The scattering amplitude of stringy hadrons with charged endpoints

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Nordita July Crete September 2019

In Memory of Ioannis Bakas

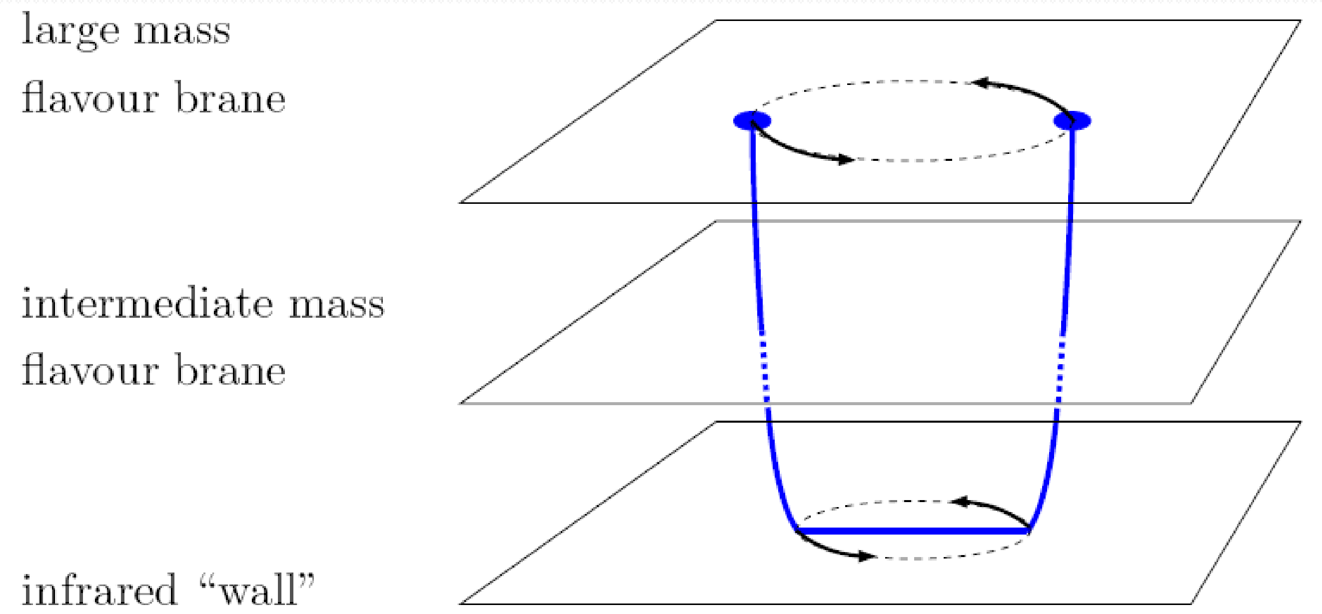
Motivation

- The spectrum and decay width of hadrons admit a clear stringy behavior.
- Are scattering processes like the p p collisions at the LHC related to scattering of strings?
- What is the corresponding scattering amplitude?
- Can we identify electro-magnetic properties of hadrons that have a stringy nature
- The stringy hadron carries electric charge on its ends. It is well known that such a system admits non-commutative geometry. What is the structure of this geometry for hadrons and can we suggest experiments to test it

Stringy holographic Hadrons

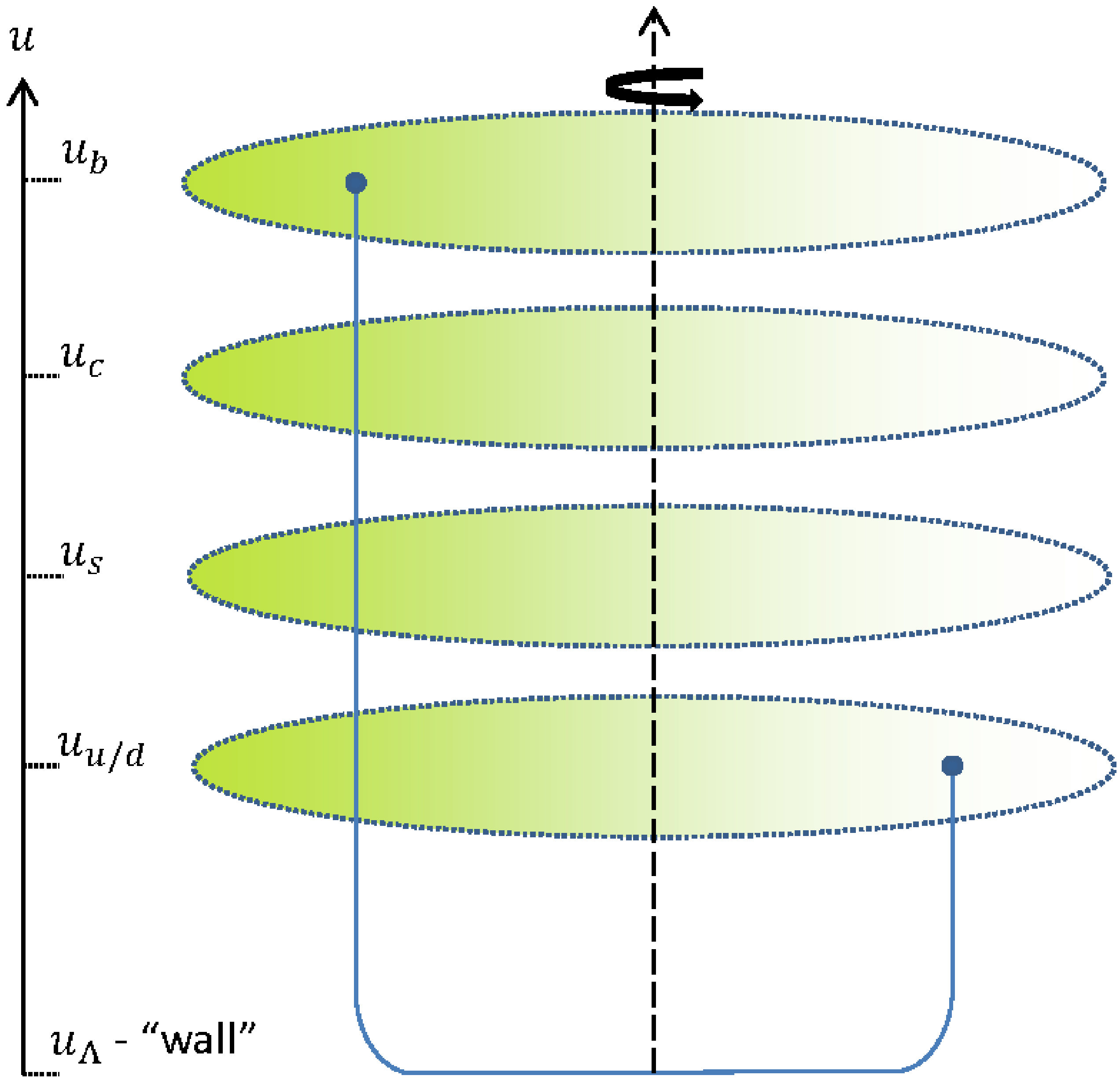
(1) The rotating holographic string meson

- The holographic meson is a string connected to flavor branes



- The string is the classical solution of the **Nambu-Goto** action defined in **confining holographic background**

Example: The B meson



(2) Stringy Baryons

- How do we identify a baryon in holography ?

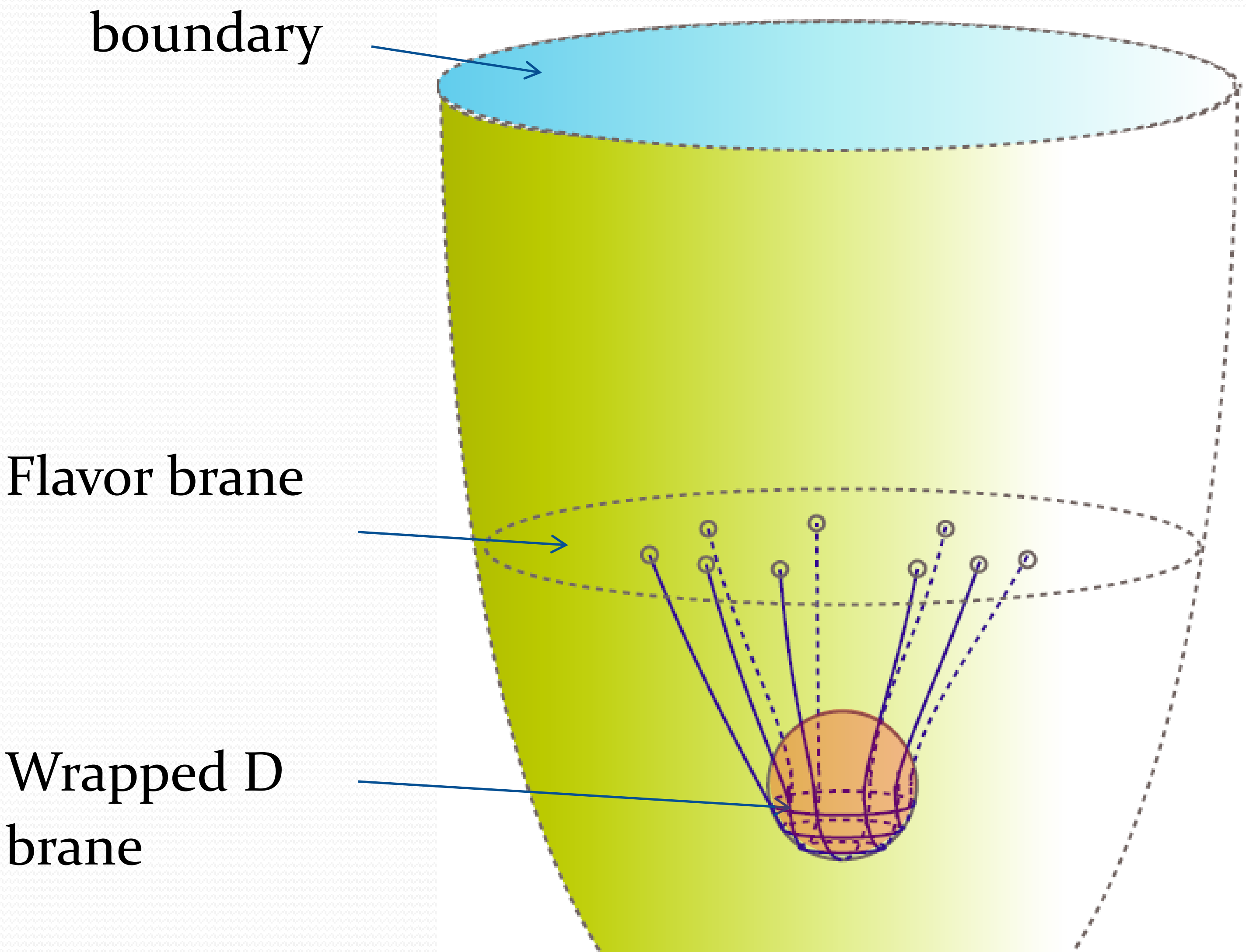
- Since a quark corresponds to an end of a string, the baryon has to be a structure with N_c strings connected to it.

