	$\mathfrak{a}_{1,2}$	$\frac{45}{2}$	$\frac{29}{16}$	$\frac{3}{2}$	45	48	$5,7,8,10$
2	$\frac{5}{2}$	$\frac{5}{16}$	$\frac{1}{2}$	10	$\mathfrak{c}_{2,1}$	$\frac{43}{2}$	$\frac{27}{16}$	$\frac{3}{2}$	86	96	$25,26,28$
3	3	$\frac{3}{8}$	$\frac{1}{2}$	15	$\mathfrak{a}_{3,1}$	21	$\frac{13}{8}$	$\frac{3}{2}$	105	120	$30,31,33-35$
4	$\frac{7}{2}$	$\frac{7}{16}$	$\frac{1}{2}$	21	$\mathfrak{b}_{3,1}$	$\frac{41}{2}$	$\frac{25}{16}$	$\frac{3}{2}$	123	144	$39,40$
5	4	$\frac{2}{5}$	$\frac{3}{5}$	24	$\mathfrak{a}_{4,1}$	20	$\frac{8}{5}$	$\frac{7}{5}$	120	144	$37,40$
6	$\frac{9}{2}$	$\frac{9}{16}$	$\frac{1}{2}$	36	$\mathfrak{b}_{4,1}$	$\frac{39}{2}$	$\frac{23}{16}$	$\frac{3}{2}$	156	192	$47,48$
7	5	$\frac{5}{8}$	$\frac{1}{2}$	45	$\mathfrak{d}_{5,1}$	19	$\frac{11}{8}$	$\frac{3}{2}$	171	216	49
8	$\frac{11}{2}$	$\frac{11}{16}$	$\frac{1}{2}$	55	$\mathfrak{b}_{5,1}$	$\frac{37}{2}$	$\frac{21}{16}$	$\frac{3}{2}$	185	240	53
9	6	$\frac{3}{4}$	$\frac{1}{2}$	66	$\mathfrak{d}_{6,1}$	18	$\frac{5}{4}$	$\frac{3}{2}$	198	264	$54,55$
10	$\frac{13}{2}$	$\frac{13}{16}$	$\frac{1}{2}$	78	$\mathfrak{b}_{6,1}$	$\frac{35}{2}$	$\frac{19}{16}$	$\frac{3}{2}$	210	288	56
11	7	$\frac{7}{8}$	$\frac{1}{2}$	91	$\mathfrak{d}_{7,1}$	17	$\frac{9}{8}$	$\frac{3}{2}$	221	312	59
12	$\frac{17}{2}$	$\frac{17}{16}$	$\frac{1}{2}$	136	$\mathfrak{b}_{8,1}$	$\frac{31}{2}$	$\frac{15}{16}$	$\frac{3}{2}$	248	384	62
13	$\frac{31}{2}$	$\frac{15}{16}$	$\frac{3}{2}$	248	$\mathfrak{e}_{8,2}$	$\frac{17}{2}$	$\frac{17}{16}$	$\frac{1}{2}$	136	384	62
14	9	$\frac{9}{8}$	$\frac{1}{2}$	153	$\mathfrak{d}_{9,1}$	15	$\frac{7}{8}$	$\frac{3}{2}$	255	408	63
15	10	$\frac{5}{4}$	$\frac{1}{2}$	190	$\mathfrak{d}_{10,1}$	14	$\frac{3}{4}$	$\frac{3}{2}$	266	456	64

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Quasi-characters and $\ell \geq 6$

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- In [Chandra-Mukhi (2018a)] we proposed a new method of quasi-characters and used them to construct all possible admissible pairs of characters for any W-index ℓ.
- Quasi-characters were indirectly anticipated in [Kiritsis (1989)].

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- Then, it turns out that there are infinitely many solutions. We call these quasi-characters.
- Example:

$$\chi_0 = q^{-\frac{25}{24}}(1 - 245q + 142640q^2 + 18615395q^3 + 837384535q^4 + \cdots)$$

with all higher coefficients positive.

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- We also constructed quasi-characters for $\ell = 2, 4$ and showed that these are building blocks for admissible characters with $\ell = 6p + 2, 6p + 4$ respectively, thus exhausting all even ℓ.
- Due to time constraints I will only discuss the $\ell = 6p$ cases in this talk.

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- Then the sum looks like:

$$\chi_0 = q^{-\frac{c}{24}-1}(1 - \cdots) + \mathcal{N}_1\, q^{-\frac{c}{24}}(1 + \cdots)$$
$$\chi_1 = q^{-\frac{c}{24}+h+1}(1 + \cdots) + \mathcal{N}_1\, q^{-\frac{c}{24}+h}(1 + \cdots)$$

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  and is admissible for large enough $\mathcal{N}_1$.
- From the leading power of q we read off that this has central charge $c + 24$ and conformal dimension $h + 1$.
- Applying Riemann-Roch, the sum is an admissible character with $\ell = 6$.

