Entanglement Wedges

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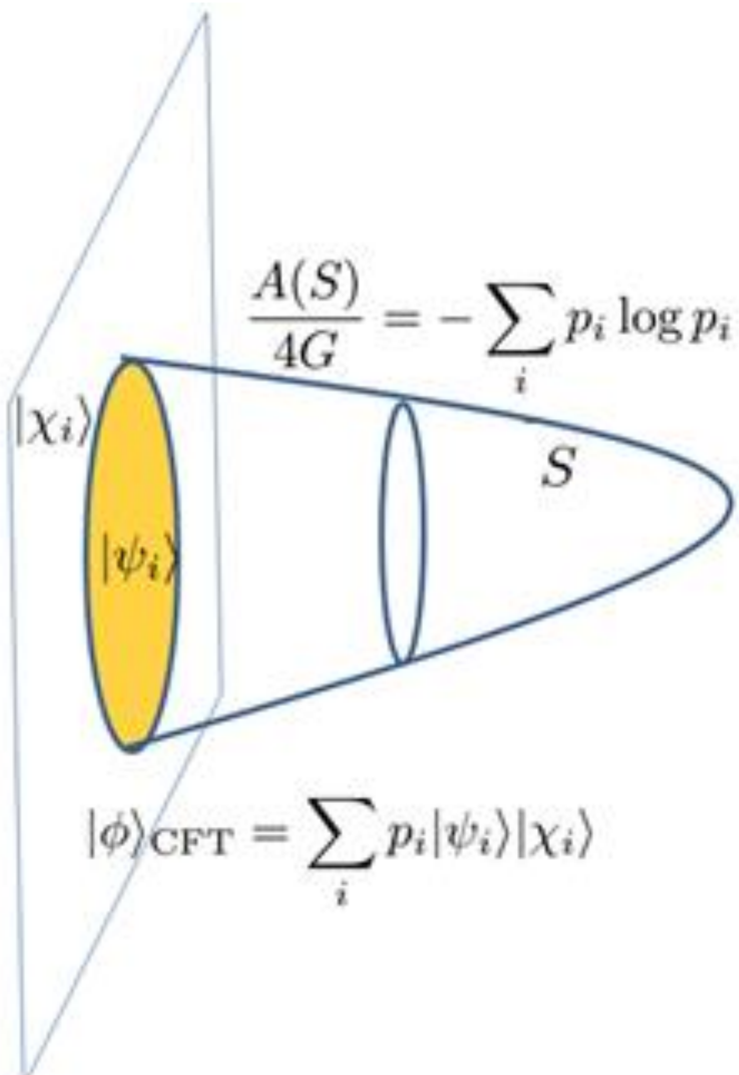
Mostly based on work with Bartek Czech, Dongsheng Ge, and Lampros Lamprou, arXiv:1903.04493 + work in progress with Lampros Lamprou (see his talk)

Kolymbari, September 17, 2019

Quantum information theory has become an important tool in AdS/CFT ever since Ryu and Takayanagi proposed that



$$S_{\mathrm{vN}} = -\mathrm{Tr}(\rho_A \log \rho_A) = \frac{A(S)}{4G_N}$$

Quantum information techniques are great for general abstract arguments and statements in “in principle proofs”.

Unfortunately, the relevant quantities are often hard to compute in practice, especially on the field theory side.

The situation is somewhat similar to the role of algebraic quantum field theory in high-energy physics.

Improvements of RT:

- Time dependent case (HRT)

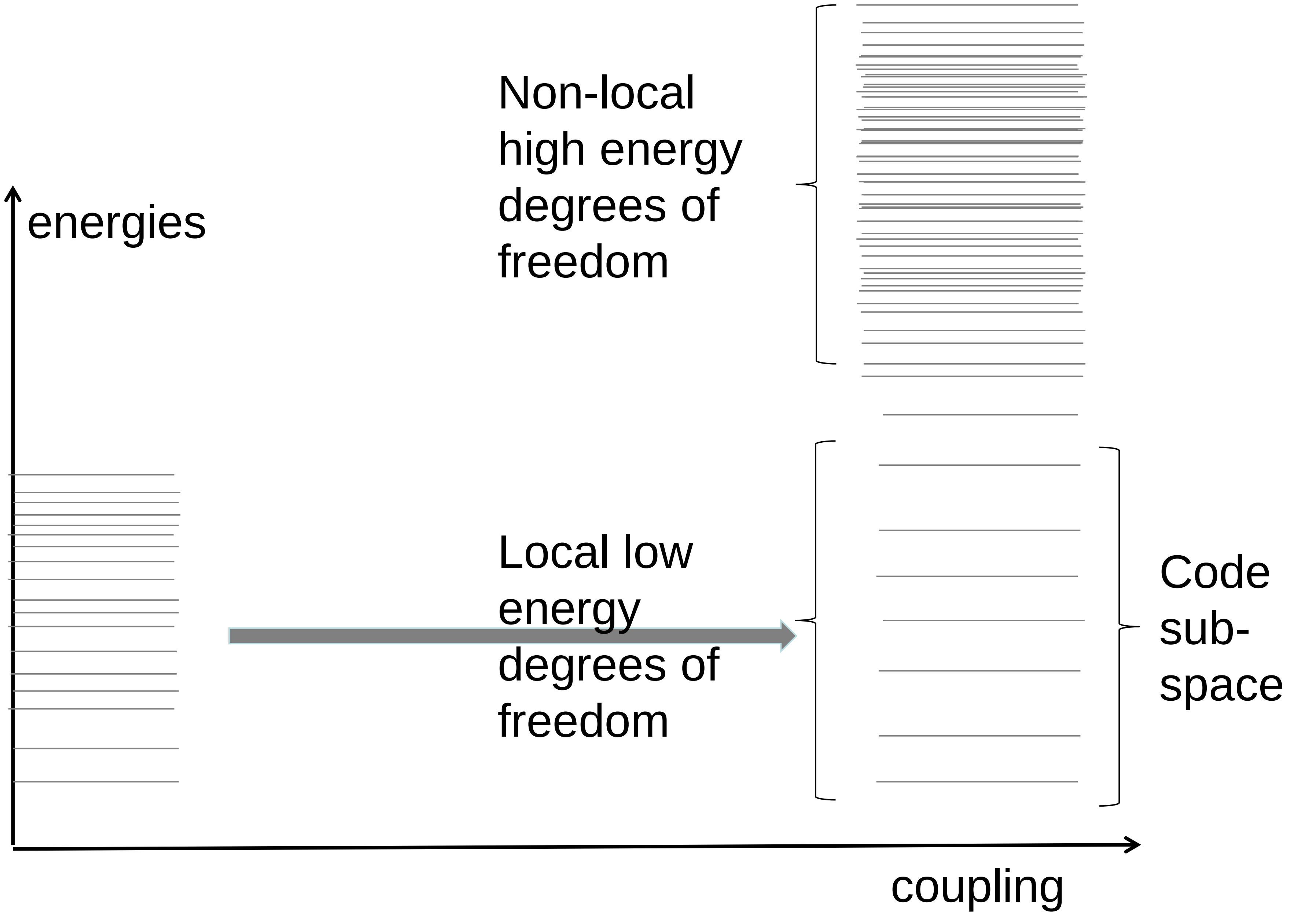
- Proof of RT using Euclidean path integrals and the replica trick (LM,FLM)