- The proposed **baryonic vertex** in holographic background is is a wrapped Dp brane over a p cycle

- Because of the RR flux in the background the wrapped brane has to be connected to Nc strings

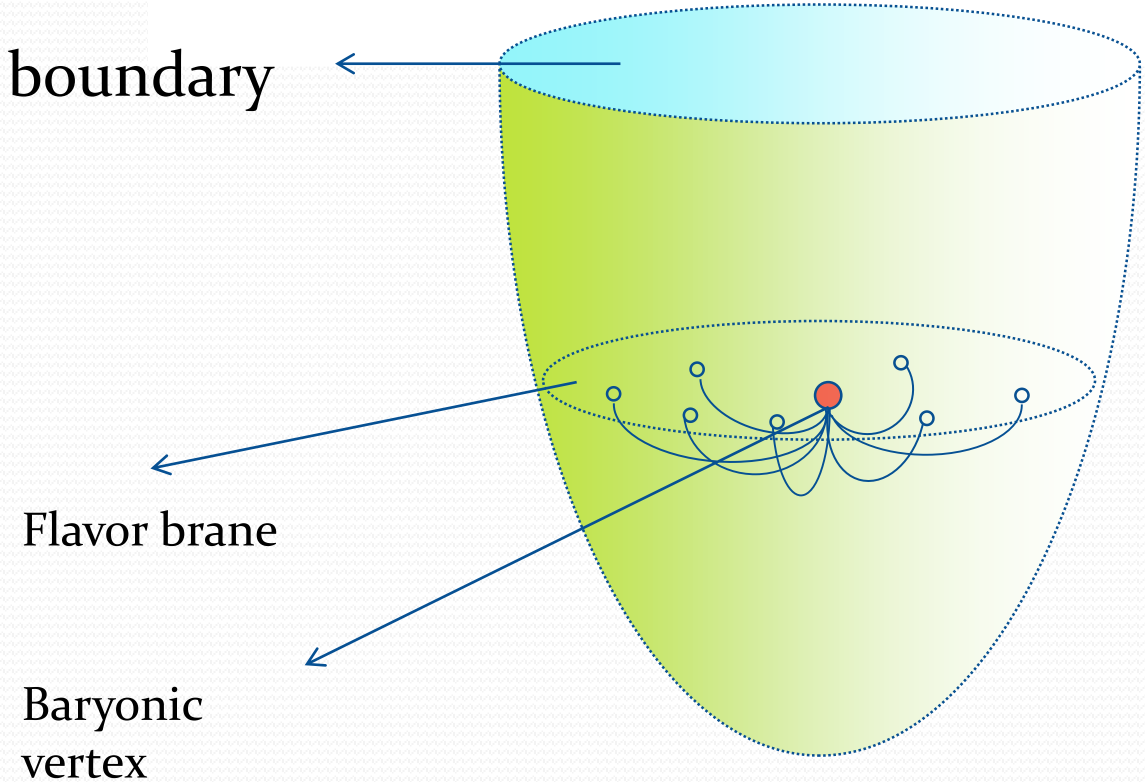
Dynamical baryon

- Dynamical baryon – Nc strings connecting the baryonic vertex and flavor branes



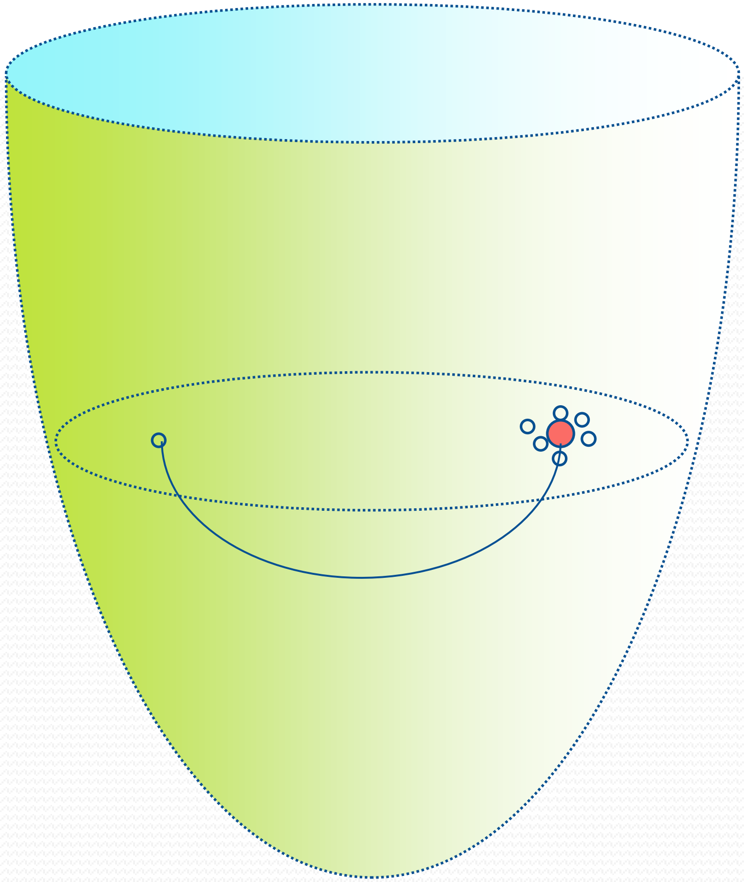
A possible baryon layout

- A possible dynamical baryon - Nc strings connected in a symmetric way to the flavor brane and to the baryonic vertex which is also on the flavor brane.



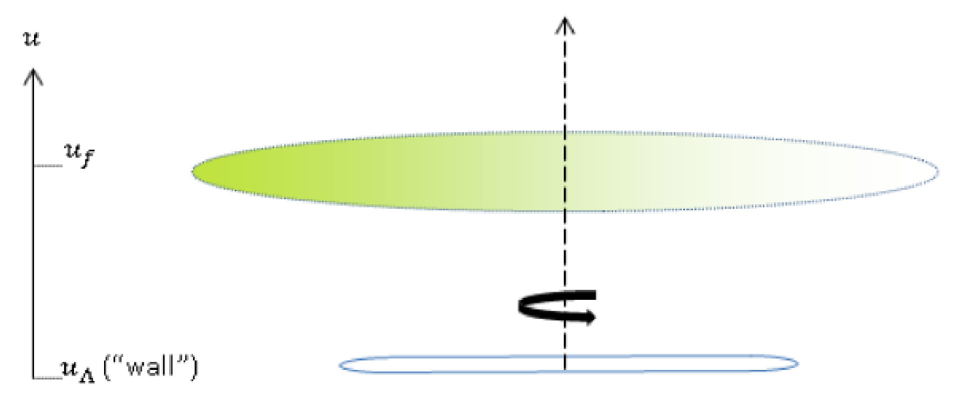
Nc-1 quarks around the Baryonic vertex

- An asymmetric possible layout is that of one quark connected with a string to the baryonic vertex to which the rest of the Nc-1 quarks are attached.



(3) Glueballs as closed strings

- Mesons are open strings connected to flavor branes.
- Baryons are Nc open strings connected to a baryonic vertex on one side and to a flavor brane on the other one.
- What are glue balls?
- Since they do not incorporate quarks it is natural to assume that they are rotating closed strings
- Angular momentum associates with rotation of folded closed strings



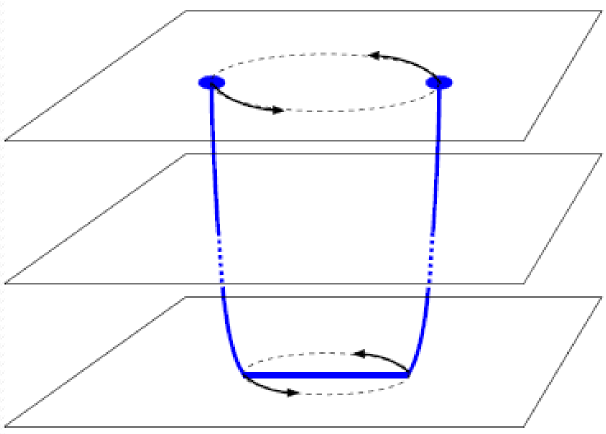
Hadrons of the (HISH) Holographic Inspired stringy model

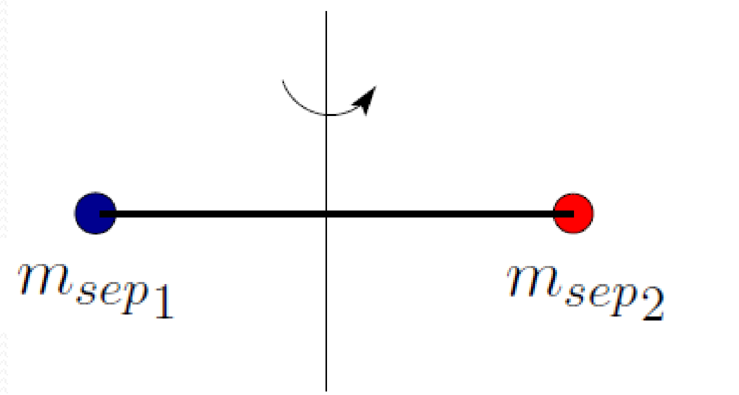
HISH- Holography Inspired Stringy Hadron

- The construction of the HISH model is based on the following steps.
- (i) Analyzing classical string configurations in confining holographic string models that correspond to hadrons.
- (ii) Performing a transition from the holographic regime (for fields) of large Nc and large λ to the real world that bypasses expansions in $\frac{1}{N_c}$ and $\frac{1}{\lambda}$
- (iii) Proposing a model of **stringy hadrons** in **flat four dimensions with massive endpoint particles** that is **inspired** by the corresponding holographic model
- (iv)Dressing the endpoint particles with structure like **baryonic vertex**, **charge, spin** etc
- (v) Confronting the outcome of the models with **experimental data** .

The HISH map of a stringy hadron

- The basic idea is to approximate the classical holographic spinning string by a string in flat space time with massive endpoints. The masses are m_{sep_1} and m_{sep_2}





String end-point mass

- We define the string end-point quark mass

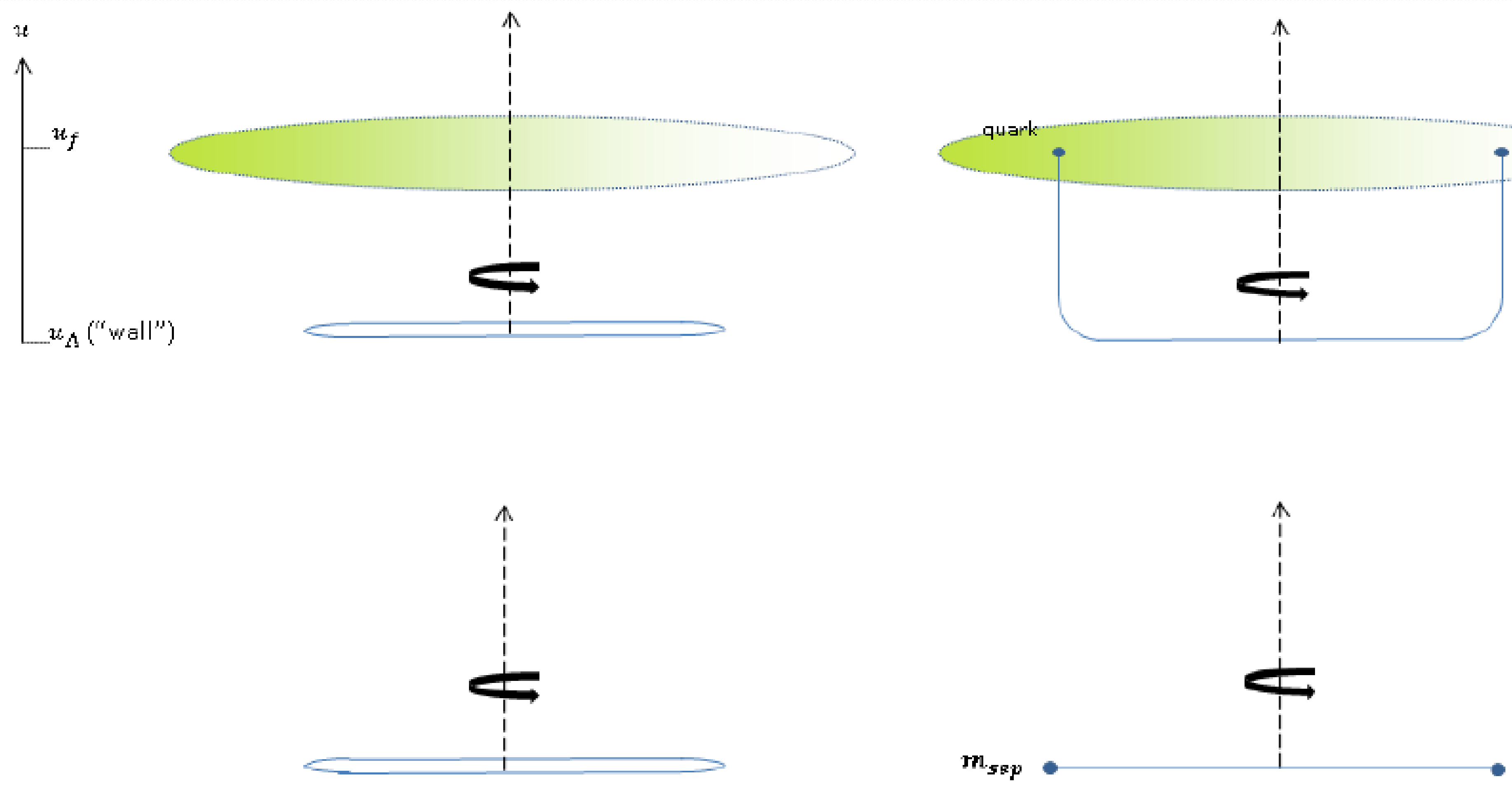
$$\boxed{m_{sep} = T\int_{u_0}^{u_f} g(u)du = T\int_{u_0}^{u_f} \sqrt{G_{00}G_{uu}}du}$$

