Path Integral Optimization for $T\overline{T}$ Deformation

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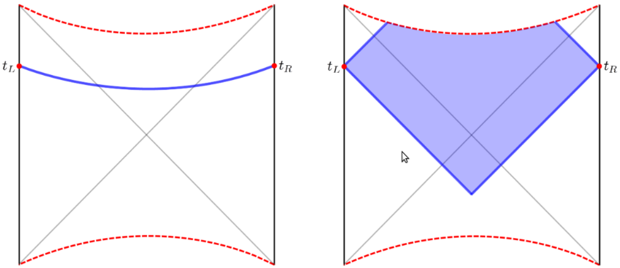
- Applications of quantum information concepts to gravity and HEP recently led to many wide-spreading developments.
- In particular quantum entanglement helps us to understand how gravity emerges from field theories.
- One concrete idea to realize this emergent spacetime is to utilize the connection between tensor networks and holographic entanglement surface. It is argued that a time slice of AdS spacetime correspond to a special tensor network called multi-scale entanglement renormalization ansatz (MERA) (Swingle-2012).
- However, when one tries to understand the interior of black hole, the entanglement is not enough (Susskind-2014). This observation has led to significant interest to the information notion of quantum state complexity which can be used as a probe to investigate the growth rate of the Einstein-Rosen bridge (ER=EPR).

- Two holographic conjectures for quantum complexity



- CV conjecture (left): The blue curve represents the maximal spacelike surface that connects the specified time slices on the left and right boundaries .

- CA conjecture (right): The shaded region is the corresponding WdW patch.

- To understand better the holographic results, recently the computational complexity for QFT states have been studied.
- In one approach which is based on the idea of (Nielsen et al.-2006),
  - One associates a geometry to the space of unitaries that connect the desired state to a reference state.
  - Then the complexity of desired state is defined as a length of a geodesic in this geometry.
  - Up to now, this approach is well established just for Gaussian states.

- In another approach which is applicable for any 2D CFTs (free or interacting) (Takayanagi et al.-2017),
  - One works with Euclidean path integral description of quantum state and perform the optimization by changing the structure (or geometry) of lattice regularization.
  - The resulting change in path integral-namely the Liouville action, was defined as a complexity of the corresponding state.
- Very interestingly, it has been shown recently that these two seemingly QFT different approaches are surprisingly connected (Heller-2019).

- We would like to apply the path integral optimization to the new class of integrable QFTs.
- These theories are introduced by (Smirnov and Zamolodchikov-2016), which they studied the deformation of 2D integrable QFTs by a special irrelevant composite operator, $T\overline{T}$, and found that the theory remains integrable even after the deformation.
- Interestingly, the resulting theory is UV complete and non-local and its energy spectrum and the S-matrix can be found exactly.
- It is proposed by (Verlinde et al.-2016) that $T\overline{T}$ deformation of 2D CFTs is dual to 3D AdS gravity with a finite bulk cut-off.

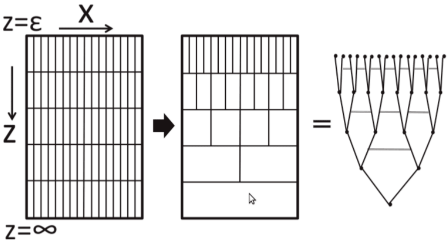
- For 2D CFTs, the ground state wave functional on $\mathbb{R}^2$ is computed by an Euclidean path integral:

$$\Psi_{\mathrm{CFT}}[\bar{\Phi}(x)] = \int \left(\prod_x \prod_{\epsilon<z<\infty} \mathcal{D}\Phi \right) e^{-S_{\mathrm{CFT}}(\Phi)} \times \\ \times \prod_x \delta\left(\Phi(\epsilon, x) - \bar{\Phi}(x)\right),$$

where the Euclidean time τ is s related to z via $z = -(\tau - \epsilon)$ and ϵ is UV cutoff (i.e. the lattice constant).

- It is worth noting that to evaluate this discretized path integral in an optimal way, one can omit any unnecessary lattice sites.

- To systematically quantify such coarse-graining, one might introduce a 2D metric (on which the path integration is performed)
$$ds^2 = \frac{1}{\epsilon^2}\left(dz^2 + dx^2\right),$$
- The optimization procedure then can be described by modifying the background metric for the path integration as
$$ds^2 = g_{zz}(z,x)dz^2 + g_{xx}(z,x)dx^2 + 2g_{zx}(z,x)dzdx.$$
- Now, the key point is that the optimized wave functional, up to a normalization factor, should be proportional to the correct ground state wavefunction.



- 2D $T\overline{T}$ deformed CFTs are described by

$$S = S_{\text{CFT}} + \mu \int dz dx \, T\overline{T}(z,x), \qquad T\overline{T} \sim T_{ww}\overline{T}_{\bar{w}\bar{w}} - \theta^2$$

- Under change of Weyl factor in $\hat{g}_{\alpha\beta} = e^{2\Omega(z,x)} g_{\alpha\beta}$ and for small μ, the partition function changes to

$$\frac{\partial \log \mathcal{Z}_\mu[\hat{g}]}{\partial \Omega} = \frac{c}{24\pi}\sqrt{\hat{g}}\hat{R}[\hat{g}] - 2\Lambda\sqrt{\hat{g}} - \mu\, \partial_\Omega \langle 0|T\overline{T}|0\rangle_{\hat{g}},$$

where constant c is the central charge of 2D CFT.