$$S_{\mathrm{vN}} = -\mathrm{Tr}(\rho \log \rho) = \lim_{n \to 1} \frac{1}{1-n} \log \mathrm{Tr} \rho^n$$

- Use of quantum extremal surface

$$S_{\mathrm{vN}}(A) = \min_S \left(\frac{A(S)}{4G_N} + S_{\mathrm{vN,\,bulk}}(S) \right)$$

- Extension to closed bulk curves through differential entropy (BCCdBH)



Important: spectrum and code subspace

Bulk reconstruction:

Reconstruct perturbative correlation functions of local low-energy bulk operators in a given semiclassical background in terms of boundary data. Can at best be an approximate notion.

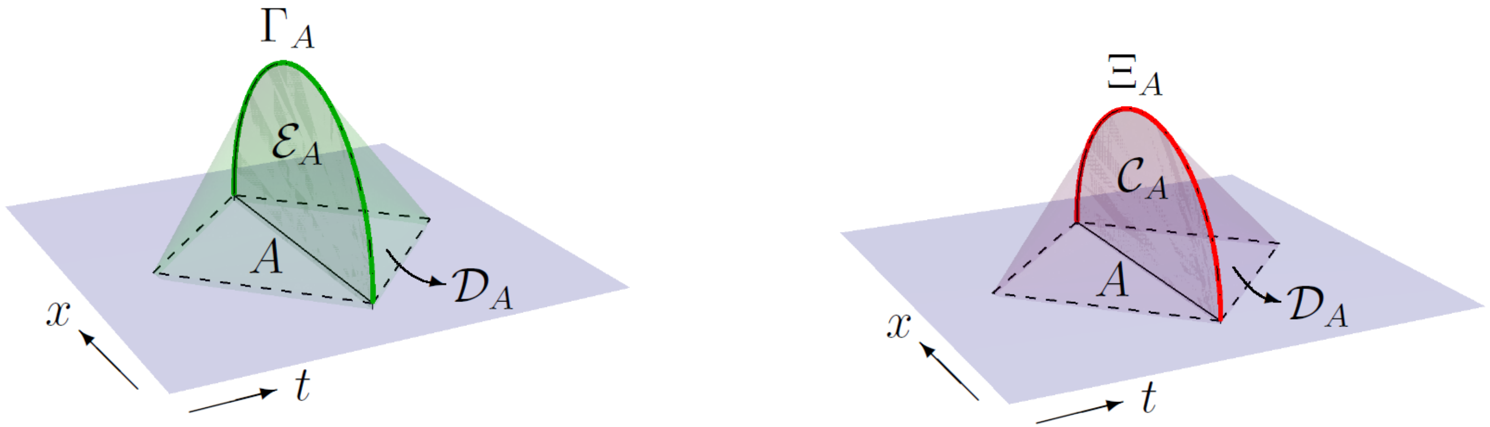
The subset of the Hilbert space that is accessible via local low-energy bulk operators is the code subspace which depends on the initial background/state in the CFT.

Standard perturbative approach involving bulk to boundary propagators etc: HKLL

$$\phi(r,x) = \int dy G(r,x,y)\mathcal{O}(y) + \int dy dz G(r,x,y,z)\mathcal{O}(y)\mathcal{O}(z) + \dots$$

Causal wedge reconstruction is the statement that perturbative bulk correlators *in the causal wedge* can be reconstructed from boundary correlators in a subregion ("subregion-subregion duality") – seems quite reasonable.