- The boundary equation of motion is

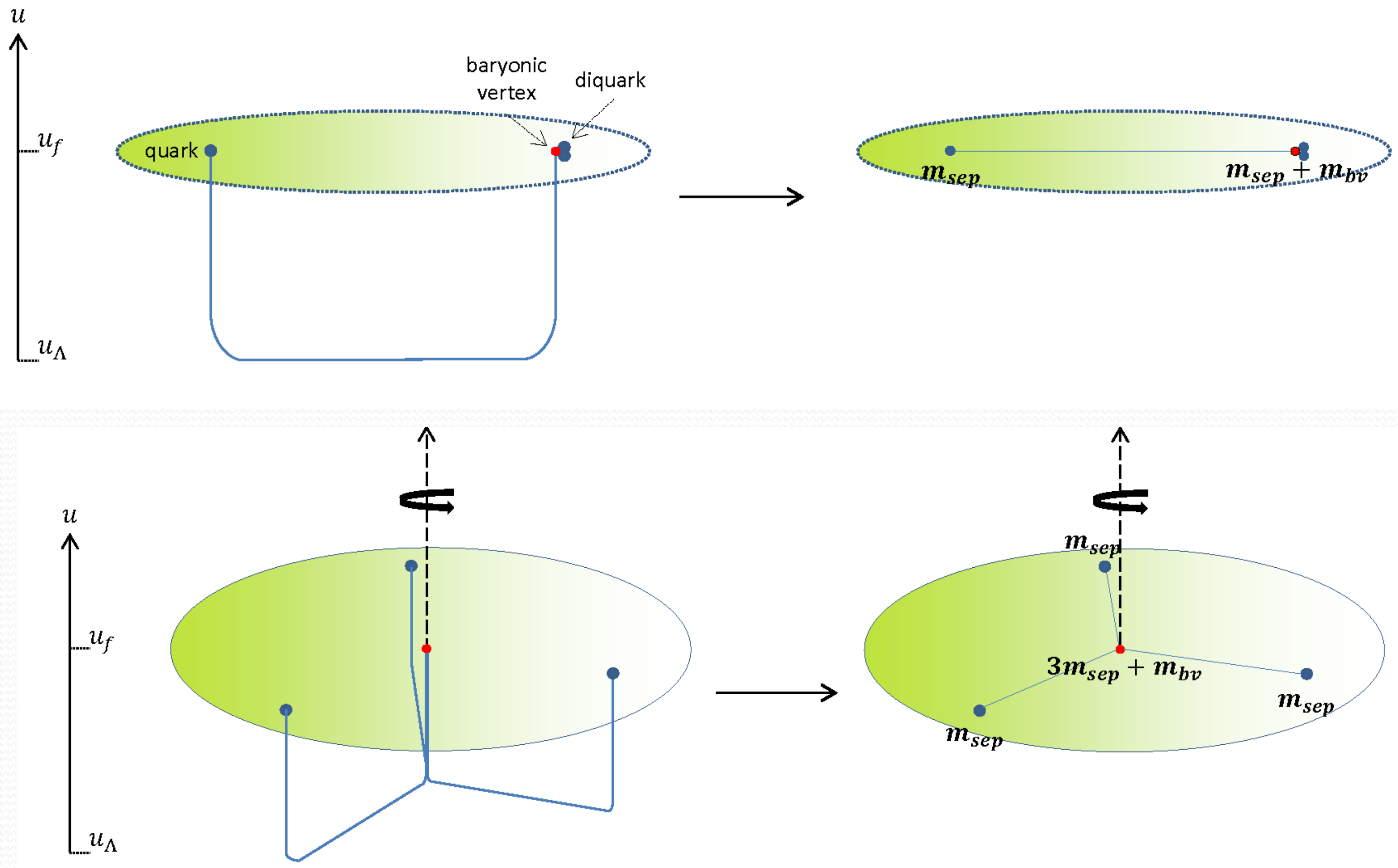
$$\boxed{\frac{T_{eff}}{\gamma} = m_{sep}\gamma\omega^2 R_0}$$

- This simply means that the tension is balanced by the (relativistic) centrifugal force.

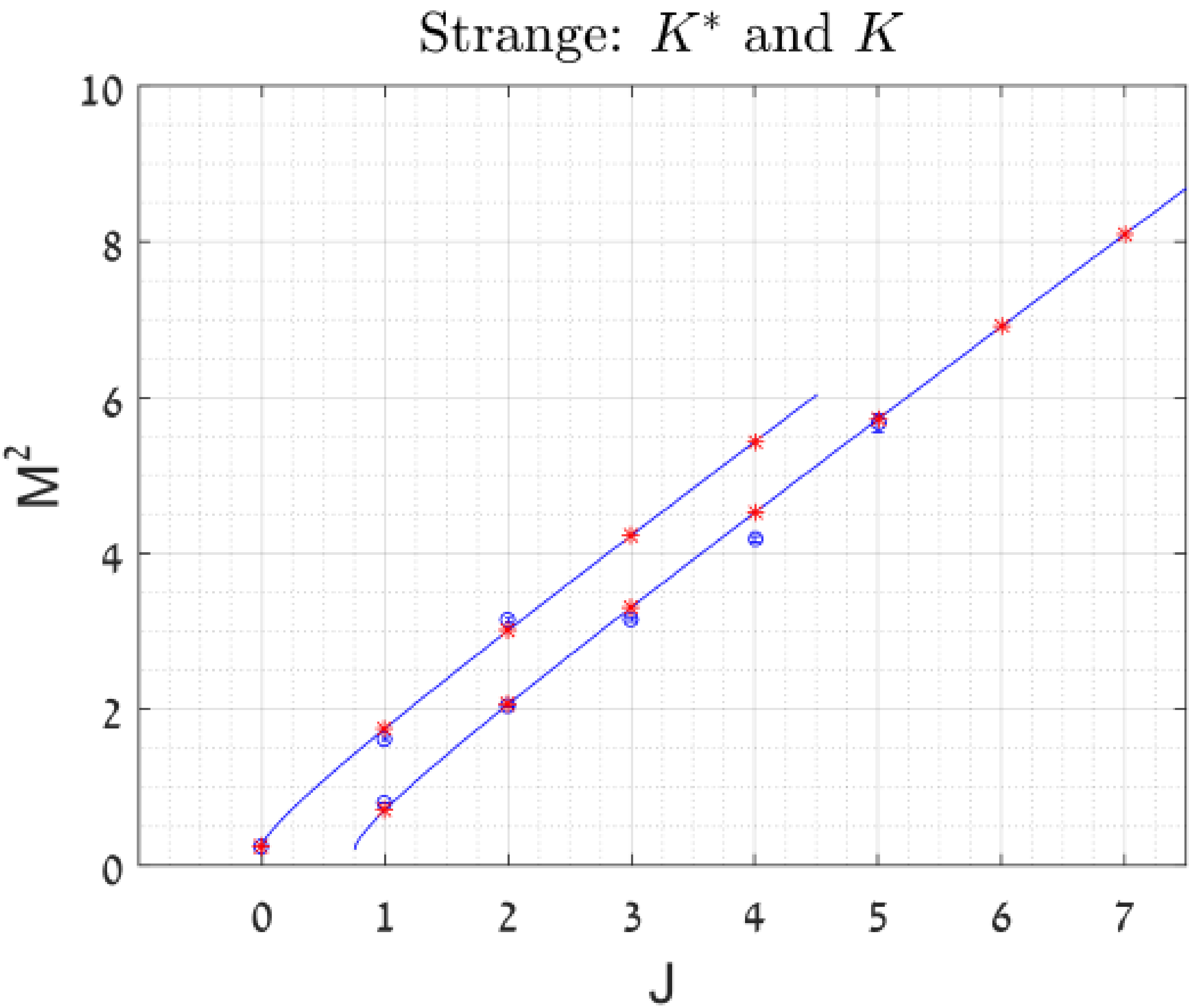
Holographic mesons and glueballs and their map