- One can now treat above equation as a differential equation for the partition function $\mathcal{Z}_\mu$ and solve it

$$\mathcal{Z}_\mu[\hat{g}] = e^{S_{GL}[\Omega,g]-S_{GL}[0,g]}\ \mathcal{Z}_\mu[g],$$

where

$$S_{GL}[\Omega,\delta] = \frac{c}{24\pi}\int d^2\omega\Big[-4\Omega\partial\bar{\partial}\Omega - \tilde{\Lambda}e^{2\Omega} \\ -\tilde{\mu}e^{-2\Omega}\left(\partial^2\Omega\bar{\partial}^2\Omega - 4(\partial\bar{\partial}\Omega)^2 + \frac{3}{16}\tilde{\Lambda}^2 e^{4\Omega}\right)\Big].$$

$\tilde{\mu} = \mu\pi c/6$ and $\bar{\Lambda} = \tilde{\Lambda} + \frac{3}{16}\tilde{\mu}\tilde{\Lambda}^2$.

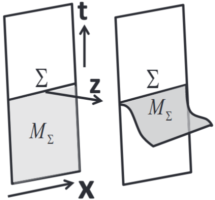
- This implies that

$$\Psi_{e^{2\Omega}\delta_{\alpha\beta}}(\bar{\Phi}(x)) = e^{S_{GL}[\Omega,\delta]-S_{GL}[0,\delta]}\ \Psi_{\delta_{\alpha\beta}}(\bar{\Phi}(x)).$$

- We consider $\Omega(\omega+\bar{\omega}) = \Omega_{\mathrm{CFT}}(\omega+\bar{\omega}) + \tilde{\mu}\,\Omega_{T\overline{T}}(\omega+\bar{\omega})$.
- Excited states created by acting a primary operator $\mathcal{O}_\alpha$ with the conformal dimension $h_\alpha = \bar{h}_\alpha$ which its behavior under the Weyl re-scaling is expressed as

$$\mathcal{O}_\alpha \sim e^{-2h_\alpha}\mathcal{O}_\alpha.$$

The deformed solution is:



$$ds^2 = \frac{a^2}{4\rho^2}d\rho^2 + \frac{l^2 a^2}{\rho}\left(1+\frac{2\tilde{\mu}c_1}{l^2}\right)\left(1-\frac{1}{2}\rho+\frac{1}{16}\rho^2\right)dx^2.$$

The gravitational energy of the this solution matches with correspondent energy of deformed primary state if $c_1 = -1/32\pi^2$.

- For finite temperature $T = 1/\beta'$ case, one can use the thermofield double (TFD) representation of wave functional.

$$\Psi_{\beta'}[\bar{\Phi}_2(x), \bar{\Phi}_1(x)] = \int \left(\prod_x \prod_{-\frac{\beta'}{4} \leq z < \frac{\beta'}{4}} \mathcal{D}\Phi \right) e^{-S(\Phi)} \times$$
$$\times \prod_x \delta\left(\Phi(-\frac{\beta'}{4}, x) - \bar{\Phi}_1(x)\right) \delta\left(\Phi(\frac{\beta'}{4}, x) - \bar{\Phi}_2(x)\right),$$

- The deformed TFD state is given by

$$ds^2 = \frac{dr^2}{f(r)} + r^2 dx^2, \;\; f(r) = \frac{r^2}{l^2} - \frac{4\pi^2 l^2}{\beta^2}(1 + \frac{2\tilde{\mu}}{l^2} c_1).$$

For $c_1 = GM_{\mathrm{BTZ}}/4\pi^2$, the deformed energy matches well with the energy spectrum of $T\overline{T}$ deformed CFT.

- The solution of 3D Einstein-Hilbert equations

$$ds^2 = \frac{l^2}{4\rho^2} d\rho^2 + \frac{l^2}{\rho} g_{\alpha\beta}(\rho, x) dx^\alpha dx^\beta,$$

$$g_{\alpha\beta}(\rho, x^\alpha) = g^{(0)}(x^\alpha) + g^{(2)}_{\alpha\beta}(x^\alpha)\rho + g^{(4)}_{\alpha\beta}(x^\alpha)\rho^2,$$

which $g^{(4)}$ and $g^{(2)}$ are determined algebraically in terms of $g^{(0)}$.

- PI conclude that the deformed theory lives on

$$g^{[\mu]}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \frac{\mu}{32\pi G l} g^{(2)}_{\alpha\beta} + \mathcal{O}(\mu^2).$$

- Another interpretation for the obtained geometry is that its conformal boundary corresponds to fixing the induced metric on a constant $\rho = \rho_c$ surface, with

$$\rho_c = \frac{\mu}{32\pi G l},$$

in agreement with the earlier proposal of (Verlinde et al.-2016).