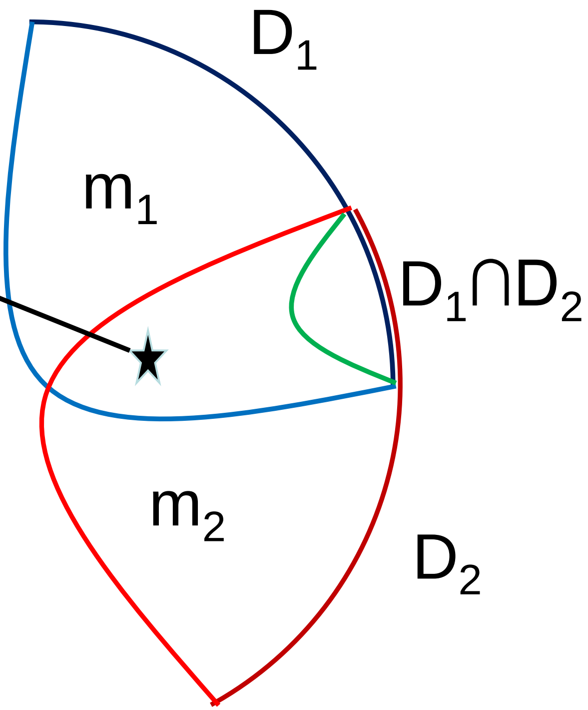
Entanglement wedge reconstructions states the same but now for the generically larger *entanglement wedge.* Less intuitive.



From 1804.05855

Entanglement (and causal) wedge reconstruction can only be approximate. If it were exact there would be a contradiction.

A local operator here would act entirely in D_1 but also entirely in D_2; but it does not act in $D_1 \cap D_2$. This is a contradiction. Exact bulk local operators do not exist.

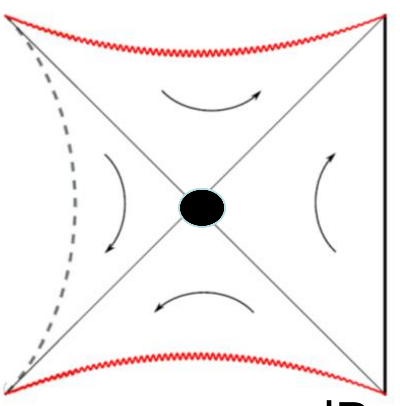


If reconstruction is only approximate and only applies in the code subspace there is no contradiction – cf quantum error correction.
(Almheiri, Dong, Harlow)

The original statement of Ryu-Takayangi is perhaps somewhat imprecise. It compares a fine-grained quantity (entanglement entropy) to a coarse grained quantity (area).

It clearly fails when A is all of space and the system is in a typical pure state.

Engelhardt-Wall: relate area of marginal trapped surfaces to outer entropy, a coarse grained notion of entropy where one extremizes over all states keeping low-energy physics on the outside fixed.



dBvBLPV

Can relate this to HRT and also works with quantum corrections.

$$S_{\mathrm{vN}}(A) = \min_S \left(\frac{A(S)}{4G_N} + S_{\mathrm{vN,\,bulk}}(S) \right)$$

Quantum entanglement wedge reconstruction also plays an important role in recent discussions of the information paradox by Almheiri, Engelhard, Marolf ,Maxfield; Almheiri, Mahajan, Maldacena, Zhao; Penington.

Unfortunately, explicit entanglement wedge reconstruction is quite hard Dong, Harlow, Wall; Cotler, Hayden, Penington, Salton, Swingle, Walter

Those papers start from the construction of Jafferis, Lewkowycz, Maldacena, Suh (JLMS) which in turn follows from Euclidean gravitational path integrals.

So somehow, Euclidean gravitational path integrals know about entanglement wedge reconstruction.

In entanglement wedge reconstruction the modular Hamiltonian plays an important role

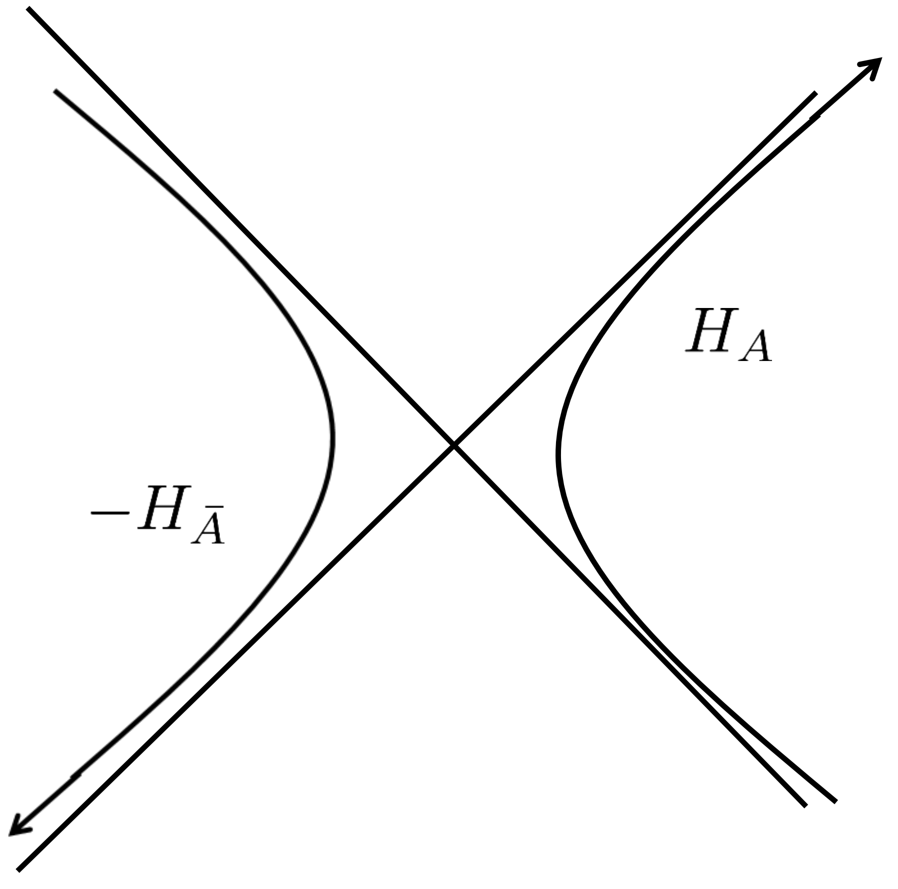
$$\rho: \ \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$$