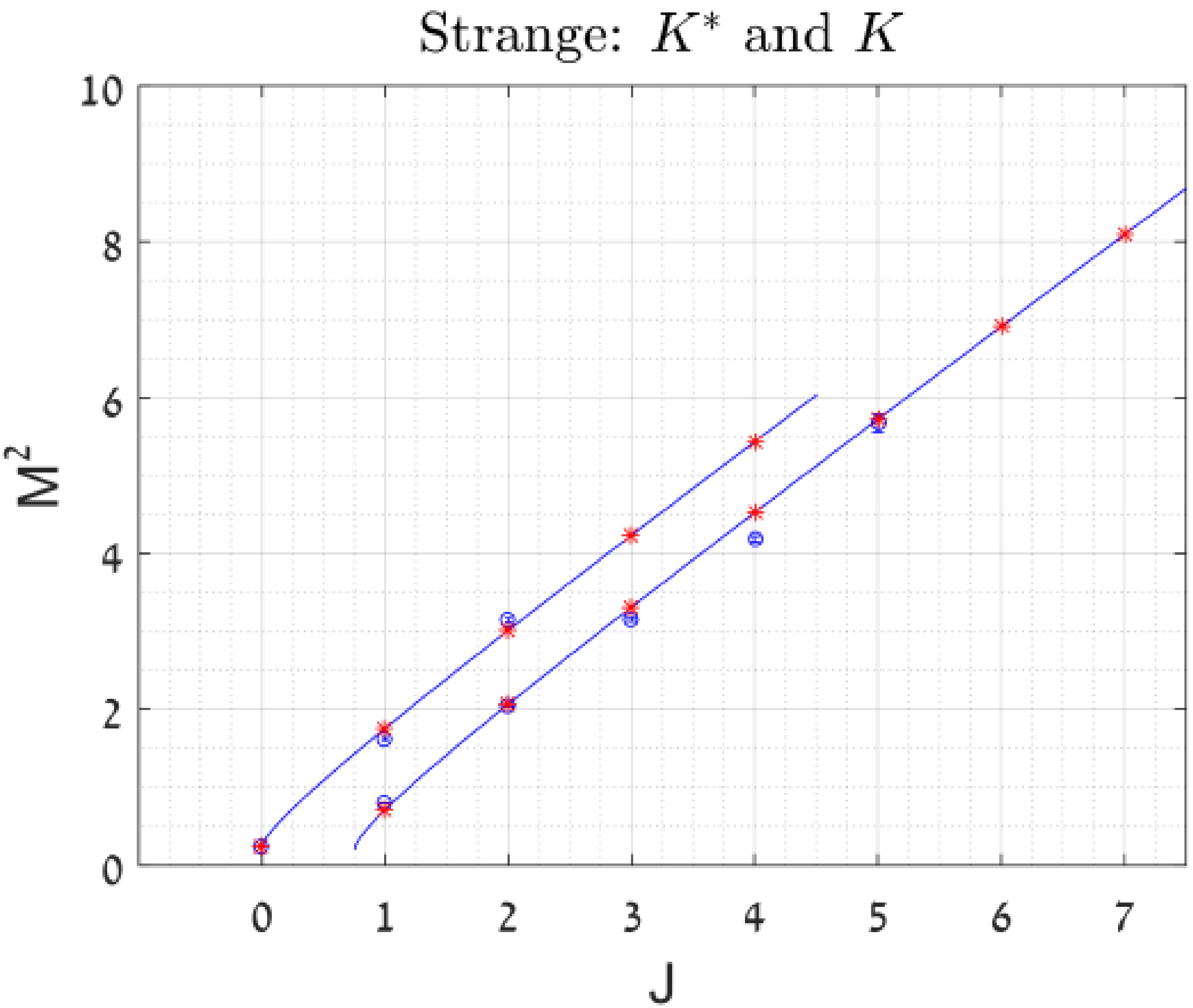
(ii) The HISH Baryons



Fitted trajectories of mesons



Fitted trajectories of mesons



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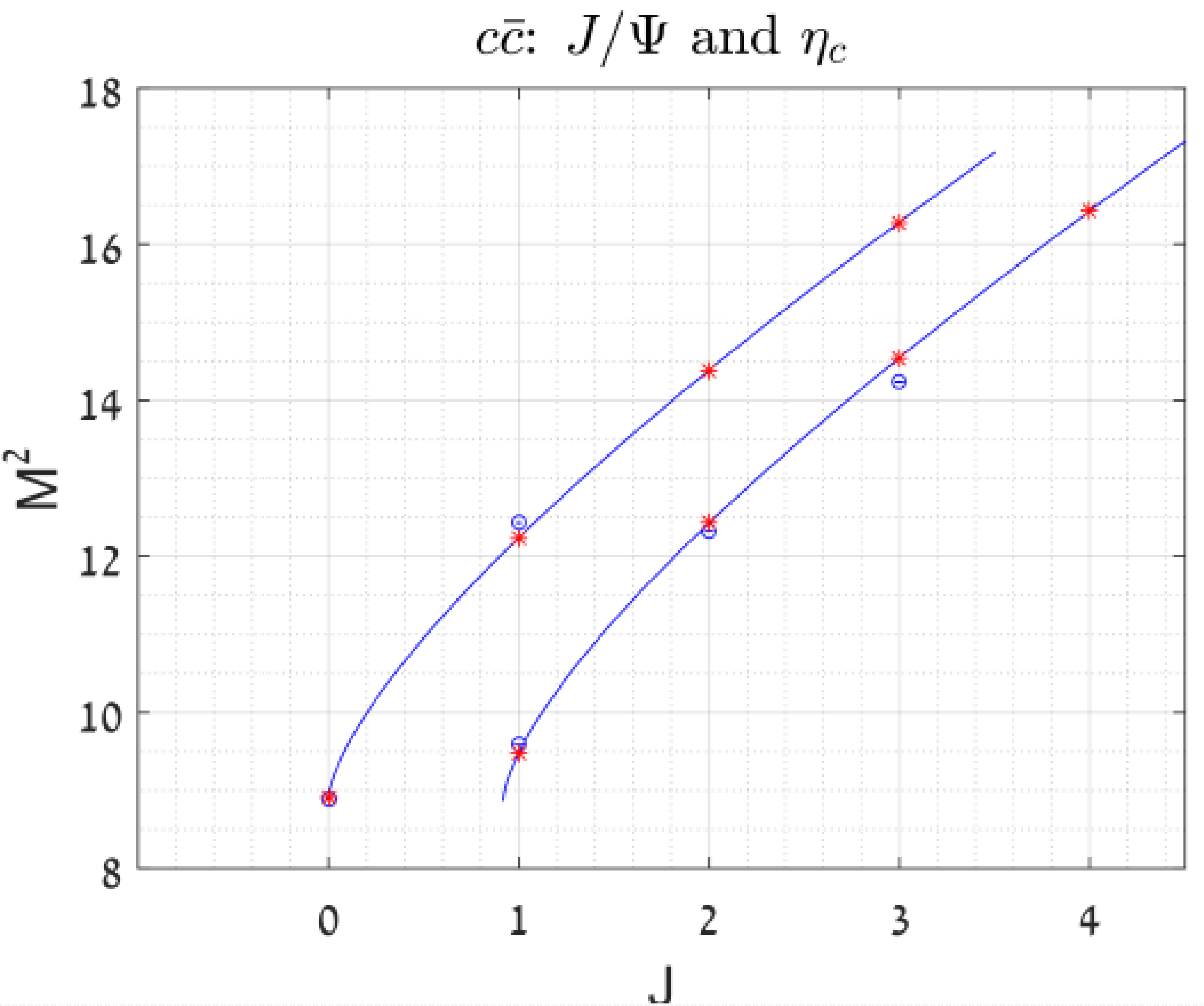
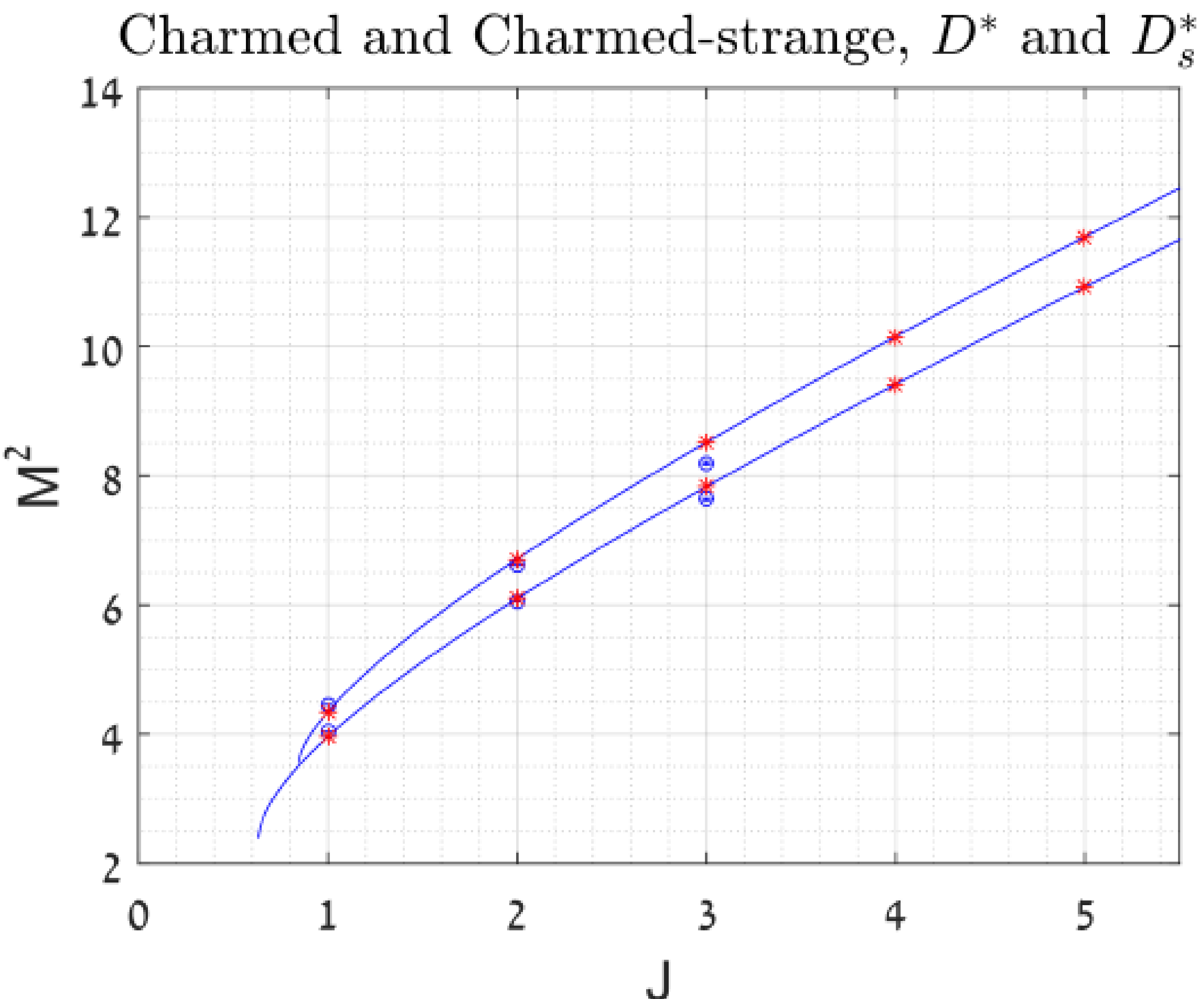


Fitted trajectories of mesons



Toward a universal model

- The fit results for several trajectories simultaneously. The (J, M^2) trajectories of ρ, ω, K^*, ϕ D, and Ψ mesons
- We take the string endpoint masses in MeV

