The L_∞-algebra of the S-matrix

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Based on

- The L_∞-algebra of the S-matrix, ASA, [arXiv:1903.05643]
- Effective field theory, L_∞-algebras, and homotopy transfer, ASA, Olaf Hohm, Chris Hull, Victor Lekeu [arXiv:19??.xxxxx]

L_∞-algebras arose in closed bosonic string field theory [Zwiebach 1993]: vertices $\leftrightarrow$ L_∞-algebra brackets

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \to |\Phi\rangle \equiv \big[|\psi_1\rangle, |\psi_2\rangle \ldots, |\psi_n\rangle\big]_{\mathbf{n}}\,.$$

L_∞ structure mostly thought to be a string theory curiosity before [Hohm Zwiebach 2017] argued its relevance to **any classical field theory**: determines action, gauge transfs, EOMs.

What we did:

1. associate an "L_∞-algebra of 1PI correlators" to any nice, anomaly-free **quantum** field theory;
2. show the LSZ formula is a *homotopy transfer* from the L_∞-algebra of correlators onto space of scattering states;
3. exhibit the vacuum moduli space as the "L_∞-algebra Maurer-Cartan moduli space" associated to either.

L_∞-algebras: generalisations of Lie with more arguments

- Lie: binary bracket $[T_{a_1}, T_{a_2}] = C^b_{a_1 a_2} T_b$, Jacobi identity;
- L_∞: **1-ary**, binary,... **n**-ary brackets, Jacobi identit**ies**:

$$[T_{a_1}, T_{a_2}, \ldots T_{a_n}]_{\mathbf{n}} \equiv C^b_{a_1 a_2 \ldots a_n} T_b \,; \quad 0 = Q^2 T_a \,,$$

$$0 = \left[QT_a, T_b, T_c\right]_{\mathbf{3}} + \left[[T_a, T_b]_{\mathbf{2}}, T_c\right]_{\mathbf{2}} + Q[T_a, T_b, T_c]_{\mathbf{3}} + \text{ perms}\,, \ldots$$

Underlying vector space $\mathcal{V} \ni T_a$ has integer *degree* deg, so e.g. $[T_a, T_b]_{\mathbf{2}} = (-1)^{(\deg T_a)(\deg T_b)} [T_b, T_a]_{\mathbf{2}}$, and all brackets increase degree by $+1$ (we find a Lie algebra if all deg $= -1$).

Classification by the cohomology $H(Q)$ of the 1-ary bracket Q:

- an L_∞-algebra is *minimal* if $H(Q) = \mathcal{V} \iff Q = 0$;
- an L_∞-algebra is *contractible* if $H(Q) = 0$;

Theorem (Minimal model theorem)

Every L_∞-algebra $\mathcal{V}$ is isomorphic to a direct sum: $\mathcal{V} \cong H(Q) \oplus \mathcal{V}_{contr}$. The L_∞ structure on $H(Q)$ is unique (up to iso).

$H(Q)$ is in particular a super Lie algebra.

TREE-LEVEL: [HOHM ZWIEBACH 2017] REDUX À LA BV

For a nice gauge theory like YM we gauge-fix as

$$S[\phi] \to S[\phi] + Q_{\mathrm{BRST}}\Psi = S[\phi] + (Q_{\mathrm{BRST}}\phi)\delta\Psi/\delta\phi$$

Introduce sources $\overset{\star}{\phi}$ for BRST transfs and antibracket $\{\phi, \overset{\star}{\phi}\} = 1$

$$S_{\mathrm{BV}} \equiv S[\phi] + (Q_{\mathrm{BRST}}\phi)\overset{\star}{\phi} \implies \{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = Q_{\mathrm{BRST}}S_0 + (Q^2_{\mathrm{BRST}}\phi)\overset{\star}{\phi}$$

If $\phi = \overset{\star}{\phi} = 0$ solves EOMs, Taylor expansion of S_{BV} yields L_∞-algebra structure consts. E.g. 1-ary bracket Q:

$$Qv \equiv (\delta^2 S_{\mathrm{BV}}/\delta\phi^2)v + (\delta^2 S_{\mathrm{BV}}/\delta\phi\delta\overset{\star}{\phi})v\,, \qquad v \in \mathcal{V}$$

Q encodes kinetic term plus *linearised* BRST.
Binary, ternary,... brackets $\leftrightarrow$ cubic, quartic,... vertices.

L_∞ Jacobis $\iff$ $\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0$ $\iff$ BRST invariance!

Tree S-matrix functional from S: recursively solve nonlinear ϕ EOMs by $\phi(\varphi) = \varphi + G\delta S_{\text{int}}/\delta\phi$, from any linearised solution φ

$$\mathcal{A}_{\text{tree}}[\varphi] \equiv S[\phi(\varphi)]$$

For $S = S_{\text{BV}}[\phi, \overset{\star}{\phi}]$, $A_{\text{tree}}[\varphi, \overset{\star}{\varphi}]$ is the homotopy-transferred L_∞-algebra structure on $H(Q)$. [Nützi Reiterer 2018, Macrelli Sämann Wolf, ASA 2019]

Quantum effects? Zinn–Justin 1PI functional **including source $\overset{\star}{\Phi}$ for BRST**: $\Gamma[\Phi, \overset{\star}{\Phi}] \equiv S_{\text{BV}}[\Phi, \overset{\star}{\Phi}] + \sum_{\ell=1} \hbar^\ell \Gamma_\ell$;

$$\boxed{\{\Gamma, \Gamma\} = 0 \iff \text{BRST Ward identities}} \qquad \text{[Zinn–Justin 1974]}$$

Γ defines **"L_∞-algebra of 1PI correlators"**.
(Assumptions: lack of tadpoles, BRST decoupling, …)

Claim [ASA 2019]:

LSZ formula realises homotopy transfer of L_∞-structure into $H(Q) \cong$ asymptotic 1-particle states (with renorm. masses)!

Concretely, **n**-ary structure constant is $(\mathbf{n}+1)$-pt amplitude.

S-matrix L_∞-algebra is a minimal model for the 1PI L_∞-algebra. So what?

Theorem (originally by Kontsevich?)

If $f : \mathcal{V}_1 \to \mathcal{V}_2$ is a morphism of L_∞-algebras "preserving the cohomologies $H(Q_{1,2})$", then $\mathcal{V}_{1,2}$ have isomorphic minimal models. They also have isomorphic "Maurer-Cartan moduli spaces".

1. If $\mathcal{V}_{\Lambda+\lambda}, \mathcal{V}_\Lambda$ are 1PI algebras at scales $\Lambda, \Lambda+\lambda$, integrating out $\Lambda < E < \Lambda+\lambda$ is a morphism $\mathcal{V}_\Lambda \hookrightarrow \mathcal{V}_{\Lambda+\lambda}$. For large Λ, obtain S-matrix equivalence. [ASA Hohm Hull Lekeu 2019?]
2. 1PI algebra MC moduli space is the vacuum moduli space by [Coleman Weinberg 1973]: vacuum moduli from S-matrix.
3. Berends-Giele recursion [1988]: BG current $J_{\mathbf{n}}$ has $\mathbf{n}$ on-shell legs and 1 off-shell one; same structure as an L_∞-morphism $H(Q) \to \mathcal{V}_{\text{tree lvl YM}}$. In fact, [Macrelli Sämann Wolf 2019] minimal model recursion = BG recursion.
4. Lie algebra structure of $H(Q)$ similar to "algebras of BPS states" [Harvey Moore 1996]; precise connection yet unclear

Thank you!