$$\rho_A = e^{-H_A} \qquad \rho_{\bar{A}} = e^{-H_{\bar{A}}}$$

$$H = H_A \otimes \mathbb{I}_{\bar{A}} - \mathbb{I}_A \otimes H_{\bar{A}}$$

Modular Hamiltonian is in general a complicated non-local operator.

Simple example: modular Hamiltonian in Rindler space



Is a good approximation to modular flow near any entangling surface.

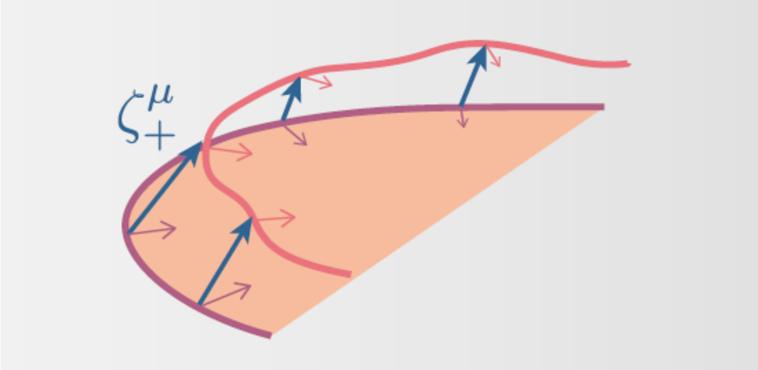
JLMS state that

$$P_{\rm code} H_{{\rm mod}\,A} P_{\rm code} = \frac{A}{4G_N} + H_{\rm bulk}$$

Modular flow near the bulk entangling surface is approximately geometric – useful in extracting local bulk physics near the entangling surface.

This is not enough to reconstruct bulk physics. Also need the translation generators of Rindler space and directions along the minimal surface.

To access the translations, need to consider deformations of extremal surfaces.



Suggests it is worth exploring the neighborhood of the extremal surface a bit more.

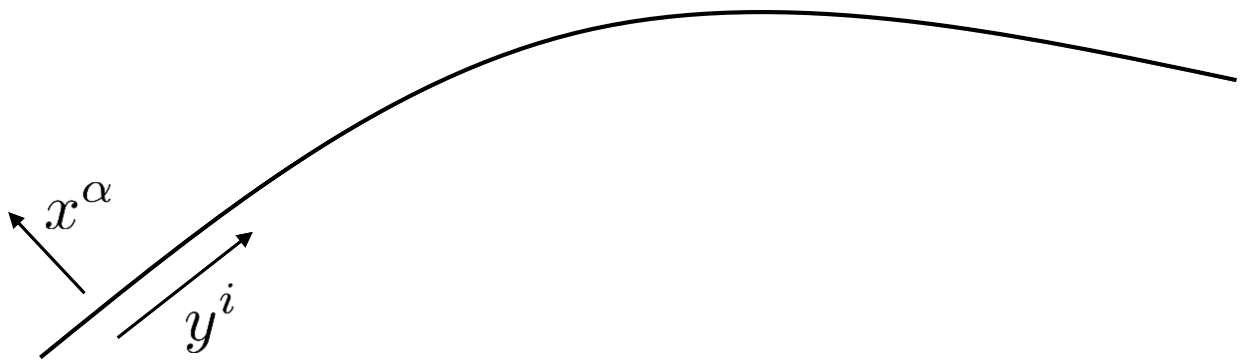
Bulk picture

It turns out that near a minimal surface there is a large new symmetry group in gravity due to the fact that we decompose the bulk geometry in two pieces.

These symmetries have been referred to as “surface symmetries”, “edge modes”, “asymptotic symmetry group”, etc (cf Donnelly, Freidel 16; Speranza 17; Camps 18).

Surface symmetries are diffeomorphisms with a non-trivial Noether charge on the minimal surface. Related to non-factorization of bulk gravitational Hilbert space.

Consider a minimal surface



Choose coordinates

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta + \gamma_{ij} dy^i dy^j + \mathcal{O}(x)$$

Modular Hamiltonian is boost in transversal plane

$$\begin{aligned} \zeta^\alpha_{\mathrm{mod}} &= 2\pi \epsilon^{\alpha\beta} x_\beta + \mathcal{O}(x^2) \\ \zeta^i_{\mathrm{mod}} &= \mathcal{O}(x^2) \end{aligned}$$

$$
\begin{aligned}
\zeta^{\alpha}_{\mathrm{mod}} &= 2\pi\epsilon^{\alpha\beta}x_{\beta} + \mathcal{O}(x^2) \\
\zeta^{i}_{\mathrm{mod}} &= \mathcal{O}(x^2)
\end{aligned}
$$

Symmetries:

$$
\begin{aligned}
\zeta^{\alpha}_{\mathrm{sym}} &= \omega(y)\epsilon^{\alpha\beta}x_{\beta} + \mathcal{O}(x^2) \\
\zeta^{i}_{\mathrm{sym}} &= \zeta^{i}_{0}(y) + \mathcal{O}(x^2)
\end{aligned}
$$

Preserve asymptotic form of metric, have non-trivial charges, and commute with modular Hamiltonian vector field.