$$m_{u/d} = 60, m_s = 220, m_c = 1500$$

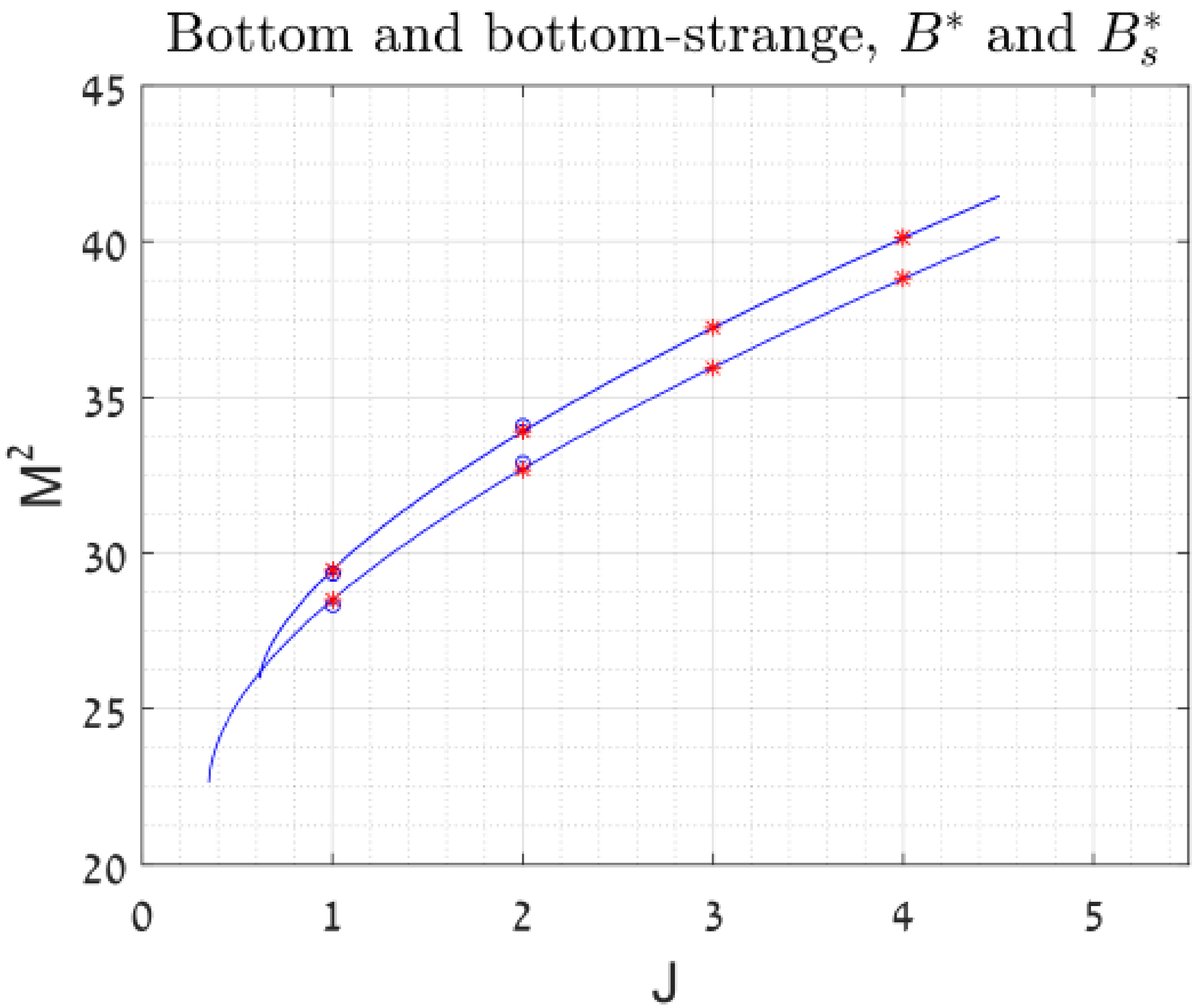
- Only the intercept was allowed to change. We got

$$\alpha' = 0.899$$

$$a_\rho = 0.51, a_\omega = 0.52, a_{K^*} = 0.49$$

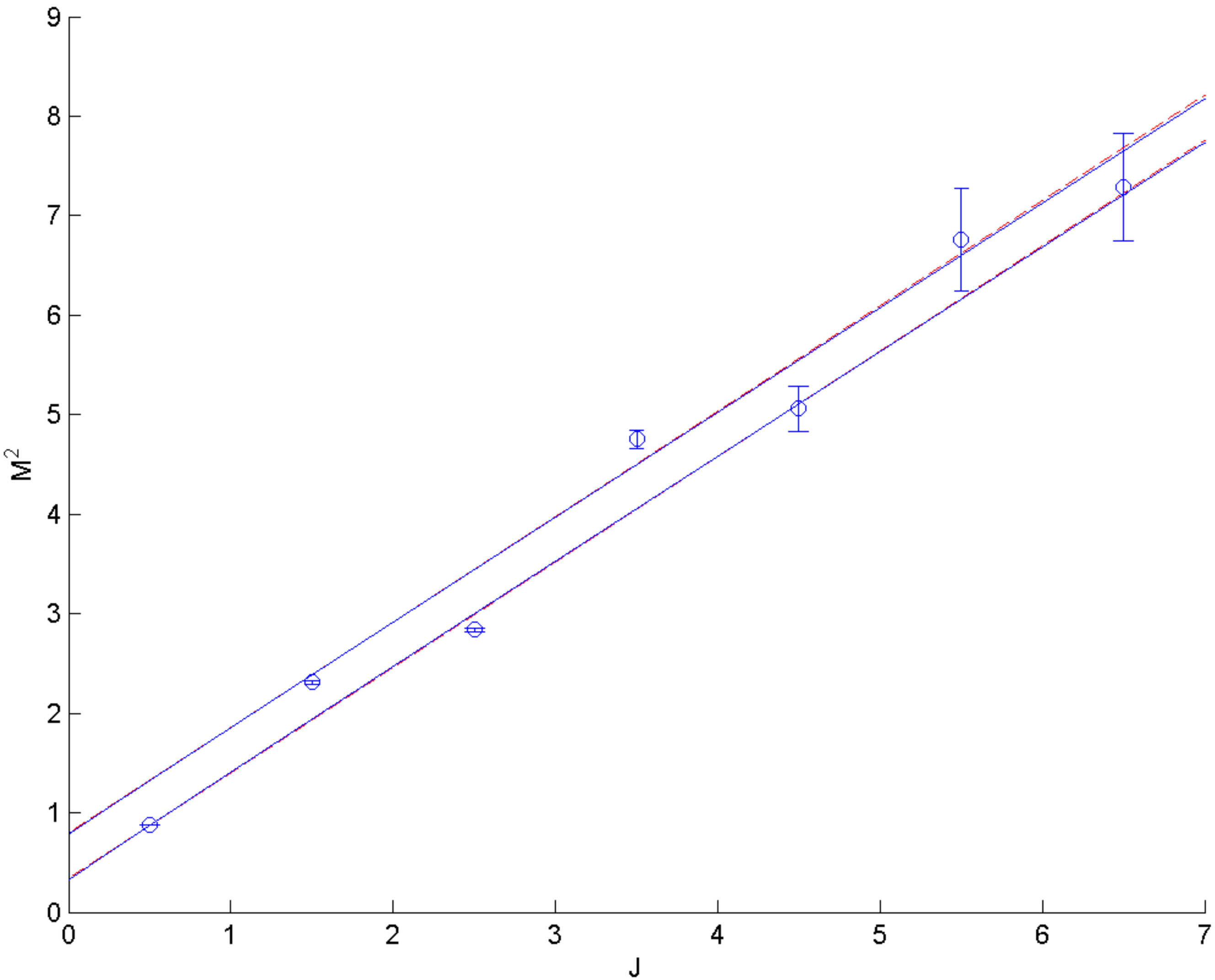
$$a_\phi = 0.44, a_D = 0.80, a_\Psi = 0.94$$