- It is worth noting that the path integral optimization solution shows that the entire spacetime should be kept especially the region outside the would-be "cutoff" surface, which in proposal (Verlinde et al.-2016) is removed.
- Since $T\overline{T}$-deformed CFTs are conjectured to contain (non-local) observables of arbitrarily high energy, it makes sense.

- The similarity between tensor network representation and Path integral representation of vacuum state, can be utilize to define computational complexity of ground state as the minimum value of the $S_{GL}[\Omega, g]$ action.

- It is convenient to look at the relative quantity $I_{GL}[g_2, g_1]$ which satisfy the following identity

$$I_{GL}[g_1, g_2] + I_{GL}[g_2, g_3] = I_L[g_1, g_3].$$

The above relation implies that

$$I_{GL}[e^{2\Omega} g, \tilde{g}] = I_{GL}[e^{2\Omega} g, g] - I_{GL}[\tilde{g}, g],$$

- $I_{GL}[g_2, g_1]$ actually measures the difference of complexity between the path-integral in g_2 and g_1.

- The general solution for the complexity functional condition is

$$I_{GL}[\Omega, g] = \int d^2\sigma \sqrt{g}\bigg(R[g]\Omega + \nabla_\alpha \Omega \nabla^\alpha \Omega + b_1 (e^{2\Omega} - 1) +$$
$$+ \tilde{\mu} e^{-2\Omega}\Big[b_2 e^{2\Omega}\left(e^{2\Omega} - 1\right) - b_3 \Box\Omega \left(R[g] - \Box\Omega\right) - \frac{b_3}{4} R^2[g]\left(e^{2\Omega} - 1\right)\Big]\bigg).$$

Comparing this action with our perturbative analysis gives

$$b_1 = -\tilde{\Lambda},\ b_2 = -3\tilde{\Lambda}^2/16,\ b_3 = 3/16.$$

- To have a well defined variational principle,

$$I_{GGH}[\Omega, \gamma] = 2\int dx \sqrt{\gamma}\bigg(K[\gamma]\Omega - \frac{3}{16}\tilde{\mu} e^{-2\Omega}\Big[K[\gamma]\Box\Omega +$$
$$+ \frac{1}{2} K[\gamma] R[g] \left(e^{2\Omega} - 1\right) - n^\alpha \nabla_\alpha \Omega \left(R[g] - \Box\Omega\right)\Big]\bigg).$$

- The path integral complexity is defined by

$$\mathcal{C}_{T\overline{T}} \equiv \frac{c}{24\pi}\big(I_{GL}[\Omega, g] + I_{GGH}[\Omega, \gamma]\big),$$

- For the deformed TFD state

$$\mathcal{C}^{\text{BTZ}}_{T\overline{T}} = \frac{c}{3}(1 + \frac{\pi c\mu}{64 l^2})\,\frac{l}{\epsilon} - \\ -\frac{\beta c}{24 l} - \frac{\pi^2 c\, l}{6\beta} - \frac{\pi c\mu}{6}\left(\frac{\beta c}{128 l^3} + \frac{\pi^2 c\, l}{48\beta^3} - \frac{c}{192\beta l}\right).$$

- The complexity for formation for deformed TFD state is finite.

- We show that the stress tensor obtained from the first law of EE is same as the one obtained from the action I_{GL}.
- The change in entanglement entropy under a small variation of a quantum state is captured by the variation in the Weyl factor field. This fact implies that for a small entangling region $A{=}[-R/2, R/2]$, the change in EE becomes

$$\Delta S_A \simeq \frac{c}{6}\int ds\, e^{\Omega(z)}\delta\Omega(z) = \frac{cR^2}{24}\partial_z^2\delta\Omega(z),$$

where for the deformed TFD solution we obtain

$$\Delta S_A \simeq \frac{\pi R^2}{3}T_{tt} \qquad \rightarrow T_{tt} = \frac{\pi c}{6\beta^2}\left(1 + \frac{2\tilde{\mu}}{L^2}c_1\right)$$

- Independently, by taking the metric variation from the action I_{GL} and using those special values of b_1, b_2, b_3, the corresponding stress tensor matches well with the above energy density.

- We have shown that the path integral optimization approach, for $T\overline{T}$ deformed 2D CFTs, implies that the dual geometry indeed capture the entire of spacetime.
- The deformed theory has stress tensor $T^{[\mu]}$ and it lives on the metric $g^{[\mu]}$,

$$g^{[\mu]}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \frac{\mu}{32\pi G l} g^{(2)}_{\alpha\beta},$$

$$T^{[\mu]}_{\alpha\beta} = \frac{1}{8\pi G l} g^{(2)}_{\alpha\beta} + \frac{\mu}{64\pi^2 G^2 l^2} g^{(4)}_{\alpha\beta},$$

In complete agreement with the analysis of (Guica-2019).

- For positive deformation coupling, the optimized solutions can be interpreted as a geometry at finite cut-off radius.
- We studied the entanglement entropy and quantum complexity for those optimized solutions. Another interesting quantities which should be studied are correlation functions.

Thank YOU