Two components: diffeomorphisms along surface and location-dependent frame rotations.

Because these symmetries commute with the bulk modular Hamiltonian, and in view of

$$P_{\rm code} H_{{\rm mod}A} P_{\rm code} = \frac{A}{4G_N} + H_{\rm bulk}$$

the prediction is that in the boundary theory one finds a new enlarged symmetry group when restricting the modular Hamiltonian to the code subspace.

One could call these the symmetries of the modular Hamiltonian.

To reconstruct the local bulk translations we have to consider deformations of the minimal surface.

The extra symmetries provide these deformations with extra structure: a notion of parallel transport which also carries important bulk information.

Parallel transport of modular Hamiltonians

$$H_{\mathrm{mod}}(\lambda) = U^\dagger(\lambda)\Delta(\lambda)U(\lambda)$$

Δ diagonal

U is ambiguous up to unitaries that leave Δ invariant. Generically $U(1)^d$. Under infinitesimal change

$$\dot{H}_{\mathrm{mod}} = [\dot{U}^\dagger U, H_{\mathrm{mod}}] + U^\dagger \dot{\Delta} U$$

Parallel transport = particular choice of $U(\lambda)$
(flatness condition)

$$\dot{H}_{\mathrm{mod}} = [\dot{U}^\dagger U, H_{\mathrm{mod}}] + U^\dagger \dot{\Delta} U$$

Proposal: parallel transport is defined by requiring that $\dot{U}^\dagger U$ does not possess a *modular zero mode*

$$P_0^\lambda[V] \equiv \lim_{\Lambda\to\infty} \frac{1}{2\Lambda}\int_{-\Lambda}^{\Lambda} ds\ e^{iH_{\mathrm{mod}}(\lambda)s}\, V e^{-iH_{\mathrm{mod}}(\lambda)s}$$

$$P_0^\lambda[\dot{U}^\dagger U] = 0$$

$$\dot{H}_{\mathrm{mod}} = [\dot{U}^\dagger U, H_{\mathrm{mod}}] + U^\dagger \dot{\Delta} U$$

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zero mode

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$$P_0^\lambda[\dot{U}^\dagger U] = 0$$

Equivalently, can write:

$$\dot{U}^\dagger U = [H_{\rm mod}, W]$$

$$\dot{H}_{\rm mod} = [\cancel{\dot{U}^\dagger U}, H_{\rm mod}] + \cancel{U^\dagger \dot{\Delta} U}$$

zero mode

zero mode

These equation may look peculiar, but are nothing but a generalization of the Berry phase:

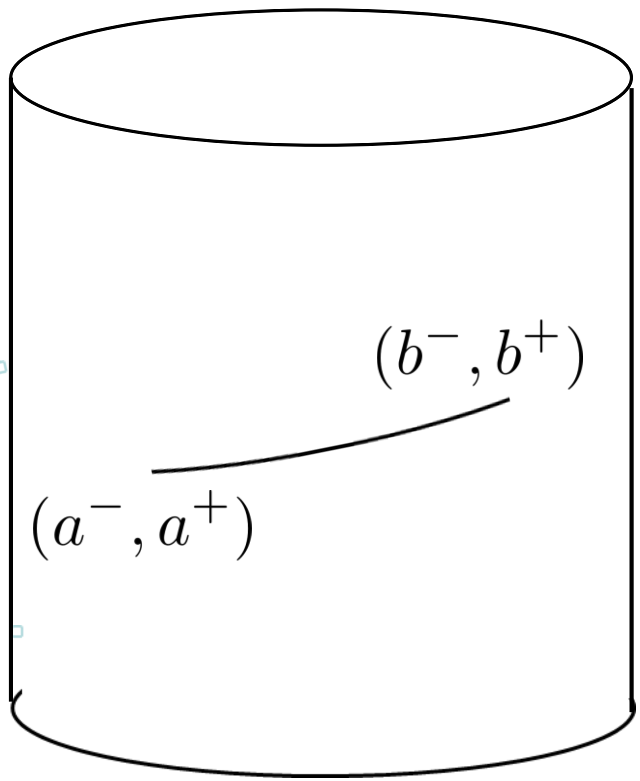
Take $\rho(\lambda) = |\psi(\lambda)\rangle\langle\psi(\lambda)|$ then

$$|\dot\psi\rangle\langle\psi| + |\psi\rangle\langle\dot\psi| = [\ |\dot\psi\rangle\langle\psi| - |\psi\rangle\langle\dot\psi|\, ,\ |\psi\rangle\langle\psi|\ ]$$

project

$$\langle\psi\ \ |\dot\psi\rangle\langle\psi| - |\psi\rangle\langle\dot\psi|\ \ \psi\rangle = \underbrace{\langle\psi|\dot\psi\rangle - \langle\dot\psi|\psi\rangle}_{\text{usual Berry connection}}$$