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- Adding p quasi-characters gives admissible characters with W-index $\ell = 6p$.
- We have proved that this procedure is complete. Of course it only classifies characters rather than actual CFT's.
- We never used use the MLDE for $\ell > 6$, which is prohibitively difficult.

- We now have new infinite families of admissible characters, but which of them are actually CFT's?

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- This is reminiscent of the one-character case at $c = 24$ (which also has $\ell = 6$, in fact).
- The admissible characters for $\ell = 6$ have central charges:

$$24 < c < 32$$

- To construct CFT's for (some of) these characters, we again use the novel coset construction of [Gaberdiel-Hampapura-Mukhi (2016)].

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- We saw that all WZW models have $\ell = 0$. Taking the quotient of an even, selfdual lattice boson theory of central charge c by a two-character WZW model, the result is a two-character CFT with:

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- We want $\ell = 6$ so we take $c = 32$. Thus we are led to consider free bosons on a 32-dimensional lattice.

- There are $> 10^{10}$ even selfdual lattices in 32 dimensions. But 132 of these are special. They have complete root systems and are called Kervaire lattices.

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- So we take the coset of a Kervaire lattice CFT by any of the WZW theories falling in the MMS series, which all have $\ell = 0$.
- The result has $\ell = 6$ and its characters are completely determined by the coset construction.Thus the arbitrary integer $\mathcal{N}_1$ is determined in each case.
- With a complete root system, one can determine the Kac-Moody algebra of the coset dual and from it the stress tensor. Thereafter the coset theory can be completely solved.

- Simple example: a 32-dimensional lattice having the complete root system A_2^{16}.

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- Scalar field theory on the torus $\mathbf{C}^{32}/\Gamma$ defines a unique $c = 32$ meromorphic CFT with $A_{2,1}^{16}$ as its Kac-Moody algebra.
- The number of spin-1 currents is the dimension of the algebra, which is 128.

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- The modular invariant (upto a phase) combination of these characters is easily found to be:

$$\begin{aligned}\chi(\tau) &= \boldsymbol{\chi}_0 + 224\boldsymbol{\chi}_2 + 2720\boldsymbol{\chi}_3 + 3360\boldsymbol{\chi}_4 + 256\boldsymbol{\chi}_5 \\ &= j(\tau)^{\frac{1}{3}}(j(\tau) - 864)\end{aligned}$$

- Now we coset the theory by the two-character $A_{2,1}$ WZW model, to get a new two-character CFT with $A_{2,1}^{15}$ as its symmetry. It has Wronskian index $\ell = 6$ as desired.

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- Its characters must be linear combinations of $\chi_0^p \chi_1^{15-p}$ whose dimensions are $m_i = \frac{15-p}{3}$. These combinations turn out to be:

$$\tilde{\chi}_0(\tau) = \boldsymbol{\chi}_0 + 140\boldsymbol{\chi}_2 + 1190\boldsymbol{\chi}_3 + 840\boldsymbol{\chi}_4 + 16\boldsymbol{\chi}_5$$
$$\tilde{\chi}_1(\tau) = 42\boldsymbol{\chi}_{\frac{5}{3}} + 765\boldsymbol{\chi}_{\frac{8}{3}} + 1260\boldsymbol{\chi}_{\frac{11}{3}} + 120\boldsymbol{\chi}_{\frac{14}{3}}$$

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- We can use methods of [Mathur-Mukhi-Sen (1989)] to compute correlation functions on the plane and torus. So the CFT is fully defined.

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- Still, given any lattice CFT with a complete root system, we can coset it by a suitable $\ell = 0$ CFT and obtain large classes of theories with various ℓ.
- For lattices with incomplete root systems, things are more complicated and not yet worked out.

- A long-standing problem, to find all admissible CFT characters of rank n, has now been solved for $n = 2$.

Conclusions and Outlook

- A long-standing problem, to find all admissible CFT characters of rank n, has now been solved for $n = 2$.
- The cases with $\ell < 6$ turns out to be extremely non-generic: there are only finite families. From $\ell = 6$ onwards there are infinite families of admissible characters.

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- Many of these (probably a large but finite number) correspond to some CFT. We found > 132 examples with $\ell \geq 6$ as cosets of even, unimodular lattices. Using higher-dimensional lattices one should be able to extend these results.

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- Many of these (probably a large but finite number) correspond to some CFT. We found > 132 examples with $\ell \geq 6$ as cosets of even, unimodular lattices. Using higher-dimensional lattices one should be able to extend these results.
- For rank 3, the $\ell = 0$ case was studied in [Mathur-Mukhi-Sen (1989), Franc-Mason (2017)], but virtually nothing is known for $\ell > 0$. The methods discussed here can surely be applied to that case.

Thank You