Fitted trajectories of mesons

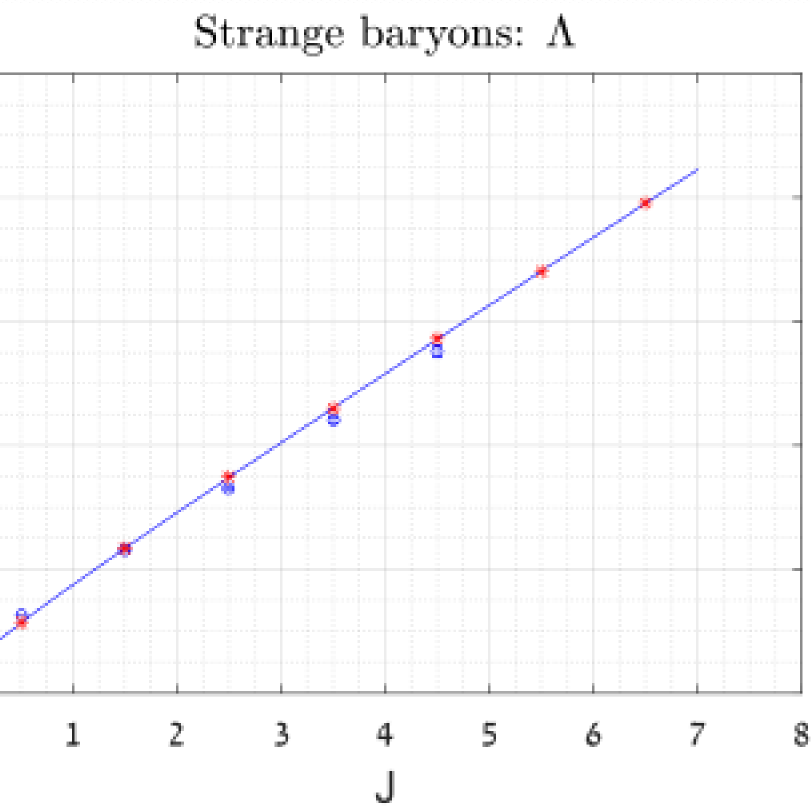


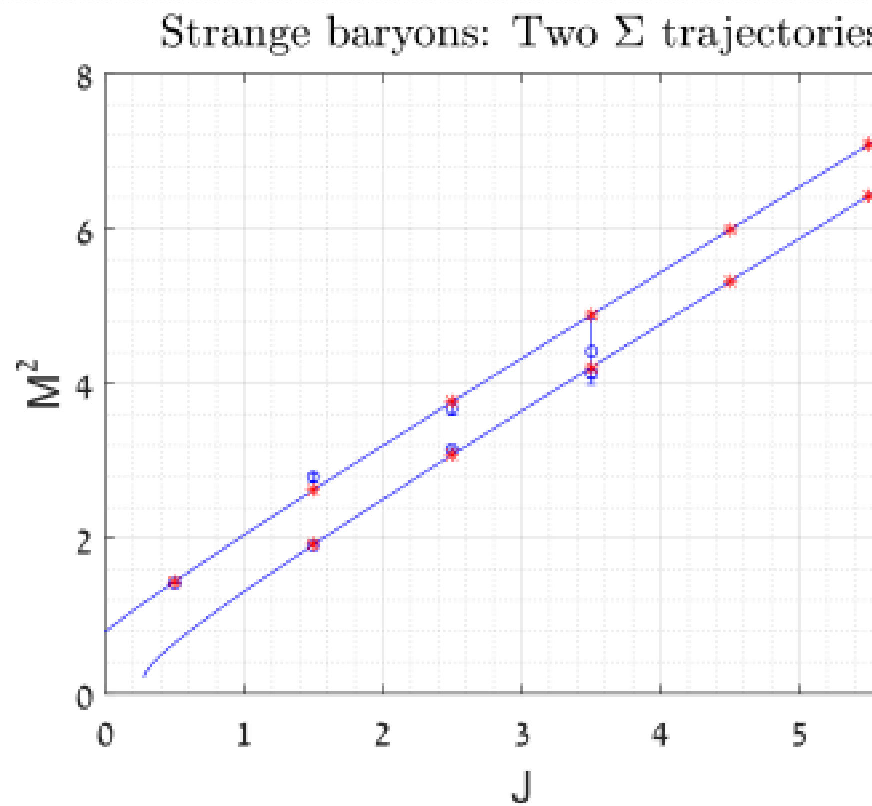
The spectra fits of Nucleons

- Trajectories for even and odd J nucleons



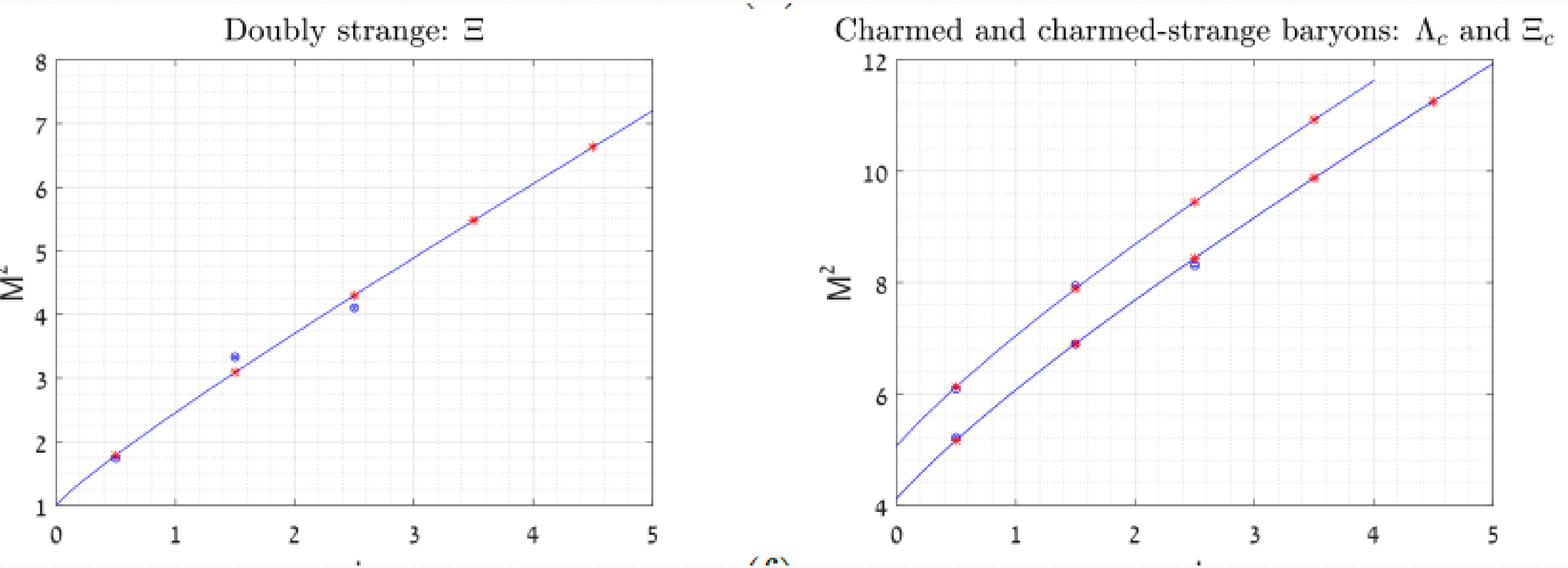
Trajectories of Λ and Σ



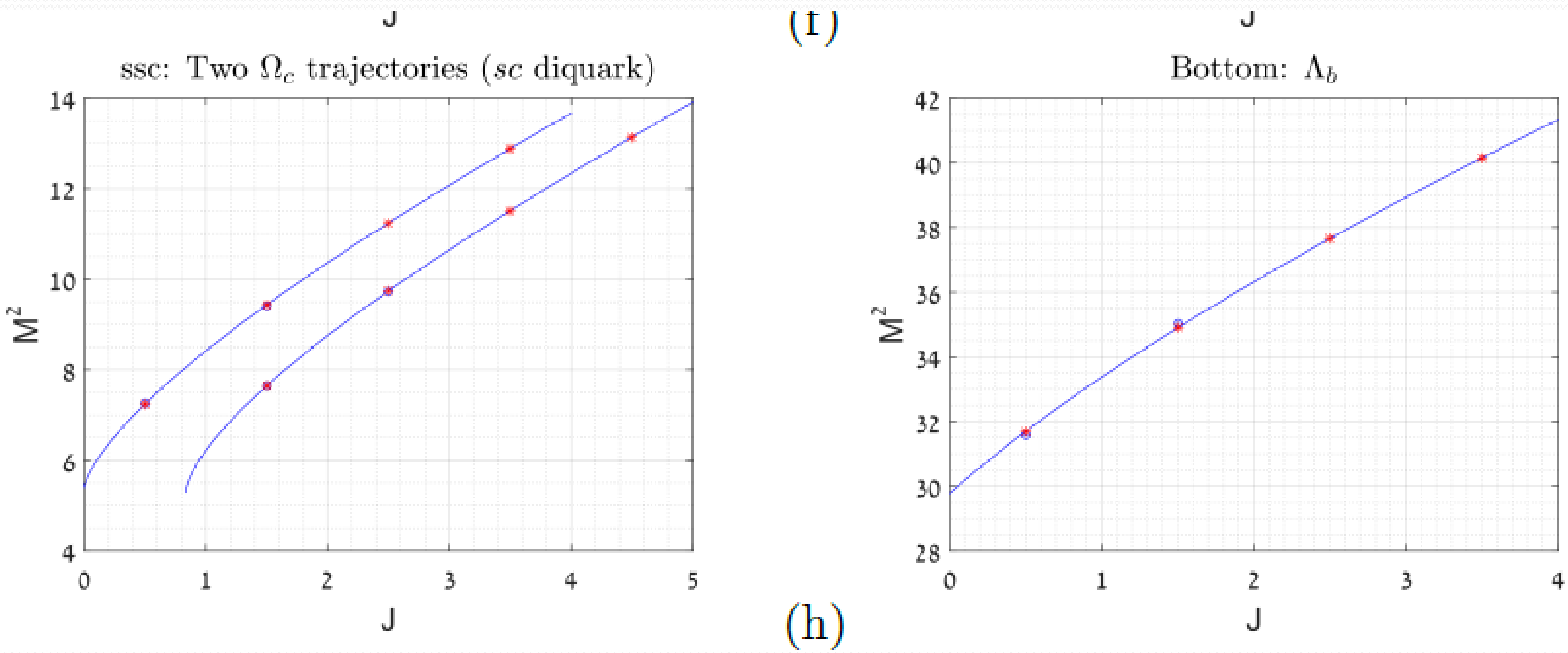


(d)

Trajectories of Ξ Λ_c and Ξ_c



Trajectories of Ω_c and Λ_b



Fit results: the total decay width of mesons

- Fits of the decay width of Mesons

$$\Gamma = \frac{\pi}{2} A T L(M, m_1, m_2, T) \,.$$

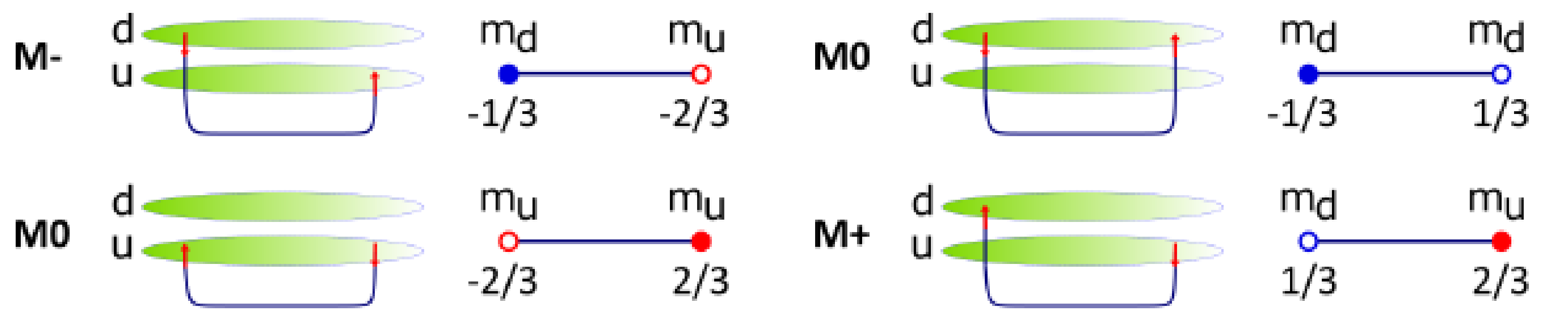
Trajectory (No. of states)		a (from spectrum)	A (fitted value)	$\sqrt{\chi^2/DOI}$
ρ	5[a]	-0.46	0.097	1.76
ω	5[a]	-0.40	0.120	2.31
ρ and ω (avg.)	6	-0.46	0.108	1.14
π	3[a]	-0.34	0.100	1.66
η	3[a]	-0.29	0.108	1.56
π and η (avg.)	4	-0.29	0.109	1.52
K^*	5	-0.25	0.098	0.77
ϕ	3	-0.10	0.074	0.50
D	2	-0.20	0.072	0.87
D_s^*	2	-0.03	0.076	1.44

Outline

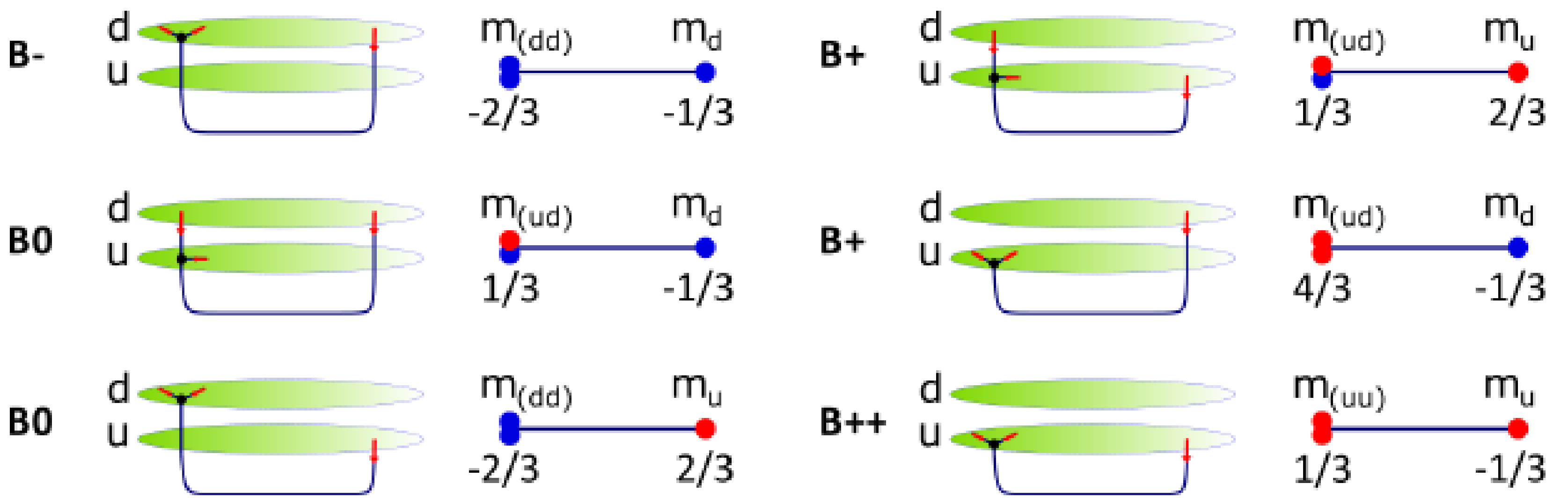
- Motivation
- Brief review of the HISH model and its fits
- Charged stringy hadrons
- Actions, equations of motion and boundary conditions
- Symmetries and conserved currents and charges
- The general mode expansion
- Classical solutions: (i) Folded rotating string in a magnetic field (ii) Stretched string in electric field.
- Canonical quantization and non-commutative geometry
- The OPE
- The energy momentum tensor and the vertex operator
- Scattering amplitude and experimental implications
- The non-critical string

Charged stringy hadrons in holography and HISH

- Mesonic strings in holography and HISH



- Baryonic strings



Charged stringy hadrons in holography and HISH

- It is important to emphasize the differences between the **hadronic strings** and the ordinary open strings.
- For the latter the spin zero state is that of a tachyon but for hadronic strings that have masses on their ends and also negative intercept it is a scalar meson
- Similarly the spin one of ordinary string is a massless gauge field and in the stringy hadron picture it is a massive vector meson.

References for strings with charges on their ends

[1] E. S. Fradkin and A. A. Tseytlin, *Nonlinear Electrodynamics from Quantized Strings*, *Phys. Lett.* **163B** (1985) 123–130.

[2] A. Abouelsaood, C. G. Callan, Jr., C. R. Nappi, and S. A. Yost, *Open Strings in Background Gauge Fields*, *Nucl. Phys.* **B280** (1987) 599–624.