Example 1: 2d CFT with intervals



$$H_{\mathrm{mod}} = s_1 L_1 + s_0 L_0 + s_{-1} L_{-1} + t_1 \bar{L}_1 + t_0 \bar{L}_0 + t_{-1} \bar{L}_{-1}$$

$$s_1 = \frac{2\pi \cot(b^+ - a^+)/2}{e^{ia^+} + e^{ib^+}} \qquad s_0 = -2\pi \cot(b^+ - a^+)/2$$ etc

$$\frac{\partial H_{\rm mod}}{\partial \lambda} = \frac{\partial a^+}{\partial \lambda}\frac{\partial H_{\rm mod}}{\partial a^+} + \frac{\partial b^+}{\partial \lambda}\frac{\partial H_{\rm mod}}{\partial b^+} + \frac{\partial a^-}{\partial \lambda}\frac{\partial H_{\rm mod}}{\partial a^-} + \frac{\partial b^-}{\partial \lambda}\frac{\partial H_{\rm mod}}{\partial b^-}$$

$$2\pi i V = \frac{\partial a^+}{\partial \lambda}\frac{\partial H_{\rm mod}}{\partial a^+} - \frac{\partial b^+}{\partial \lambda}\frac{\partial H_{\rm mod}}{\partial b^+} - \frac{\partial a^-}{\partial \lambda}\frac{\partial H_{\rm mod}}{\partial a^-} + \frac{\partial b^-}{\partial \lambda}\frac{\partial H_{\rm mod}}{\partial b^-}$$

then

$$\partial_\lambda H_{\rm mod} = [V, H_{\rm mod}]$$

This reproduces the results of Czech, Lamprou, McCandlish and Sully 17. In particular, the holonomy is related to differential entropy.

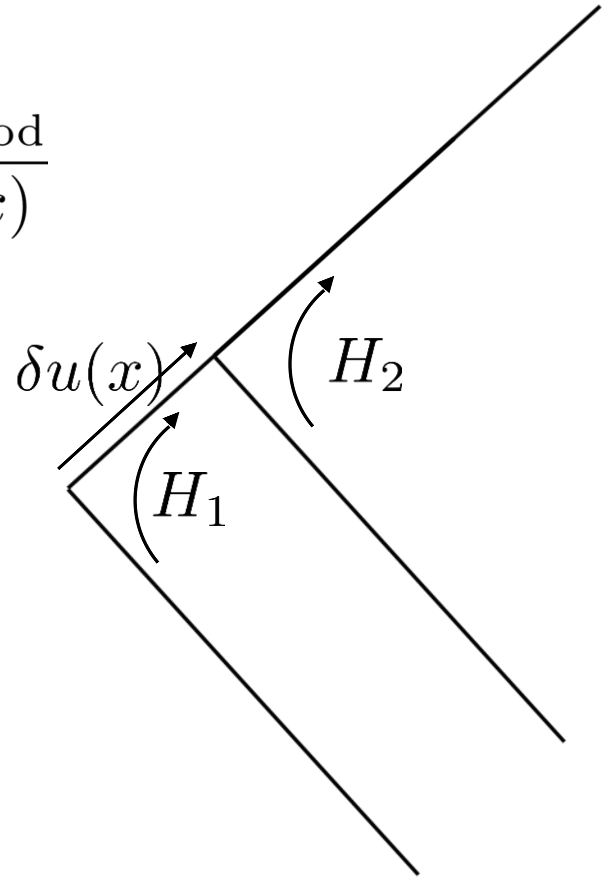
Example 2: modular inclusions (Casini, Teste, Torroba 17)

$$[H_2, H_1] = 2\pi i (H_2 - H_1)$$

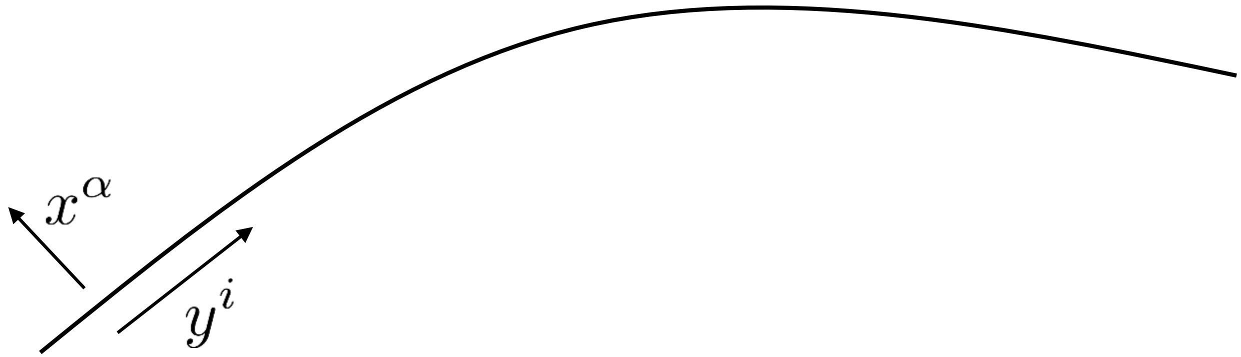
$$\left[\frac{\delta H_{\text{mod}}}{\delta u(x)}, H_{\text{mod}}\right] = 2\pi i \frac{\delta H_{\text{mod}}}{\delta u(x)}$$

$\downarrow$

connection



Bulk picture? Recall that



Choose coordinates

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta + \gamma_{ij} dy^i dy^j + \mathcal{O}(x)$$

$$\begin{aligned} \zeta^\alpha_{\rm mod} &= 2\pi \epsilon^{\alpha\beta} x_\beta + \mathcal{O}(x^2) \\ \zeta^i_{\rm mod} &= \mathcal{O}(x^2) \end{aligned} \qquad \begin{aligned} \zeta^\alpha_{\rm sym} &= \omega(y) \epsilon^{\alpha\beta} x_\beta + \mathcal{O}(x^2) \\ \zeta^i_{\rm sym} &= \zeta^i_0(y) + \mathcal{O}(x^2) \end{aligned}$$

Parallel transport should provide:

- a relation between the coordinates y of nearby minimal surfaces
- a relation between choices of orthonormal bases for the normal plane

One can repeat analysis:

$$\frac{d\zeta_{\rm mod}}{d\lambda} = [\xi, \zeta_{\rm mod}]$$

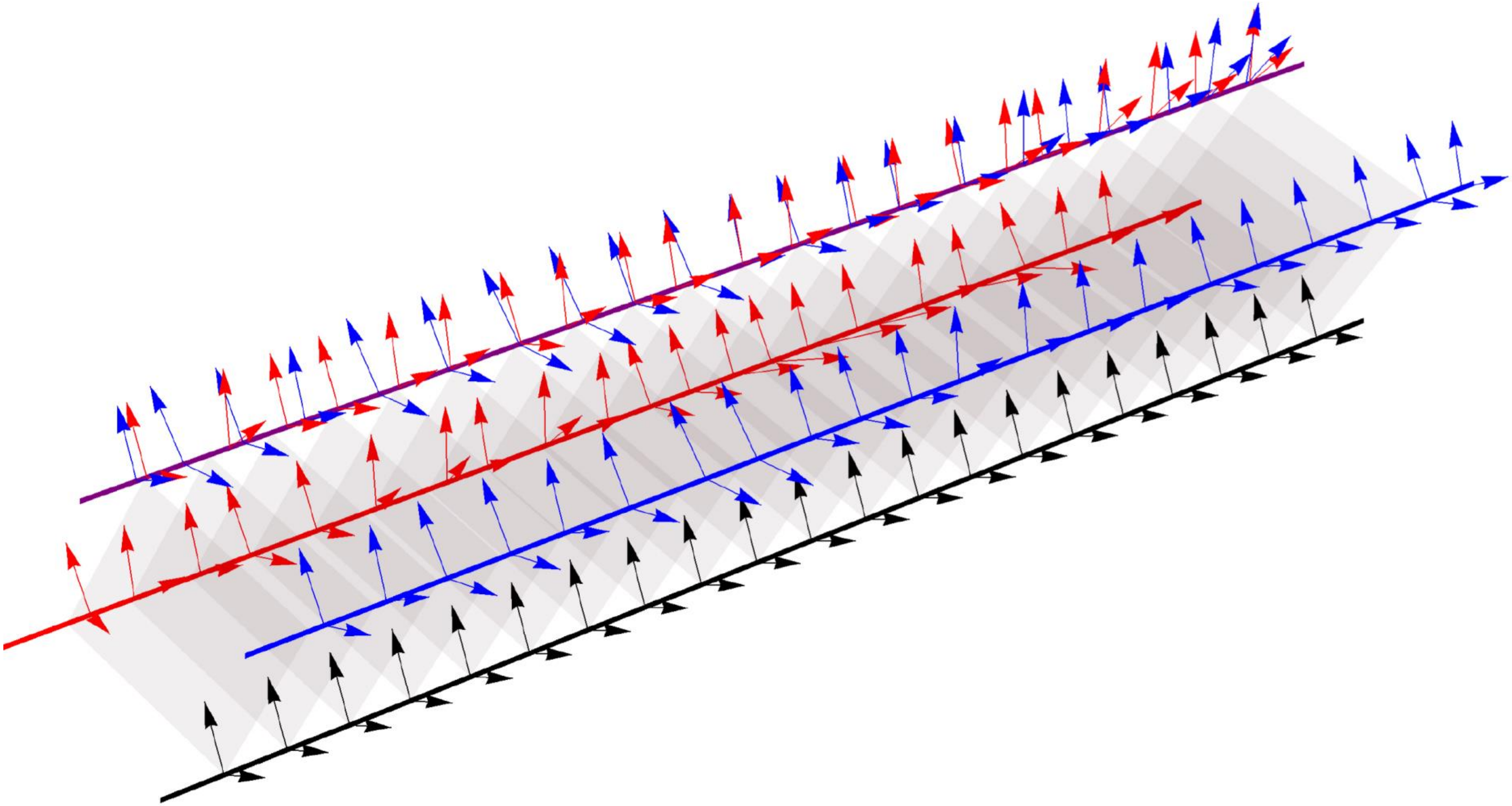
where on minimal surface ξ should not have modular zero modes:

$$\begin{aligned} \epsilon^{\alpha\beta}\partial_\alpha \xi_\beta &= 0 \\ \xi^i &= 0 \end{aligned}$$

Qualitative picture:

- To transport a point on a minimal surface to a point on a nearby minimal surface: separation should be orthogonal to the minimal surfaces

- To transport normal frame: parallel transport in orthogonal direction and project into new normal plane



Holonomy: surface diffeomorphism + frame rotation

Surface diffeomorphism illustrated

Endpoint of line gets displaced by length of path. Can use this to reconstruct length of curves.

(Balasubramanian, Chowdhury, Czech, JdB, Heller 13)

Interpretation of frame rotation? Closely related to bulk curvature.

Consider e.g. 2d case where minimal surfaces are points.

Procedure reduces to ordinary parallel transport of tangent vectors, and curvature of connection = geometric curvature.