[3] C. Bachas and M. Porrati, *Pair creation of open strings in an electric field*, *Phys. Lett.* **B296** (1992) 77–84, [hep-th/9209032].

[4] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032, [hep-th/9908142].

Action and equations of motion

- The action describing a stringy hadron

$$\boxed{S = S_{st} + (S_{pm} + S_{pq})\,|_{\sigma=0} + (S_{pm} + S_{pq})\,|_{\sigma=\ell}}$$

- The string action

$$S_{st} = -T\int d\tau d\sigma\sqrt{-h} = -T\int d\tau d\sigma\sqrt{\dot{X}^2 X'^2 - (\dot{X}\cdot X')^2}$$

Where $\mu,\nu = 0,\ldots D-1$. $-\infty < \tau < \infty$ and $0 \le \sigma \le \ell$.

- The endpoint actions

$$\boxed{S_{pm} = m_i\int d\tau\sqrt{-\dot{X}^2} \qquad S_{pq} = Tq_i\int d\tau A_\mu(X)\dot{X}^\mu}$$

Action and equations of motion

- One can consider the interaction between the charges by turning on
$$S \to S - \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}$$
- We consider here only the interaction with a background electromagnetic field.
- For the neutral case $q_1 = -q_2 = q$ S_{pq} can be written as a bulk action

$$\boxed{S_{sq} = -\frac{T}{2}\int d\tau d\sigma\left(qF_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu)\right)}$$

- The bulk equation of motion

$$\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu) = 0 \qquad X''^\mu - \ddot{X}^\mu = 0$$

Action and equations of motion

- The boundary conditions read

$$T X'^{\mu} + m_1 \partial_\tau \frac{\dot{X}^{\mu}}{\sqrt{-\dot{X}^2}} + T q_1 F^{\mu}{}_{\nu} \dot{X}^{\nu} = 0 \qquad \sigma = 0$$

$$T X'^{\mu} - m_2 \partial_\tau \frac{\dot{X}^{\mu}}{\sqrt{-\dot{X}^2}} - T q_2 F^{\mu}{}_{\nu} \dot{X}^{\nu} = 0 \qquad \sigma = \ell$$

- For the neutral case and with no masses

$$\boxed{X'^{\mu} + q F^{\mu}{}_{\nu} \dot{X}^{\nu} = 0 \qquad \sigma = 0,\ \ell}$$

Symmetries and conserved currents

- **World sheet reparameterization invariance**.- The corresponding energy momentum tensor is classically unaffected by the charges

$$T(z) = -\frac{1}{\alpha'}\eta_{\mu\nu}\partial X^{\mu}\partial X^{\nu}(z) \qquad \bar{T}(\bar{z}) = -\frac{1}{\alpha'}\eta_{\mu\nu}\bar{\partial} X^{\mu}\bar{\partial} X^{\nu}(\bar{z})$$

- **Weyl invariance**- Vanishing beta function requires

$$\partial^{\nu} F_{\mu}^{\lambda}\left(\frac{1}{1-q^2F^2}\right)_{\lambda\nu} = 0$$

- Obviously obeyed for constant F
- **Space-time translation**-The momenta get contributions from the boundaries

$$P_{\mu} = T\int d\sigma \dot{X}_{\mu} - TqF_{\mu\nu}X^{\nu}|_{\sigma=0} + TqF_{\mu\nu}X^{\nu}|_{\sigma=\ell}$$

- Lorentz invariance and gauge invariance

The general soltuion

- The general solution is a sum of left and right modes

$$X^\mu(\tau,\sigma) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$$

$$X^\mu_R = x^\mu_R + \alpha^\mu_0(\tau-\sigma) + i\sqrt{N}\sum_n \frac{\alpha^\mu_n}{\omega_n} e^{-i\frac{\pi}{\ell}\omega_n(\tau-\sigma)}$$

$$X^\mu_L = x^\mu_L + \tilde{\alpha}^\mu_0(\tau+\sigma) + i\sqrt{N}\sum_n \frac{\tilde{\alpha}^\mu_n}{\omega_n} e^{-i\frac{\pi}{\ell}\omega_n(\tau+\sigma)}$$

- The boundary conditions

$$\boxed{\begin{aligned} &X'^\mu + q_1 F^\mu{}_\nu \dot{X}^\nu = 0\,, \qquad && \sigma = 0 \\ &X'^\mu - q_2 F^\mu{}_\nu \dot{X}^\nu = 0\,, \qquad && \sigma = \ell \end{aligned}}$$

- In Matrix notation

$$M_1 = (1+q_1F)^{-1}(1-q_1F)$$

$$\tilde{\alpha}_n = M_1\alpha_n \quad M_2M_1\alpha = e^{2i\pi\omega_n}\alpha$$

General solution

- For the neutral case the general solution

$$\boxed{\begin{aligned} X^\mu(\tau,\sigma) &= x^\mu + \alpha_0^\mu \tau - qF^\mu{}_\nu \alpha_0^\nu(\sigma - \frac{\ell}{2}) \\ &+ i\sqrt{N}\sum_n \frac{\alpha_n^\nu}{n} e^{-i\frac{\pi}{\ell}n\tau}\left(e^{i\frac{\pi}{\ell}n\sigma}\delta_\nu^\mu + e^{-i\frac{\pi}{\ell}n\sigma}M^\mu{}_\nu\right) \end{aligned}}$$

- Example (1) Magnetic field M is a rotation

$$M = \frac{1}{1+q^2B^2}\begin{pmatrix} 1-q^2B^2 & 2qB \\ -2qB & 1-q^2B^2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \qquad \sin\alpha = \frac{2qB}{1+q^2B^2}$$

- (ii) Electric field M is a boost

$$M = \frac{1}{1-q^2E^2}\begin{pmatrix} 1+q^2E^2 & -2qE \\ -2qE & 1+q^2E^2 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \qquad \beta = \frac{2qE}{1+q^2E^2}$$

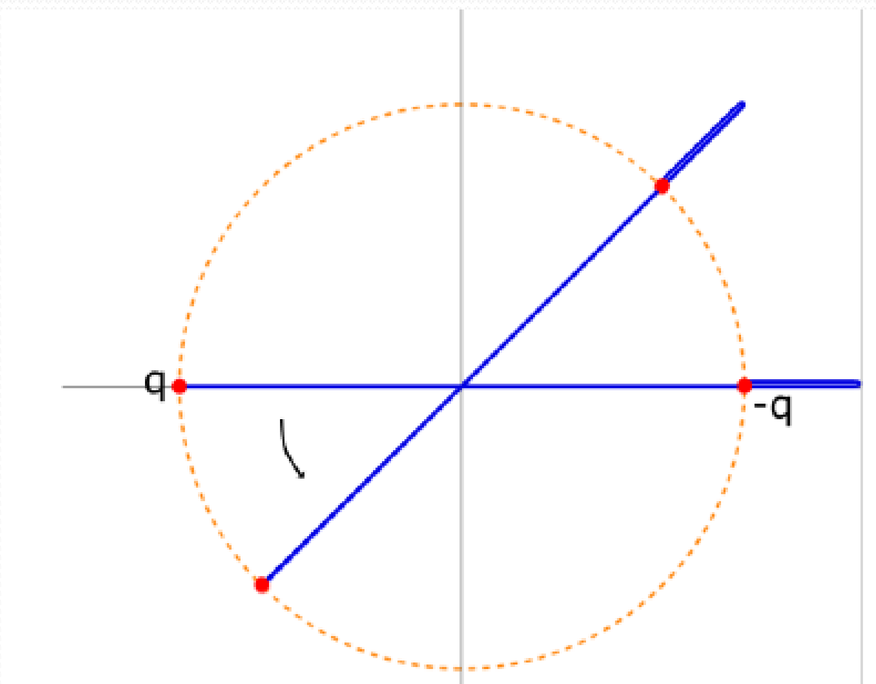
A rotating folded string in a magnetic field

- A **folded neutral string rotating** in a magnetic field

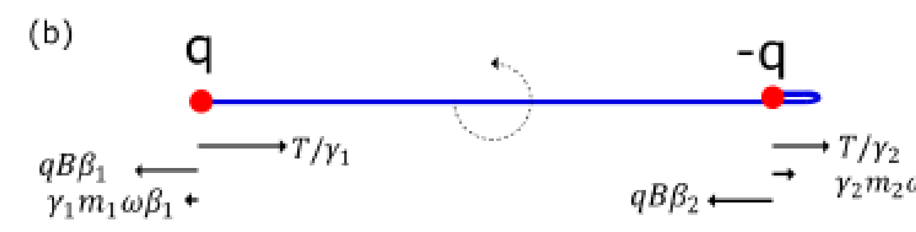
$$X^0 = e\tau \qquad X^1 = \frac{e}{\omega}\cos(\omega\sigma + \phi)\cos(\omega\tau) \qquad X^2 = \frac{e}{\omega}\cos(\omega\sigma + \phi)\sin(\omega\tau)$$

provided that $\omega = \frac{\pi}{\ell} n \qquad \phi = \arctan(qB)$

- These solutions are folded n times. For n=1 it is



when we add masses the forces are



Rotating folded string in a magnetic field

- The **classical Regge** trajectory is

$$\boxed{J = \frac{1}{2\pi T n} E^2}$$

- So the effective tension in n times the ordinary tension
- The charged endpoints move with a speed of

$$\beta = |\cos\phi|$$

- But the folding point at a speed of light. It has a divergent 2d scalar curvature which will have to be renormalize when we quantize this string

Canonical quantization

- We quantize by imposing the equal time commutators

$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = 0 \qquad [X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$$

- We impose the following algebra

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \qquad [\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n} \qquad [x^\mu, \alpha^\nu_{n\neq 0}] = 0$$

- Together with

$$[x^\mu, \alpha^\nu_0] = \frac{1}{T\ell}g^\nu{}_\rho[x^\mu, P^\rho] = \frac{i}{T\ell}g^{\nu\mu} = \frac{i}{T\ell}g^{\mu\nu}$$

- Where

$$g_{\mu\nu} = \left(\frac{1}{1-q^2F^2}\right)_{\mu\nu}$$

Non-commutative geometry

- The canonical quantization commutators hold only if and only if the zero-modes admit a **non-commutative algebra**

$$\boxed{[x^\mu, x^\nu] = i\theta^{\mu\nu} = i\frac{q}{T}F^\mu{}_\rho g^{\rho\nu}}$$

- For electric field E

$$\boxed{[x^0, x^1] = -\frac{i}{T}\frac{qE}{1-q^2E^2}}$$

- For magnetic field B

$$\boxed{[x^1, x^2] = \frac{i}{T}\frac{qB}{1+q^2B^2}}$$

The spectrum and the intercept

- Upon quantization **world sheet Hamiltonian** is

$$\frac{l}{\pi}H = \alpha' g_{\mu\nu}p^\mu p^\nu + \frac{1}{2}\sum_{n\neq 0}\eta_{\mu\nu}\alpha^\mu_{-n}\alpha^\nu_n$$

For the neutral case $w_n = n$ and $a = -\frac{D-2}{2}\sum_{n=1}^{\infty} n = \frac{D-2}{24}$

- This leads to a **spectrum of states** with

$$M^2 = -\eta_{\mu\nu}p^\mu p^\nu = \frac{1}{\alpha'}(N-a) + q^2 F_{\mu\alpha}g^{\alpha\beta}F_{\beta\nu}p^\mu p^\nu$$

- So in a magnetic field

$$M^2 = \frac{1}{\alpha'}(N-a) - \frac{q^2B^2}{1+q^2B^2}(p_1^2 + p_2^2)$$

- in electric field $M^2 = \frac{1}{\alpha'}(N-a) - \frac{q^2E^2}{1-q^2E^2}(p_0^2 - p_1^2)$

The OPE

- The propagator is the singular part of

$$\langle X^\mu(\tau,\sigma)X^\nu(\tau',\sigma')\rangle = T[X^\mu(\tau,\sigma)X^\nu(\tau',\sigma')] - :X^\mu(\tau,\sigma)X^\nu(\tau',\sigma'):$$

- The normal ordering is defined in the usual way
- After a lengthy calculation we get

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) - :X^\mu(z,\bar{z})X^\nu(w,\bar{w}): =$$

$$=[x^\mu,x^\nu] - \frac{\alpha'}{2}\left(\eta^{\mu\nu}\log|z-w|^2 + \left(\frac{1-qF}{1+qF}\right)^{\mu\nu}\log(z-\bar{w}) + \left(\frac{1+qF}{1-qF}\right)^{\mu\nu}\log(\bar{z}-$$

- On the boundary with $z = y_1,\ w = y_2$ we get

$$G^{\mu\nu}(y_1,y_2) = -\alpha' g^{\mu\nu}\log(y_1-y_2)^2 + \frac{1}{2}i\theta^{\mu\nu}(sign(y_1-y_2)+1)$$

$$g^{\mu\nu} = (\frac{1}{1-q^2F^2})^{\mu\nu} \qquad i\theta^{\mu\nu} = [x^\mu,x^\nu] = i\frac{q}{T}F^\mu{}_\rho g^{\rho\nu}$$

The boundary energy momentum tensor

- We have seen that classically the energy momentum tensor is not affected by the endpoint charges.
- QM we found out that on the boundary $z = \bar{z} = y$ the energy momentum tensor must have the form

$$T(y) = -\frac{1}{2\alpha'}(g^{-1})_{\mu\nu}{:}\partial_y X^\mu \partial_y X^\nu(y){:}$$

- So that using the boundary OPE

$${:}X^\mu(y_1)X^\nu(y_2){:} = X^\mu(y_1)X^\nu(y_2) + 2\alpha' g^{\mu\nu}\log|y_1 - y_2|$$

- One has the required OPE of T with primary fields

$$T(y_1)\mathcal{O}(y_2) \sim \frac{2h_y}{(y_1-y_2)^2}\mathcal{O}(y_2) + \frac{2}{y_1-y_2}\partial_y\mathcal{O}(y_2)$$

The vertex operator

- We take a general ansatz for the gs vertex operator

$$\boxed{V_k(y) = :e^{iv_{\mu\nu}k^\mu X^\nu}(y):}$$

- In order that this Vertex operator is a (1,1) operator under the OPE with T we must take

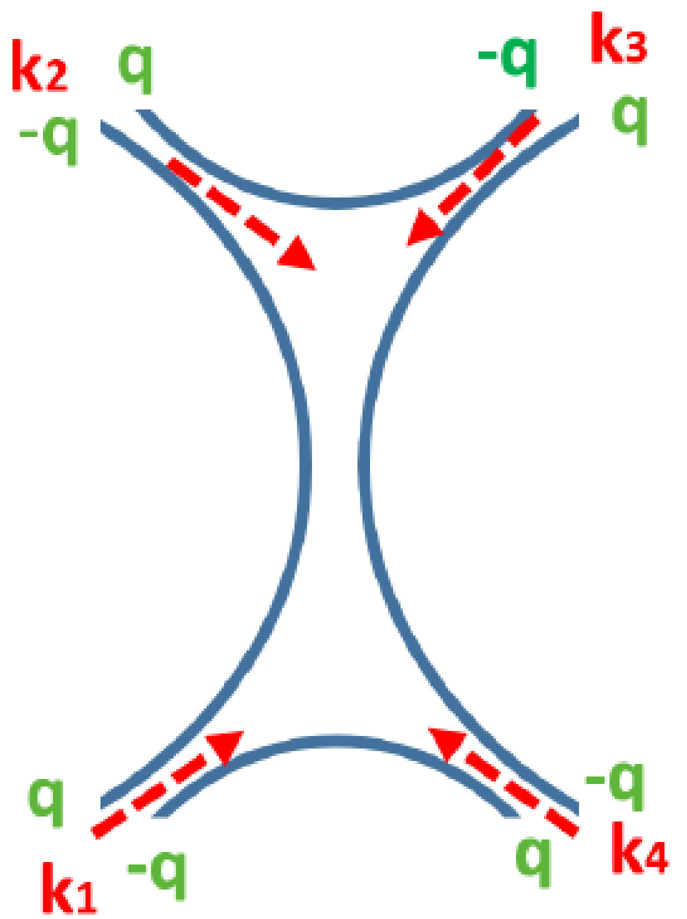
$$\boxed{v_{\mu\nu} = \eta_{\mu\nu}.}$$

- And not the modified metric $g_{\mu\nu}$
- This ensures that the VO transforms correctly also under space time translations

$$X^\mu \rightarrow X^\mu + a^\mu.$$

The scattering amplitude

- Now we would like to compute the scattering amplitude of 2->2 strings with opposite charges in their ground state



Scattering amplitude

- We use the basic OPE on the boundary to compute

$$:e^{ik_1\cdot X}(y_1)::e^{ik_2\cdot X}(y_2): \sim e^{-\frac{i}{2}\theta_{\mu\nu}k_1^\mu k_2^\nu \mathrm{sign}(y_1-y_2)}|y_1-y_2|^{2\alpha' k_1\odot k_2} :e^{i(k_1+k_2)\cdot X}(y_2):$$

- Where $a\odot b \equiv g_{\mu\nu}a^\mu b^\nu$
- The expectation value of a product of n VOs reads

$$\left\langle \prod_{i=1}^{n} :e^{ik_i\cdot X^\mu}(y_i): \right\rangle_{D_2} = \prod_{i<j} e^{-\frac{i}{2}\theta_{\mu\nu}k_i^\mu k_j^\nu \mathrm{sign}(y_i-y_j)}|y_i-y_j|^{2\alpha' k_i\odot k_j}$$

- For the four tachyon scattering

$$S_{D_2}(k_1,k_2,k_3,k_4) = \int_{-\infty}^{\infty} dy_4 \left\langle \prod_{i=1}^{3} :c^1 e^{ik_i\cdot X}(y_i)::e^{ik_4\cdot X}(y_4): \right\rangle + (k_2 \leftrightarrow k_3)$$

The scattering amplitude

- We can fix now $y_1 = 0,\ y_2 = 1,\ y_3 \to \infty$
- We integrate over y_4 and sum over the 6 cyclic ordering

$$S_{D_2}(k_1, k_2, k_3, k_4) = \int_{-\infty}^{\infty} dy_4 e^{i\Theta(y_4)} |y_4|^{2\alpha' k_1 \odot k_4} |1 - y_4|^{2\alpha' k_2 \odot k_4} + (k_2 \leftrightarrow k_3)$$

- The final **scattering amplitude** takes the form

$$S_{D_2}(k_1, k_2, k_3, k_4) = \left[e^{i\Theta_{st}} I(\tilde{s}, \tilde{t}) + e^{i\Theta_{su}} I(\tilde{s}, \tilde{u}) \right.$$

$$\left. + e^{i\Theta_{tu}} I(\tilde{t}, \tilde{u})\right] + (k_2 \leftrightarrow k_3)$$

where the modified Mandelstam variable are

$$\tilde{s} = -(k_1 + k_2) \odot (k_1 + k_2) \qquad \tilde{t} = -(k_1 + k_3) \odot (k_1 + k_3) \qquad \tilde{u} = -(k_1 + k_4) \odot (k_1 + k_4)$$

Scattering amplitude

- In terms of the beta function

$$I(\tilde{s},\tilde{t}) = B(-\alpha'\tilde{s}-1,-\alpha'\tilde{t}-1)$$

- The phases are given by

$$\Theta_{st} = \frac{1}{2}\theta_{\mu\nu}(k_1^\mu k_2^\nu - k_3^\mu k_4^\nu)$$

$$\Theta_{su} = -\frac{1}{2}\theta_{\mu\nu}(k_1^\mu k_4^\nu - k_2^\mu k_3^\nu)$$

Experimental implications

- A way to confront the theoretical results is to look for zeros of the scattering amplitude which are not zeros without the EM field.
- This is the case if

$$\cos(\Theta_{st}) = 0, \qquad \cos(\Theta_{su}) = 0, \qquad \cos(\Theta_{tu}) = 0$$

- For the st amplitude to vanish we need to obey

$$\boxed{\frac{q}{T}\frac{B}{1+q^2B^2}\left[(k_1{}^1k_2{}^2) - (k_1{}^2k_2{}^1) - (k_3{}^1k_4{}^2) + (k_3{}^1k_4{}^2)\right] = \pi}$$

- For a projectile on a fixed target

$$\boxed{\vec{k}_1 = (\tilde{k},0,0), \qquad \vec{k}_2 = (0,0,0), \qquad \vec{k}_3 = (k_x,k_y,0), \qquad \vec{k}_4 = (\tilde{k}-k_x,-k_y,0)}$$

- The condition of vanishing st amplitude is $\frac{q}{T}\tilde{k}k_y\frac{B}{1+q^2B^2} =$

Non-critical strings with endpoint opposite charges

- For non-critical long strings we use the **effective string action** of Polchinski and Strominger

$$S_{PS} = \int d\tau \mathcal{L}_{PS} = \frac{26-D}{24\pi} \int d\tau d\sigma \frac{(\partial_+^2 X \cdot \partial_- X)(\partial_-^2 X \cdot \partial_+ X)}{(\partial_+ X \cdot \partial_- X)^2}$$

- We examine this for the folded rotating string in a magnetic field
- For this case we get that the PS action diverges

$$E_{PS} = -\int_0^\ell d\sigma \mathcal{L}_{PS}(X_{rot}) = \frac{26-D}{24\pi}\omega \int_0^\pi dx \cot^2(x+\phi)$$

- Due to the fact that the folding point moves at the speed of light.
- We can regulate this by adding a mass at the fold

Non-critical strings with endpoint opposite charges

- The PS energy is then

$$E_{PS} = \frac{26-D}{12\pi}\left(\frac{4T}{\gamma m} - \frac{4(\arcsin\beta)^2}{\tilde{L}}\right)$$

- As we did for strings with massive endpoints we now subtract the result for an infinitely long string to find that

$$a_{PS} = -\frac{1}{\omega}E_{PS}^{(ren)} = \frac{26-D}{24}$$

- So the total intercept is

$$a = \frac{D-2}{24} + \frac{26-D}{24} = 1$$

Comment on the non-critical scattering amplitude

- The boundary vertex operator for a tachyon in non-critical dimensions was discussed by Hellerman et al
- The corresponding VO reads

$$\boxed{V_k = :e^{i\eta_{\mu\nu}k^\mu X^\nu} e^{\gamma\varphi}(y):}$$

- Where $\varphi = -\frac{1}{2}\log\left(\partial_\alpha X^\mu \partial^\alpha X_\mu\right)$ is the Liouville mode and gamma is determined by the requirement of an appropriate OPE with T.
- At leading order

$$\boxed{V_k = :e^{i\eta_{\mu\nu}k^\mu X^\nu}\left(\partial_\alpha X^\mu \partial^\alpha X_\mu\right)^{-\alpha' k\odot k+1}(y):}$$

- But for the tachyon $\alpha' k \odot k = 1$ so **no dressing**.

Future directions

- The case of a charged string
- The non-commutative Poincare algebra
- The scattering amplitude for charged strings
- Zeros of the scattering amplitudes for protons in EM field.
- The quantization and renormalization of folded strings
- The scattering of string with charges and masses on its endpoints