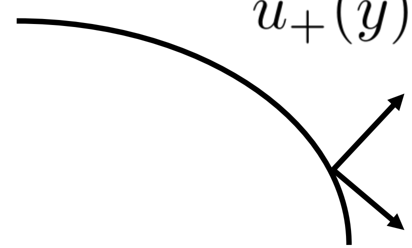
Connection to curvature only clear in case $K^2, \partial K \ll R$

For AdS3, boundary discussion can be directly translated into bulk (symmetries=Killing vectors) and agrees with above picture.

Examples show that there are special situations in which

$$\pm 2\pi i\, \delta H_{\rm mod} = [\delta H_{\rm mod}, H_{\rm mod}]$$

In the bulk, these correspond to deformations in light cone directions. One can think of these vector fields as “shock waves”



$$u_+(y)\frac{\partial}{\partial x^+} + \dots$$

$$u_-(y)\frac{\partial}{\partial x^-} + \dots$$

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta + \gamma_{ij} dy^i dy^j + \mathcal{O}(x)$$

In the bulk, these modes always exist. But what about the boundary theory?

Conjecture:

If $[H_{\rm mod}, \delta H_{\rm mod}] = i\lambda \delta H_{\rm mod}$ then $|\lambda| \leq 2\pi$. This is closely related to the bound on chaos. If there is a semiclassical gravitational dual the bound is saturated.

cf modular chaos of
Faulkner, Li, Wang 18

Better formulation (see talk by Lampros):

$$\frac{d}{ds} \log |\langle \chi_i | \delta H_{\rm mod}(s) | \chi_j \rangle| \leq 2\pi \quad \text{for:} \quad s \gg 1$$

where: $\delta H_{\rm mod}(s) = e^{-iH_{\rm mod} s} \delta H_{\rm mod} e^{iH_{\rm mod} s}$

Can be a bit more precise:

If $\mathcal{N} \subset \mathcal{M}$ then $\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it}$ is analytic for $0 < \mathrm{Imt} < 1/2$ and obeys $|\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it}| \leq 1$. (Borchers 99)

If $\phi \leq \psi$ then $\Delta_{\phi,\psi}^{-it}\Delta_{\psi}^{it}$ is analytic for $0 < \mathrm{Imt} < 1/2$ and obeys $|\Delta_{\phi,\psi}^{-it}\Delta_{\psi}^{it}| \leq 1$ (Araki 76)

$$\log(\Delta + \delta\Delta) - \log\Delta = \frac{\pi}{2}\int_{-\infty}^{\infty}\frac{dt}{\cosh^2 \pi t}\Delta^{-it-1/2}\delta\Delta\Delta^{it-1/2} + \ldots$$

(Sarosi, Ugajin 17; Lashkari, Liu, Rajagopal 18)

$$[H_{\text{mod}}, \delta H_{\text{mod}}] = i\lambda\delta H_{\text{mod}}$$

Example: take a 2d CFT on a circle and as subspace half of the circle.

$$H_{\text{mod}} = \pi i(L_1 - L_{-1})$$

$$[H_{\text{mod}}, L_1 + L_{-1} \pm 2L_0] = \pm 2\pi i(L_1 + L_{-1} \pm 2L_0)$$

These are the shockwave operators corresponding to ANEC operators in the CFT.

ANEC is the statement that $L_1 + L_{-1} + 2L_0 \geq 0$ which indeed holds in 2d CFT…

- Conjecture also holds for Virasoro deformations of AdS3
- does not hold in higher spin theories in AdS3 – maximal value of λ agrees with result of Perlmutter 16

$$\delta H_{\text{mod}} = W_{-2} + 4W_{-1} + 6W_0 + 4W_1 + W_2$$

$$[H_{\text{mod}}, \delta H_{\text{mod}}] = 4\pi i\, \delta H_{\text{mod}}$$

The “shock wave” vector fields have an interesting commutator

$$Q([V_+,V_-]) \sim \int_\Sigma \sqrt{\gamma}(\frac{1}{2}\gamma^{ij}\partial_i u_+ \partial_j u_- - 2R_{-+-+}u_+u_-)$$

which is somewhat similar to the expression for Planckian scattering found by ‘t Hooft 90; Verlinde, Verlinde 90…

Key in this computation is to pick the right vector fields, i.e. where the modular zero mode has been projected out.

This is also an explicit expression for part of the Berry curvature.

Have not used Einstein equations or extremality of surface in above computation yet, still missing some ingredients… (cf Lewkowycz, Parrikar 18)

Outlined a program to reconstruct the bulk using

- the modular Hamiltonian
- special “modular scrambling modes”
- the symmetries of the modular Hamiltonian in the code subspace.

Several ingredients have not fallen into place: e.g. the use of the bulk Einstein equations and the butterfly velocity in chaos.

The role of state deformations is also less clear, as is the corresponding bulk interpretation. There may possibly be interesting connections to complexity.

Many questions:

- Clarify the precise relation to standard OTOC chaos
- Connect to recent work on shock waves (Kologlu, Kravchuk, Simmons-Duffin, Zhiboedov 19; Belin, Hofman, Mathys 19)
- Connection to other recent appearances of the Berry connection? (Belin, Lewkowycz, Sarosi 18)
- Is all of this useful to get a handle on the code subspace?
- Connection to recent discussions of soft hair? (…)
- Include other fields (e.g. gauge fields) with non-trivial edge modes?
- Generalize to other spacetimes?
- Can we get dynamics in this framework?
- Applications to two-sided case?